Quantum theory of propagation of elliptically polarized light through a Kerr medium

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We consider quantum-mechanically partially polarized light propagating through a Kerr-like medium. Using the usual form of the induced polarization $\mathbf{P}=A(\mathbf{E}\cdot\mathbf{E}^*)\mathbf{E}+B(\mathbf{E}\cdot\mathbf{E})\mathbf{E}^*$, the theory is formulated in terms of an effective Hamiltonian which is quartic in terms of the operators for two orthogonally polarized modes. Exact solutions in closed form for the Heisenberg equations of motion are obtained. These solutions are used to evaluate the physical behavior of various observables as the field propagates through a nonlinear medium. We also present explicit results for the time evolution of the input coherent and Fock states of the field. We show the generation of states that are macroscopic superposition of coherent states. We also find that if the input field is completely polarized, then due to quantum effects the output field becomes partially polarized. This is in contrast to the classical prediction and can have an important bearing on questions like topological phases of light propagating through a nonlinear medium. Numerical results for the energy in each mode, the correlation between two modes, and the higher-order correlations are presented. The input photon statistics is found to make a considerable difference in the dynamics.

I. INTRODUCTION

Propagation of light through a Kerr medium has been very extensively studied. Most of the work concerns the classical behavior of the field amplitudes as the light propagates through the nonlinear medium. The quantum fluctuations in the field amplitude have also been examined[1] in the context of the generation of squeezed light. What one does is, first, obtain the classical solution and then study *small* quantum-mechanical fluctuations around classical solutions. It is, of course, desirable to have a fully quantized theory so that one need *not* consider the limit of small fluctuations. This is, in fact, the purpose of this paper. We show that one-dimensional propagation can be formulated in terms of an effective Hamiltonian involving the two polarization modes. We further show that this Hamiltonian can be solved exactly to obtain field dynamics. It may be added that the case of a single mode through a Kerr medium has been analyzed quantum mechanically.[2-6] Our analysis differs from the existing one since we properly treat the changes in the polarization characteristics as the field propagates through the Kerr medium. The outline of this paper is as follows. In Sec. II we consider classically one-dimensional propagation of elliptically polarized light through a Kerr medium. If the field is written as a superposition of two circular components, then classically each circular component acquires an intensity dependent phase which is different for the two components. In Sec. III, we quantize the field and find an effective Hamiltonian that characterizes the interaction between two polarization modes a and b. Explicit time-dependent solutions for Heisenberg operators a and b are given. In Sec. IV we present the time-dependent states of the field for two different initial states of the field. In Sec. V we evaluate various physical observables and several aspects of photon statistics. We present numerical results for (i) the mean number of photons in the two modes and (ii) fluctuations in photon numbers. Finally in Sec. VI we examine the changes in the degree of polarization of light as it propagates through the Kerr medium.

II. SUMMARY OF CLASSICAL RESULTS FOR LIGHT PROPAGATION THROUGH FIBER

The induced polarization in a Kerr medium can be written in the form[7]

$$\mathbf{P}=\chi\mathbf{E}+A(\mathbf{E}\cdot\mathbf{E}^*)\mathbf{E}+B(\mathbf{E}\cdot\mathbf{E})\mathbf{E}^* , \quad (2.1)$$

where χ is the linear susceptibility of the medium and A and B characterize the nonlinearity of the medium. We consider one-dimensional propagation and thus express the electric field in the form

$$\mathbf{E}=(\varepsilon_1\hat{\mathbf{x}}+\varepsilon_2\hat{\mathbf{y}})e^{i(kz-\omega t)}, \quad k=(\omega/c)\sqrt{1+4\pi\chi} , \quad (2.2)$$

where ε_1 and ε_2 are the components of the field envelope. The Maxwell equations in slowly varying approximation lead to the following system of coupled differential equations.[8]

$$\frac{\partial\varepsilon_1}{\partial z}=\frac{2\pi i\omega^2}{c^2k}[A(|\varepsilon_1|^2+|\varepsilon_2|^2)\varepsilon_1+B(\varepsilon_1^2+\varepsilon_2^2)\varepsilon_1^*] , \quad (2.3)$$

$$\frac{\partial\varepsilon_2}{\partial z}=\frac{2\pi i\omega^2}{c^2k}[A(|\varepsilon_1|^2+|\varepsilon_2|^2)\varepsilon_2+B(\varepsilon_1^2+\varepsilon_2^2)\varepsilon_2^*] . \quad (2.4)$$

