# Dynamics of water droplets on a window pane

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We performed experiments and computer simulations in order to investigate whether the rain drops on a window pane are in a "self-organized critical state," as proposed by Bak, Tang, and Wiesenfeld [Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); J. Stat. Phys. **51**, 797 (1988)]; and Tang and Bak [Phys. Rev. Lett. **60**, 2347 (1988)]. In contrast to the expected behavior, we found that the coverage fluctuations exhibit an $f^{-2}$ power spectrum, and the characteristic distribution functions of the elementary events have no power-law dependence.

## I. INTRODUCTION

Recently Bak, Tang and Wiesenfeld[1–4] (BTW) introduced the idea of "self-organized criticality" in terms of a model of how they expected some dynamical systems to behave. These extended dissipative dynamical systems evolve into structures with long-range fractal spatial correlations[5,6] and/or long-range temporal correlations with $1/f$ power spectrum.[7] They have suggested that this behavior may be caused by the self-organization of the systems into a critical state. Systems that exhibit spatial or temporal power-law correlations naturally evolve into this critical state.[2] Tang and Bak[4] have pointed out the analogy with traditional critical phenomena by defining several critical exponents and deriving scaling relations between them. Hwa and Kardar[8] have performed a dynamic renormalization-group calculation to determine various critical exponents in $d \leq 4$ dimensions. They have found slightly different noise exponent values for the energy dissipation function defined by BTW in Ref. 1, and they have pointed out an important principle, namely that the dynamics should satisfy a conservation law to ensure the self-similarity of the steady-state configurations.

Several real systems have been suggested as possible candidates for this behavior, for example, the sand flow in an hour glass, properties of earthquakes,[9] raindrops running down a window pane, the flows of rivers such as the Nile, motion of dislocations in a resistor, or even interactive economical systems.[1,2] Jaeger, Liu, and Nagel[10] have performed experiments to describe the nature of sand flow along the free surface of sandpiles. In contradiction to the above-mentioned model proposed also for the dynamics of granular systems, they did not observe power-law distributions of elementary events, and they found a simple model to explain their results.

Motivated by the above developments we have investigated another possible system: water droplets running down a window pane. It is an everyday observation that droplets and streams of rain on window panes tend to evolve into a stationary state. This means that the average covering of the glass is seemingly almost independent of the "rain-power." When a little drop falls onto the dry pane, it may stick to the glass. If the mass of the droplet is larger than some critical value, or joining to another drop the mass exceeds the critical value, a stream runs down. This stream or "water avalanche" may stop when the window is dry enough because there remains a thin water film behind the running drop, so the mass of the stream continually decreases. More often the flow meets other droplets and "eats" them, so the running continues across the full pane.

Our aim was to collect some quantitative information about the dynamical behavior of this system, therefore we have performed experiments and numerical simulations. Section II will be concerned with the experiments, the results will be presented in Sec. III. In Sec. IV we give a simple model for the dynamical behavior of rain droplets. Finally, in Sec. V, we give a short discussion of our results. We use the notations of BTW (Ref. 1-4) in this paper.

## II. EXPERIMENTS

Our experimental arrangement consisted of an optically smooth silicate glass plate which was 70 cm wide and 100 cm long. The slope of the pane was adjustable between 40° and 90°. We have sprayed destilled water droplets onto the glass from $\approx 1$ m distance on a ballistic orbit. The discharge of the water was pulsed with a frequency of 1–2 Hz, so the system had enough time to relax between the successive charges. Behind the "window" a videocamera was used to record the events. Dark background and light sources were applied at the edge of the glass pane, so the droplets gave sharp contrasts on the screen. To evaluate the recorded result the VITAL© image processing system was used. A typical digitalized output is shown in Fig. 1.