In order to solve these equations, it is convenient to go to a circular basis

$$\varepsilon_\pm=\frac{\varepsilon_1\pm i\varepsilon_2}{\sqrt{2}} ,$$
$$|\varepsilon_1|^2+|\varepsilon_2|^2=|\varepsilon_+|^2+|\varepsilon_-|^2 , \tag{2.5}$$
$$2\varepsilon_+\varepsilon_-=\varepsilon_1^2+\varepsilon_2^2 .$$

The equations for the amplitudes $\varepsilon_\pm$ are easily obtained

$$\frac{\partial\varepsilon_+}{\partial z}=i\beta[A(|\varepsilon_+|^2+|\varepsilon_-|^2)+2B|\varepsilon_-|^2]\varepsilon_+ , \tag{2.6}$$

$$\frac{\partial\varepsilon_-}{\partial z}=i\beta[A(|\varepsilon_+|^2+|\varepsilon_-|^2)+2B|\varepsilon_+|^2]\varepsilon_- ,$$
$$\beta=\frac{2\pi\omega^2}{kc^2} . \tag{2.7}$$

Clearly $|\varepsilon_+|^2$ and $|\varepsilon_-|^2$ do not change with z and thus

$$\varepsilon_\pm(z)=\varepsilon_\pm e^{i\beta z\Phi_\pm} ,$$
$$\Phi_\pm=A(|\varepsilon_+|^2+|\varepsilon_-|^2)+2B|\varepsilon_\mp|^2 . \tag{2.8}$$

The Cartesian components of the field can be obtained by combining (2.5) and (2.8) and can be written in the matrix form as

$$\begin{bmatrix}\varepsilon_1(z)\\ \varepsilon_2(z)\end{bmatrix}=\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}^{-1}\begin{bmatrix}e^{iz\Phi_+\beta} & 0\\ 0 & e^{iz\Phi_-\beta}\end{bmatrix}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}\begin{bmatrix}\varepsilon_1(0)\\ \varepsilon_2(0)\end{bmatrix} , \tag{2.9}$$

i.e.,

$$\begin{bmatrix}\varepsilon_1(z)\\ \varepsilon_2(z)\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^{i\phi_+z\beta}+e^{i\phi_-z\beta} & i(e^{i\phi_+z\beta}-e^{i\phi_-z\beta})\\ -i(e^{i\phi_+z\beta}-e^{i\phi_-z\beta}) & e^{i\phi_+z\beta}+e^{i\phi_-z\beta}\end{bmatrix}\begin{bmatrix}\varepsilon_1(0)\\ \varepsilon_2(0)\end{bmatrix} . \tag{2.10}$$

Note that we can rewrite

$$\Phi_\pm=(A+B)(|\varepsilon_+|^2+|\varepsilon_-|^2)\mp B(|\varepsilon_+|^2-|\varepsilon_-|^2) . \tag{2.11}$$

It is clear that if z is chosen such that

$$\sin[\beta Bz(|\varepsilon_+|^2-|\varepsilon_-|^2)]=0 , \tag{2.12}$$

then

$$\begin{bmatrix}\varepsilon_1(z)\\ \varepsilon_2(z)\end{bmatrix}=\exp[i\beta(A+B)(|\varepsilon_+|^2+|\varepsilon_-|^2)z]\begin{bmatrix}\varepsilon_1(0)\\ \varepsilon_2(0)\end{bmatrix} , \tag{2.13}$$

where z is given by (2.12). Thus, for values of z given by (2.12), the field distribution returns to the original value except for an overall phase factor. This overall phase factor is important in considerations of Berry's phase.[9]

In general, when the field is treated classically its intensity at some point in the medium can be obtained from (2.10), for example

$$|\varepsilon_1(z)|^2=\tfrac{1}{2}\{|\varepsilon_1(0)|^2+|\varepsilon_2(0)|^2+\cos(\Phi z\beta)[|\varepsilon_1(0)|^2-|\varepsilon_2(0)|^2]+2|\varepsilon_1(0)||\varepsilon_2(0)|\cos(\theta)\sin(\Phi z\beta)\},\quad e^{i\theta}=\alpha\beta^*/|\alpha||\beta| , \tag{2.14}$$

where $\Phi=4B|\varepsilon_1(0)||\varepsilon_2(0)|\sin(\theta)$. Thus, in this case, the field exhibits simple sinusoidal oscillations except when it is circularly polarized.

Finally the interaction energy between the polarization and the electric field can be written as

$$H=-\int^E \mathbf{P}\cdot\delta\mathbf{E} . \tag{2.15}$$

This interaction energy enables us to write the effective quantum-mechanical Hamiltonian in the next section.

III. QUANTUM THEORY: BASIC HAMILTONIAN AND SOLUTIONS OF HEISENBERG EQUATIONS FOR FIELD OPERATORS

In order to describe the light propagation through a Kerr medium, we express the electric field operator E in the form

$$E^{(+)}(z,t)=i\left[\frac{2\pi\hbar\omega}{n^2V}\right]^{1/2}(a\hat{\mathbf{x}}+b\hat{\mathbf{y}})e^{i(kz-\omega t)} ,$$
$$k=\frac{\omega}{v}=\frac{\omega}{c}n , \tag{3.1}$$

where n is the linear refractive index, V is the quantization volume. The localized annihilation and creation operators obey the commutation relations

$$[a,a^\dagger]=[b,b^\dagger]=1,\quad [a,b]=[a,b^\dagger]=0 . \tag{3.2}$$

The interaction Hamiltonian can now be written in the form

$$H=\hbar\frac{p}{2}[(a^\dagger)^2+(b^\dagger)^2](a^2+b^2)+\hbar\frac{q}{2}:(a^\dagger a+b^\dagger b)^2: , \tag{3.3}$$

where : : stand for the normal ordering of operators. The parameters $p(q)$ are proportional to the coefficient $B(A)$ in Eq. (2.1). The Heinsenberg equations of motion for a and b are easily found

$$\dot{a} = -ipa^{\dagger}(a^2+b^2) - iq(a^{\dagger}a + b^{\dagger}b)a \;, \tag{3.4}$$

$$\dot{b} = -ipb^{\dagger}(a^2+b^2) - iq(a^{\dagger}a + b^{\dagger}b)b \;. \tag{3.5}$$

These equations can be converted to the traveling wave case by replacing t by $-z/v$. On comparing (3.1) with (2.3) we find that

$$q = -\frac{4\pi^2\hbar\omega^2}{Vn^4}A, \quad p = -\frac{4\pi^2\hbar\omega^2}{Vn^4}B \;. \tag{3.6}$$

In order to solve the Heisenberg equations (3.4) and (3.5) we introduce the Heisenberg operators analogous to (2.5),

$$a_{\pm} = \frac{a \pm ib}{\sqrt{2}} \;,$$
$$a_+^{\dagger}a_+ + a_-^{\dagger}a_- = a^{\dagger}a + b^{\dagger}b \;, \tag{3.7}$$
$$2a_+a_- = a^2 + b^2 \;.$$

Using (3.4), (3.5), and (3.7), we get the equations

$$\dot{a}_{\pm} = -i[(a_+^{\dagger}a_+ + a_-^{\dagger}a_-)q + 2pa_{\mp}^{\dagger}a_{\mp}]a_{\pm} \;. \tag{3.8}$$

Since p and q are real it is easily seen that $a_+^{\dagger}a_+, a_-^{\dagger}a_-$ are constants of motion

$$a_{\pm}^{\dagger}(t)a_{\pm}(t) = a_{\pm}^{\dagger}(0)a_{\pm}(0) \;. \tag{3.9}$$

In view of these conservation laws, the integration of (3.8) is now straightforward[10]

$$a_{\pm}(t) = \exp\{-it[q(a_+^{\dagger}a_+ + a_-^{\dagger}a_-) + 2pa_{\mp}^{\dagger}a_{\mp}]\}a_{\pm} \;. \tag{3.10}$$

For brevity the operators $a_{\pm}$ will stand for $a_{\pm}(0)$. The Heinsenberg operators $a(t)$ and $b(t)$ can be obtained from (3.7) and (3.10), i.e., from

$$a(t) = \frac{a_+(t) + a_-(t)}{\sqrt{2}}, \quad b(t) = \frac{a_+(t) - a_-(t)}{\sqrt{2}i} \;. \tag{3.11}$$