Figure 2 shows the distribution of the droplet size. We probed "rains" with different average size of droplets. If the mean value was essentially smaller than $\approx 1$ mm (Fig. 2), the image processing became difficult because of the

 

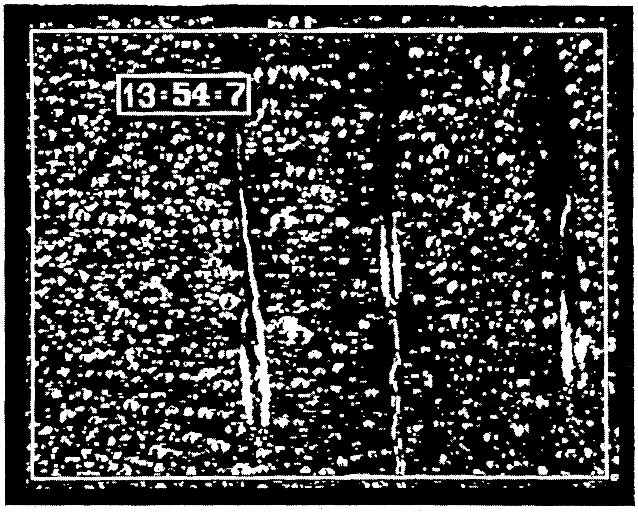


FIG. 1. Typical digitalized output from the image-processing system. The double frame indicates the area, which was the basis of the coverage calculation.

limited resolution of the video camera. Using much bigger droplets, almost all of them immediately ran down. The density of the spray was fixed about 1 drop/$cm^2$.

Sticking, spreading, and flowing of liquid droplets on solid surfaces often appears to be a complicated process, where many (not necessarily known) parameters play a role.[11] During the spreading or moving of a droplet even traces of impurities in the phases may drastically change the observed phenomena. The roughness of the solid surface is also of importance. The interaction between the liquid and solid is characterized by the contact angle $\theta$ of the drop[11] resting on the horizontal plate.

Different qualities of glass panes were probed during the experiments. Finally, a silicate glass was used that could be characterized by the contact angle $\theta \approx 75°-85°$. In this case the remaining water film behind the running drop splitted up into small drops in a very short time, so there was no macroscopically visible wet between the individual drops. With other type of glasses (with characteristic contact angle $\theta \approx 40°-45°$), the behavior of the streams considerably changed. The lifetime of the meandering water films were much longer (up to 1–2 min) because of the good wetting, so any small droplet ran down if falling onto this route.

At the next step we examined the effect of the pane slope on the dynamics. We fixed the angle of incidence of droplets nearly perpendicular to the surface of the glass. The critical size of the sticking droplets and the coverage of the glass slightly changed, nevertheless, the main features of the dynamics were found independent of the slope, at least between 40° and 90°. For example, the average velocity of the running droplets seemed to be independent of the inclination angle. So this angle was fixed at 70°.

A dry and clean surface was carefully prepared for each experiment. We recorded several experiments, each lasted about 15 min. After the recording we evaluated the results by the computer-aided image-processing system.

## III. EXPERIMENTAL RESULTS

The idea of self-organized criticality rests upon the assumption that the coverage of the surface has a critical value. If the coverage is increased continuously (e.g., by adding more water droplets to the window pane), the system will organize itself in such a way that its average coverage will be the critical value by unloading excess water through streams. The theory of BTW predicts a self-organized state at this critical coverage characterized by long-range spatial and temporal correlations and giving rise to a typical $1/f$ power spectrum of the fluctuations around the steady-state coverage value.

We evaluated the coverage of the window pane at the steady state in every $\frac{1}{10}$ sec. Figure 3 shows the typical time trace of the fluctuations. The corresponding power spectrum (Fig. 4) was obtained by Fourier transforma-

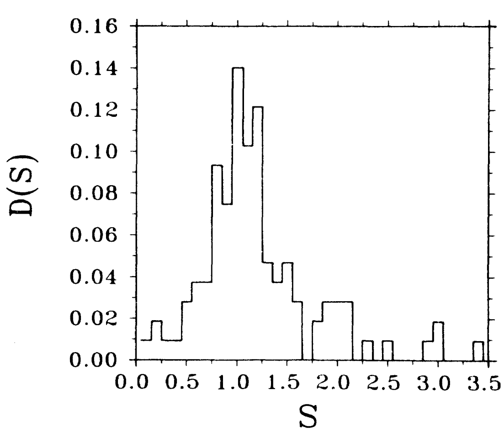


FIG. 2. Distribution of the droplet size. The $x$ axis shows the diameters of the contact area of the droplets resting on the horizontal glass pane (in mm).

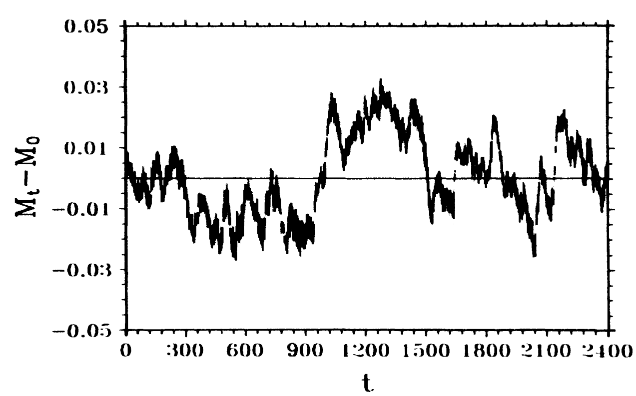


FIG. 3. Characteristic time trace of the coverage fluctuations around an average value. $M_0 = 0.8123$. The time is measured in $\frac{1}{10}$ sec units.

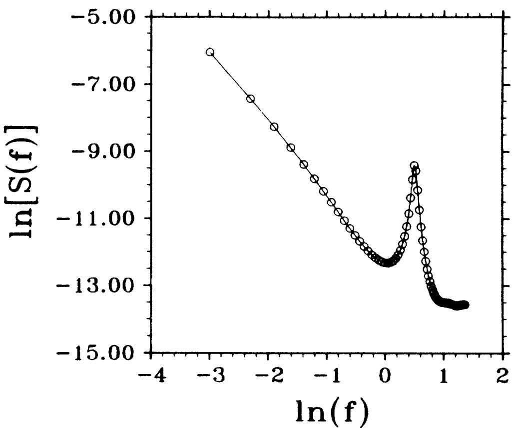


FIG. 4. Calculated power spectrum of the noise plotted in Fig. 3.

tion.[12] The small-frequency part of the power spectrum in the log-log plot has the free surface of sandpiles. In contradiction to the slope $\Phi=-2.15\pm0.09$, which significantly differs from the expected nearly $1/f$ behavior. The narrow peak at the higher-frequency part of the spectrum is trivially caused by the experimental setup, namely by the pulsed water discharge with 1–2-Hz frequency. The highest observable frequency component was 5 Hz as follows from the $\frac{1}{10}$-sec time resolution of the video camera and from the Debye condition.[12] The lowest observable frequency is determined by the recording time. Considering the log-log scale of Fig. 4, we have concluded that the overall shape of the power spectrum is nearly $f^{-2}$, at least in the $10^{-3}$–$10^{0}$ Hz frequency range. Changing the slope of the window pane, the mean coverage of the glass changed between 65–80 %, but the power spectrum of the fluctuations remained $f^{-2}$ in all cases. We would like to point out that the Fourier spectrum of a random walker is $A(f)\propto f^{-H-1/2}$ where $H=\frac{1}{2}$ for ordinary Brownian motion.[7] The power spectrum of this distribution is same as the power spectrum of the coverage fluctuations described above. Nevertheless, the time average of the last one must be less than 1. It follows from Parseval's theorem that this is possible only if the power spectrum is normalizable. It means that a characteristic corner frequency must exist for all $f^{-\alpha}$ power spectra, where $\alpha\geq 1$. Below this frequency the spectrum must turn over to a regime with an exponent less than one. Since the recording time determines the lowest observable frequency, we must observe this crossover when the recording time is longer than the system relaxation time. We can estimate this time using the discrete Parseval formula applied to our data and we find that the crossover time is approximately 39 hours.