The action of Heisenberg operator $a_{\pm}(t)$ on a coherent state $|\alpha_+,\alpha_-\rangle$ is quite illuminating and useful in the computation of the expectation values

$$a_{\pm}|\alpha_+,\alpha_-\rangle = \alpha_{\pm}|\alpha_+,\alpha_-\rangle \;,$$
$$|\alpha_+,\alpha_-\rangle = e^{-1/2|\alpha_+|^2 - 1/2|\alpha_-|^2} \times \sum_{n_+,n_-} \frac{(\alpha_+)^{n_+}(\alpha_-)^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle \;. \tag{3.12}$$

On using (3.12) in (3.10) and the fact that $|n_+,n_-\rangle$ is an eigenstate of both $a_+^{\dagger}a_+$, $a_-^{\dagger}a_-$, we obtain

$$a_{\pm}(t)|\alpha_+,\alpha_-\rangle = \alpha_{\pm}e^{-1/2|\alpha_+|^2 - 1/2|\alpha_-|^2} \times \sum_{n_+,n_-} \frac{(\alpha_+)^{n_+}(\alpha_-)^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle \times \exp\{-it[q(n_+ + n_-) + 2pn_{\mp}]\}, \tag{3.13}$$

which can be summed up

$$a_{\pm}(t)|\alpha_+,\alpha_-\rangle = \alpha_{\pm}|\alpha_+e^{-i\theta_{\pm}t}, \alpha_-e^{-i\theta_{\mp}t}\rangle \;, \tag{3.14}$$

where

$$\theta_+ = q, \quad \theta_- = (q+2p) \;. \tag{3.15}$$

Thus, the Heisenberg operators $a_{\pm}(t)$ transform the initial coherent states into new coherent states *whose amplitudes are related to the old amplitudes via the phase factors.*

IV. QUANTUM THEORY: EVOLUTION OF THE STATES OF THE FIELD AND THE GENERATION OF MACROSCOPIC SUPERPOSITION OF STATES

In this section we examine the dynamical evolution of the states of the system. The Hamiltonian (3.3) can be written in terms of the operator $a_{\pm}$:

$$H = \frac{\hbar}{2}q{:}(a_+^{\dagger}a_+ + a_-^{\dagger}a_-)^2{:} + 2\hbar pa_+^{\dagger}a_-^{\dagger}a_+a_- \;. \tag{4.1}$$

If, at $t=0$, the field *is in a coherent state* $|\alpha_+,\alpha_-\rangle$, then the field at time t can be obtained by using the expansion (3.12):

$$|\alpha_+,\alpha_-\rangle_t = e^{-iHt/\hbar}|\Psi(0)\rangle = e^{-iHt/\hbar}|\alpha_+,\alpha_-\rangle$$
$$= e^{-iHt/\hbar}e^{-1/2|\alpha_+|^2 - 1/2|\alpha_-|^2}\sum_{n_+,n_-}\frac{(\alpha_+)^{n_+}(\alpha_-)^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle$$
$$= e^{-1/2|\alpha_+|^2 - 1/2|\alpha_-|^2}\sum_{n_+,n_-}\frac{(\alpha_+)^{n_+}(\alpha_-)^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle \times \exp\left[-2iptn_+n_- - \frac{iqt}{2}(n_+ + n_-)^2 + \frac{iqt}{2}(n_+ + n_-)\right]|n_+,n_-\rangle \;. \tag{4.2}$$

The series (4.2) cannot be summed up except for certain special values of t. For $pt=qt=\pi$ it turns out that

$$|\Psi(t)\rangle=\frac{e^{-i\pi/4}}{\sqrt{2}}(|i\alpha_+,i\alpha_-\rangle+i|-i\alpha_+,-i\alpha_-\rangle) . \quad (4.3)$$

Thus the state of the system at time t given by $pt=qt=\pi$ is a macroscopic superposition of two coherent states.[11] The detailed derivation of (4.3) is rather complicated. However, one can see the form (4.3) from (3.14) which for $pt=qt=\pi$ leads to

$$\begin{aligned} a_+(t)|\alpha_+,\alpha_-\rangle&=\alpha_+|-\alpha_+,-\alpha_-\rangle ,\\ a_-(t)|\alpha_+,\alpha_-\rangle&=\alpha_-|-\alpha_+,-\alpha_-\rangle , \end{aligned} \quad (4.4)$$

and, hence,

$$\begin{aligned} a_+^2(t)|\alpha_+,\alpha_-\rangle&=-\alpha_+^2|\alpha_+,\alpha_-\rangle ,\\ a_-^2(t)|\alpha_+,\alpha_-\rangle&=-\alpha_-^2|\alpha_+,\alpha_-\rangle ,\\ a_-(t)a_+(t)|\alpha_+,\alpha_-\rangle&=-\alpha_-\alpha_+|\alpha_+,\alpha_-\rangle . \end{aligned} \quad (4.5)$$

These equations can also be written in terms of $|\alpha_+,\alpha_-\rangle_t$ as

$$a_\pm^2|\alpha_+,\alpha_-\rangle_t=-\alpha_\pm^2|\alpha_+,\alpha_-\rangle_t ,$$

$$a_\pm|\alpha_+,\alpha_-\rangle_t=\alpha_\pm|-\alpha_+,-\alpha_-\rangle_t , \quad (4.6a)$$

$$a_-a_+|\alpha_+,\alpha_-\rangle_t=-\alpha_+\alpha_-|\alpha_+,\alpha_-\rangle_t . \quad (4.6b)$$

Equation (4.6) shows that $|\Psi(t)\rangle$ is an eigenstate of $a_\pm^2$ with eigenvalues $-\alpha_\pm^2$. It is known that if $|\alpha\rangle$ is an eigenstate of the boson operator a

$$a|\alpha\rangle=\alpha|\alpha\rangle , \quad (4.7)$$

then the eigenstates of a^2 can be written as

$$a^2(|\alpha\rangle+e^{i\phi}|-\alpha\rangle)=\alpha^2(|\alpha\rangle+e^{i\phi}|-\alpha\rangle) , \quad (4.8)$$

where ϕ is a phase factor. Using (4.8) it is clear that $|\alpha_+,\alpha_-\rangle_t$ must have the form

$$|\alpha_+,\alpha_-\rangle_t=\mathcal{N}(|i\alpha_+,i\alpha_-\rangle+e^{i\phi}|-i\alpha_+,-i\alpha_-\rangle) , \quad (4.9)$$

where $\mathcal{N}$ is a normalized constant. The phase factpr ϕ can be fixed by the condition (4.6b).

Note further that the state of the mode a_+ is not a pure superposition state. In fact, using (4.3) the density matrix for mode a_+ can be written as

$$\begin{aligned}\rho_+(t)=\tfrac{1}{2}[&|i\alpha_+\rangle\langle i\alpha_+|+|-i\alpha_+\rangle\langle -i\alpha_+|\\ &-ie^{-2|\alpha_-|^2}(|i\alpha_+\rangle\langle -i\alpha_+|-\text{c.c.})] . \end{aligned} \quad (4.10)$$

Next we find the state of the field at time t given that at $t=0$, the two modes a and b are in the *Fock state* $|n_a,n_b\rangle$. Note that the time evolution of the initial Fock state $|n_+,n_-\rangle$ ($a_\pm^\dagger a_\pm|n_+,n_-\rangle=n_\pm|n_+,n_-\rangle$) is simple as the large parentheses in (4.2) already show

$$|n_+,n_-\rangle_t=\exp\left[-2iptn_+n_- -it\frac{q}{2}(n_++n_-)^2 +it\frac{q}{2}(n_++n_-)\right]|n_+,n_-\rangle . \quad (4.11)$$

In order to obtain $|n_a,n_b\rangle_t$ we will relate $|n_a,n_b\rangle$ to $|n_+,n_-\rangle$. Note that since

$$a^\dagger a+b^\dagger b=a_+^\dagger a_+ +a_-^\dagger a_- ,$$

it is clear that the states involved must be such that $n_a+n_b=n_++n_-$. Such a relation can be obtained from the defining relations[12] between a, b, and $a_\pm$:

$$|n,N-n\rangle_{a,b}=2^{-N/2}\sum_{m=n}^{N}\sum_{l=m\neq n}^{m}\frac{(-i)^{N-n}\sqrt{(N-n)!n!}\sqrt{(N-l)!l!}}{(m-l)!(N-m)!(l-m+n)!(m-n)!}|N-l,l\rangle_\pm . \quad (4.12)$$

On combining (4.11) and (4.12), we obtain the time evolution of the field initially in a Fock state

$$\begin{aligned}|n,N-n\rangle_{a,b,t}=&2^{-N/2}\exp(-iqtN^2/2+iqtN/2)\\ &\times\sum_{m=n}^{N}\sum_{l=m-n}^{m}\frac{(-i)^{N-n}\sqrt{(N-n)!n!}\sqrt{l!(N-l)!}}{(m-l)!(N-m)!(l-m+n)!(m-n)!}\exp[-2ipt(N-l)l]|N-l,l\rangle_\pm . \end{aligned} \quad (4.13)$$

Thus, whenever pt is an integral multiple of π, the state at time t is same as the initial Fock state except for a phase factor.