The question naturally arises why we have chosen the coverage instead of the mass. It is clear that the sticking and the motion of a droplet is determined by two main factors: gravity and the interaction between the liquid and the surface of the solid. The gravitational force is proportional to the droplet mass $m$, the cohesion is proportional to the area of the interface layer $F$. These two quantities are not independent of each other, however, an exact functional interdependence is not known. It is because of the existence of the contact angle hysteresis[13] and the absence of self-similarity of the different size droplets. However, an approximate relation may be valid for small $m$:

$$m\propto F^{\alpha}\ , \tag{1}$$

where $1<\alpha\leq\frac{3}{2}$ ($\alpha=1$ is the "flat" droplet limit, while $\alpha=\frac{3}{2}$ gives the hemisphere shape). We can estimate simply the mass fluctuations from the measured contact area fluctuations. When we rescaled the measured contact area fluctuations according to (1), there was a slight slope change in the small-frequency part of the power spectrum, but the value of the exponent remains very close to $\Phi=-2$.

From another point of view, BTW claimed that the main fluctuating quantity that would exhibit $1/f$ power spectrum is the so-called energy dissipation function.[2] During flow of a stream energy dissipation occurs at the border of the drop and the air, in the bulk of the droplets because of the viscosity, but the dominant part comes from the contact surface between the drop and the glass caused by the friction. Consequently, the energy dissipation is directly related to the changes of coverage. A good estimation of the last one is the time derivative of the coverage. For the random-walk regime we have obtained a spectrum corresponding to $f^{0}$, with good agreement with the expected behavior.

## IV. MODELING

In order to simulate the $f^{-2}$ power spectrum behavior we have constructed a simple, discrete model. The algorithm of our model is as follows. An $n\times n$ square lattice is first established and lattice sites are allowed to be occupied by "drops" of different size,

$$0\leq m_{i,j}\leq m_c^s\ , \tag{2}$$

where $i,j$ denotes the position of the site, $m_{i,j}$ is some characteristic quantity of the drop (mass or contact surface area), and $m_c^8$ is the static critical value depending on the "slope." At each step one position is randomly chosen and a unit mass is added to the existing drop,

$$m_{i,j}\rightarrow m_{i,j}+1\ . \tag{3}$$

If the mass of the $(i,j)$th resting "droplet" exceeds the static critical value $m_c^s$, a "stream" runs down.

It is a well-known fact that streams of rain on window panes tend to meander even on seemingly smooth and clean surfaces. The causes of meandering may be some microscopical impurities on the surface or in the liquid, and instabilities in the stream.[14] If the window is densely covered with droplets, the effect of instabilities is negligible because of the short "mean-free path" between the individual droplets. I this case the meandering occurs mostly because of the meetings of the stream with the standing droplets. If the "collision" is noncentral, which means that the direction of the flow is not identical with the vertical symmetry axis of the standing drop, the steam will deviate from the vertical flowing direction toward the droplet. However, the dynamics of this joining and bending may be difficult, so the meandering behavior can be built into the model in a much simpler way.

The stream algorithm is the following. If the avalanche reaches the $(i,j)$th position, the mass of the stream will increase with the mass of the droplet, but a little amount of water will remain on the site modeling the wetting phenomenon,

$$m_{i,j} \rightarrow m_0 \ , \tag{4a}$$

$$M \rightarrow M + m_{i,j} - m_0 \ , \tag{4b}$$

where $M$ denotes the mass of the avalanche and $m_0$ is the remaining mass. At the next step the $y$ coordinate (or the column index) will decrease modeling the vertical motion, but the $x$ coordinate (the row index) may change due to the meandering phenomenon. This deviation depends on the size of the bottom droplets, the stream will bend toward the largest drop,

$$\begin{aligned} &i \rightarrow i-1 \ , \\ &j \rightarrow j' : m_{i,j'} \ , \\ &m_{i,j'} = \max(m_{i,j-1}, m_{i,j}, m_{i,j+1}) \ . \end{aligned} \tag{5}$$

When the masses of the bottom droplets are equal, the vertical direction is preferred. These steps are repeated until the stream reaches the bottom of the window pane, or the mass of the avalanche becomes smaller than the "dynamical critical value" $m_c^d$. In the latter case the running of the drop can be stopped by the collective effects of the friction and the decrease of the moving mass. The dynamical critical value is much smaller than the static one, but we could not find any well-established estimation of this quantity. The overall effect of this event is negligible because of the relatively dense coverage as it can be checked easily by looking at a rainy window pane.