V. DYNAMICAL EVOLUTION OF OBSERVABLES AND PHOTON STATISTICS

The evolution of photon statistics $p(n_a,n_b)$ can be studied by using the relation (3.14) if the input state of the field is a coherent state $|\alpha,\beta\rangle$:

$$a|\alpha,\beta\rangle=\alpha|\alpha,\beta\rangle,\quad b|\alpha,\beta\rangle=\beta|\alpha,\beta\rangle , \quad (5.1)$$

which, in turn, implies that

$$a_\pm|\alpha_+,\alpha_-\rangle=\alpha_\pm|\alpha_+,\alpha_-\rangle,\quad \alpha_\pm=\frac{\alpha\pm i\beta}{\sqrt{2}} . \quad (5.2)$$

From (3.14) it is clear that

$$\langle \alpha_+,\alpha_-|a_\pm^\dagger(t)a_\pm(t)|\alpha_+,\alpha_-\rangle=|\alpha_\pm|^2 , \tag{5.3}$$

$$\begin{aligned}\langle \alpha_+,\alpha_-|a_+^\dagger(t)a_-(t)|\alpha_+,\alpha_-\rangle&=\alpha_+^*\alpha_-\langle \alpha_+e^{-i\theta_+t},\alpha_-e^{-i\theta_-t}|\alpha_+e^{-i\theta_-t},\alpha_-e^{-i\theta_+t}\rangle\\&=\alpha_+^*\alpha_-\exp[-|\alpha_+|^2(1-e^{+i(\theta_+-\theta_-)t})-|\alpha_-|^2(1-e^{i(\theta_--\theta_+)t})] ,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\langle \alpha_+,\alpha_-|a_+^2(t)|\alpha_+,\alpha_-\rangle&=\alpha_+^2e^{-i\theta_+t}\langle \alpha_+,\alpha_-|\alpha_+e^{-2i\theta_+t},\alpha_-e^{-2i\theta_-t}\rangle\\&=\alpha_+^2e^{-i\theta_+t}\exp[-|\alpha_+|^2(1-e^{-2i\theta_+t})-|\alpha_-|^2(1-e^{-2i\theta_-t})] .\end{aligned} \tag{5.5}$$

Note that

$$\theta_--\theta_+=2p . \tag{5.6}$$

On using (3.11), (5.4), and (5.6) we find that mean number of photons in the two Cartesian modes

$$\begin{aligned}\langle a^\dagger(t)a(t)\rangle&=|\alpha|^2+|\beta|^2-\langle b^\dagger(t)b(t)\rangle\\&=\tfrac{1}{2}(|\alpha_+|^2+|\alpha_-|^2+\{\alpha_+^*\alpha_-\exp[-|\alpha_+|^2(1-e^{-2ipt})-|\alpha_-|^2(1-e^{2ipt})]+\text{c.c.}\})\\&=\tfrac{1}{2}(|\alpha|^2+|\beta|^2+\{(|\alpha|^2-|\beta|^2)\cos[x(t)]-2|\alpha||\beta|\cos(\theta)\sin[x(t)]\}\exp[-2(|\alpha|^2+|\beta|^2)\sin^2(pt)]) ,\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}&x(t)=2|\alpha||\beta|\sin\theta\sin(2pt) ,\\&\exp(i\theta)=\alpha\beta^*/|\alpha||\beta| .\end{aligned} \tag{5.8}$$

This shows that the number of photons in each mode oscillates with frequency $2p|\alpha|\,|\beta|\sin(\theta)$ about the average of the number of photons in the two modes. For $\theta=0,\pi/2$ these oscillations are centered at $t_n=n\pi/p$ and are enveloped by approximately a Gaussian of width