We performed simulations on $n = 20,30,50,100$ size square lattices. The boundary condition was cyclic in the horizontal direction. We found that the results were not sensitive to the size of the system or to the value of the $m_c^s$ critical mass.[15] Figure 5 shows the power spectrum of the simulated coverage fluctuations around the average value. The slope of the straight line in the log-log plot is $s = 1.91 \pm 0.006$, which approximates well the measured behavior. The peak in the high-frequency range is trivially absent because the discharge of "water" was continuous in the model.

We evaluated the characteristic distribution functions of the elementary events. Figure 6 shows the stream mass distribution, which exhibits a peaked shape around an average value. This curve is similar to the sand avalanche mass distribution measured by Jaeger, Liu, and Nagel.[10] The reason for the asymmetry can be understood on the basis of the running time distribution (Fig. 8) and will be explained later.

In contrast to the sand flow case,[10] the distribution of the waiting time between the successive events exhibits an exponential behavior (Fig. 7),

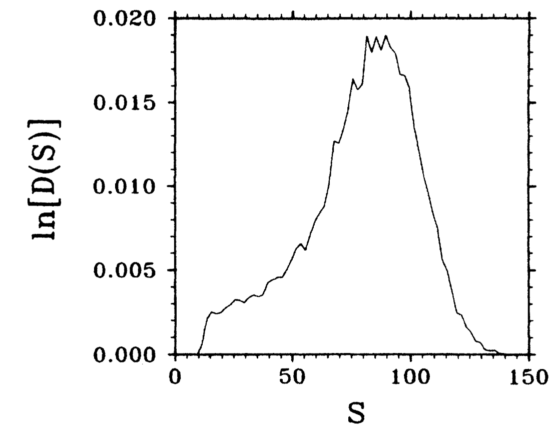


FIG. 6. Distribution of the stream masses. $n = 20$, $m_c = 10$, the number of the MC steps is 1 500 000.

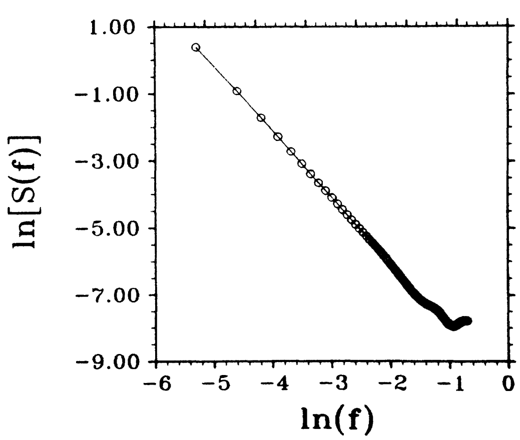


FIG. 5. Calculated power spectrum of a simulated coverage fluctuation. $n = 30$, $m_c = 10$, the number of the Monte Carlo (MC) steps is 1 500 000 (in arbitrary units).

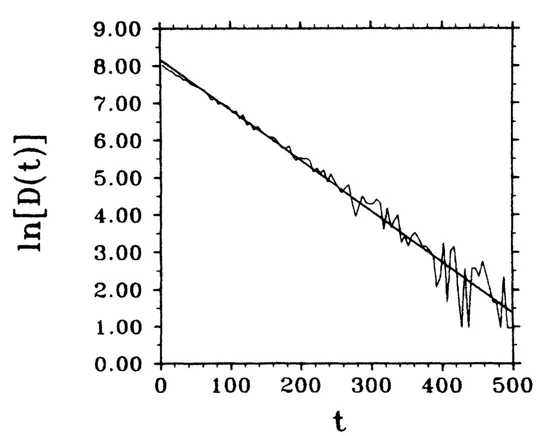


FIG. 7. Distribution of the waiting time between the successive streams. $n = 30$, $m_c = 10$, the number of MC steps is 1 000 000. The $y$ axis has a logarithmic scale, $t$ is in MC step unit. The slope of the straight line is $a = 0.0012 \pm 0.0002$.