$$t_\gamma=\sqrt{2}/\{p[|\alpha|^2+|\beta|^2)]^{1/2}\} .$$

Thus, during the quiscent period, there is an equal number of photons in the two modes. For $x=0$, i.e., for $\theta=0$ and also for $|\alpha|$ or $|\beta|=0$, the frequency of oscillations is zero. Hence, in this case, these are pulses of photons of height $|\alpha|^2$, centered at t_r, having base value $(|\alpha|^2+|\beta|^2)/2$ and width t_c. These characteristics[13] are apparent in Figs. 1–3 where we have plotted $\langle a^\dagger a\rangle$ as a function of pt for various values of α and β. Also note that, if the field is initially circularly polarized (i.e., if $\theta=\pi/2$ and $|\alpha|=|\beta|$), then its intensity does not change as it propagates through the medium. The quantum features in the light propagation can be seen from a comparison of these figures with the behavior that follows from (2.14). For example, for $\theta=0$, the solution (2.14) shows that the light intensity in the mode a does not change as the light propagates through the medium. The quantum theory (curves A in Figs. 1–3) shows a definite periodic evolution. For other values of θ, the classical theory shows a simple periodic evolution. However, the quantum theory shows oscillations around $pt=n\pi$ enveloped approximately by a Gaussian. Note that the classical limit is obtained by letting $p\to 0$ as p is proportional to $\hbar$. Thus $x(t)\to 4|\alpha||\beta|pt\sin\theta$, which would be independent of $\hbar$ because $\hbar|\alpha||\beta|$ is proportional to the square roots of the intensities of the beams along two axes.

Next, the correlation between the two quantized modes is found to be

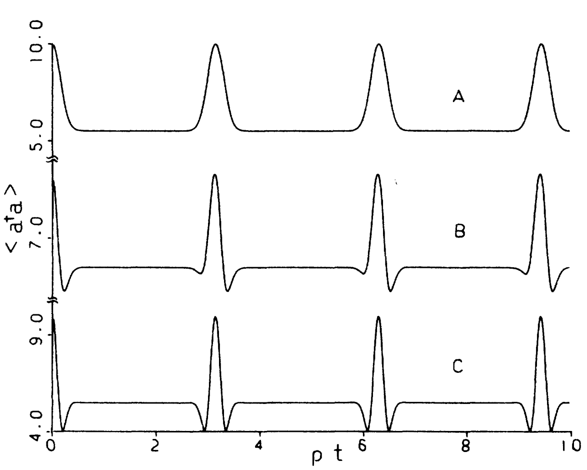

FIG. 1. $\langle a^\dagger(t)a(t)\rangle$ as a function of pt for the two modes in the coherent state $|\alpha,\beta\rangle$ with $|\alpha|^2=10$, $|\beta|^2=1$, and $\theta=0$ (curve A), $\theta=\pi/4$ (curve B), and $\theta=\pi/2$ (curve C).

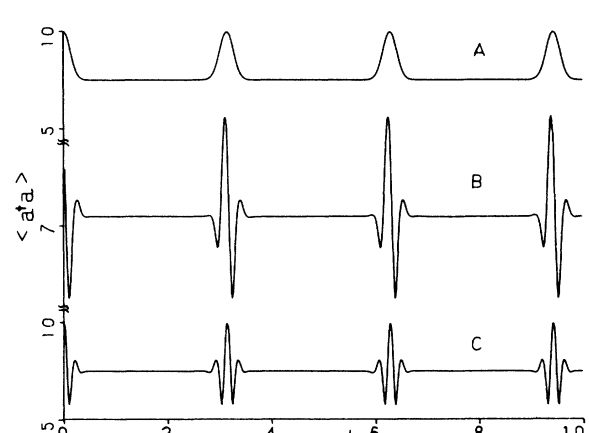

FIG. 2. Same as Fig. 1 but for $|\alpha|^2=10$, $|\beta|^2=5$.

$$\langle a^{\dagger}(t)b(t)\rangle=\frac{1}{2i}(|\alpha_+|^2-|\alpha_-|^2-\{\alpha_+^*\alpha_-\exp[-|\alpha_+|^2(1-e^{-2ipt})-|\alpha_-|^2(1-e^{2ipt})]-\text{c.c.}\}) \; . \tag{5.9}$$

The higher-order moments of the field operators can be obtained by repeated application of (3.14). For example, one has the result

$$\langle \alpha_+,\alpha_-|a_+^m(t)|\alpha_+,\alpha_-\rangle=\alpha_+^m e^{-i\theta_+ t(m-1)}\langle \alpha_+,\alpha_-|\alpha_+ e^{-im\theta_+ t},\alpha_- e^{-im\theta_- t}\rangle \; . \tag{5.10}$$