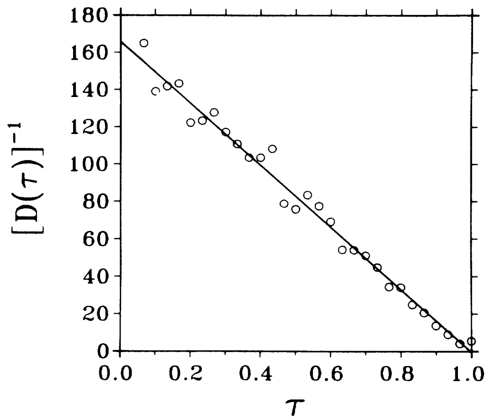


FIG. 8. Inverse frequency of the stream life time. $n=30$, $m_c=10$, the number of MC steps is 1 500 000. The $x$ axis is in $\tau_{max}$ unit, $c=0.0062$ (see text).

$$D(t) \propto \exp(-at) , \quad (6)$$

where the value of the coefficient is $a=0.0012\pm0.0003$ on an $n=30$ lattice. Simple considerations give $a=1/n^2$, i.e., the time constant of this exponential distribution is equal to the probability per unit time for a droplet to arrive at a critical site. This behavior is related to the full independence of the individual events.

A very interesting result was obtained for the distribution of the "cluster lifetime," that is, for the running time of the streams. We supposed in the model that the average velocity of the running droplets is constant and does not depend strongly on the slope of the window. During the flow the discharge of the water was interrupted, letting the system relax. Figure 8 shows the singular distribution of the lifetime

$$D(\tau) \propto \frac{c}{\tau_{max}-\tau} , \quad (7)$$

where $\tau_{max}$ is the largest possible lifetime in the finite-size system and $c$ is a slightly size-dependent constant. The transformed form of this distribution shows that this approximation is valid even in the small $\tau$ range. This behavior is only qualitatively understandable. Because of the constant velocity assumption the probability of the $\tau_l$ running time is proportional to the height of the originating place of the stream. The nature of the flow assures that the higher origins have larger probabilities because from the $0<k\leq l$ sites the water flows down along the stream route. But the form of the distribution is unusual. It is certain that no real distribution has this form as follows, for example, from the normalization condition. We plan large-scale simulations and analytical calculations to clear the origin of this behavior.[15]

In terms of this distribution we can explain the mass distribution of the streams (Fig. 6). Since the probabilities of the longer flows are larger, the average mass shifts toward the maximal possible value. The steep cutoff of the right-hand side of the peaked curve is a boundary effect, and it occurs in all finite-size systems.

## V. DISCUSSION

We performed experiments and computer simulations to investigate the dynamics of raindrops running down a window pane. From the experiments we can conclude that the coverage fluctuates with $f^{-2}$ power spectrum. We found a very simple model to simulate this behavior, and the results approximate well the observed behavior. The distribution of the stream size has an asymmetrical peaked shape. The waiting time between the successive events has an exponential form, which shows the absence of correlations. The avalanche lifetime distribution may be approximated by an unusual singular functional form whose explanation requires further investigations. The stream mass and the stream lifetime distributions cannot be approximated by a power law, consequently, no scaling relations are valid in this case. In contrast to the case of the sand pile, the kinetic friction is rather large in this model (the droplets do not accelerate, and the velocities are quite small). So in this case, the deviation from the power-law dependence cannot be explained by the smallness of kinetic friction.[9]

In conclusion, the $f^{-2}$ power spectrum of the coverage fluctuations we obtained shows that the dynamics of our system is different from that proposed by BTW. In order to find a real physical system in a self-organized critical state, it is primordial to verify the power-law dependence of the distribution of the elementary events rather than the resulting fractal structure or the $1/f$ noise.


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