Thus, quantities like Q and I can be computed

$$Q=\frac{\langle (a^{\dagger}(t))^2a^2(t)\rangle-\langle a^{\dagger}(t)a(t)\rangle^2}{\langle a^{\dagger}(t)a(t)\rangle}, \quad I=\frac{(\langle (a^{\dagger})^2a^2\rangle\langle (b^{\dagger})^2b^2\rangle)^{1/2}}{|\langle a^{\dagger}ab^{\dagger}b\rangle|}-1 \; . \tag{5.11}$$

The quantity Q is a measure of the sub-Poissonian ($Q<0$) or super-Poissonian ($Q>0$) statistics of the field. The quantity I is also a measure of the nonclassical nature[14] of the field. A negative value of I will imply that the underlying P distribution has nonclassical properties.

We can evaluate $Q(t)$ and $I(t)$ by using Eq. (5.7) and the following results:

$$\begin{aligned}\langle a^{\dagger}ab^{\dagger}b\rangle=&\tfrac{1}{8}[(|\alpha|^2+|\beta|^2)^2+4|\alpha|^2|\beta|^2\sin^2(\theta)]\\&-\tfrac{1}{8}\exp[-2(|\alpha|^2+|\beta|^2)]\{[(|\alpha|^2-|\beta|^2)^2-4|\alpha|^2|\beta|^2\cos^2(\theta)]\cos(y)\\&-4|\alpha|\beta|\cos(\theta)(|\alpha|^2-|\beta|^2)\sin[y(t)]\} \; ,\end{aligned} \tag{5.12}$$

$$\langle (a^{\dagger})^2a^2\rangle=A+B, \quad \langle (b^{\dagger})^2b^2\rangle=A-B \; , \tag{5.13}$$

where

$$\begin{aligned}A=&\tfrac{1}{8}[3(|\alpha|^2+|\beta|^2)^2-4|\alpha|^2|\beta|^2\sin^2(\theta)]\\&-\tfrac{1}{8}\exp[-2(|\alpha|^2+|\beta|^2)\sin^2(2pt)][\{4|\alpha||\beta|(|\alpha|^2-|\beta|^2)^2\cos(\theta)\sin[y(t)]\\&-[(|\alpha|^2-|\beta|^2)^2-4|\alpha|^2|\beta|^2\cos^2(\theta)]\cos[y(t)]\} \; ,\end{aligned} \tag{5.14}$$

$$\begin{aligned}B=&\tfrac{1}{2}\exp[-2(|\alpha|^2+|\beta|^2)\sin^2(pt)]((|\alpha|^2+|\beta|^2)(|\alpha|^2-|\beta|^2)\cos[x(t)+2pt]\\&-2|\alpha||\beta|\cos(\theta)\{(|\alpha|^2+|\beta|^2)\sin[x(t)]\cos(2pt)+2|\alpha||\beta|\sin(\theta)\cos(x)\sin(2pt)\}) \; ,\end{aligned} \tag{5.15}$$

$$y(t)=2|\alpha||\beta|\sin(\theta)\sin(4pt) \; . \tag{5.16}$$

In Fig. 4 we have plotted $Q(t)$ as a function of pt. We find that $Q(t)$ exhibits antibunching $[Q(t)<0]$ in a small neighborhood of $pt=n\pi$ for $\theta\neq 0$. This can also be seen by analyzing the analytic expression for $Q(t)$. We have also numeri-

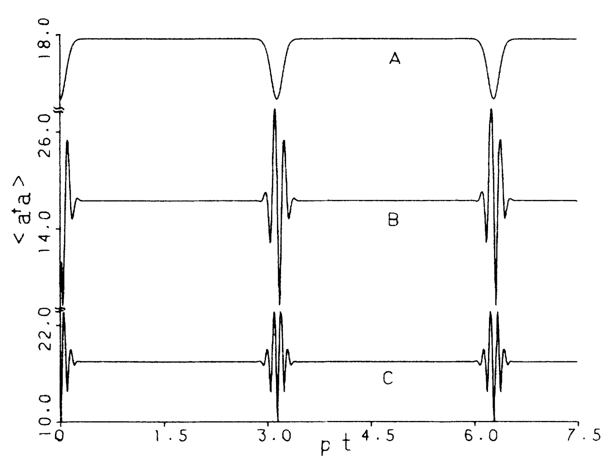

FIG. 3. Same as Fig. 1 but for $|\alpha|^2=10$, $|\beta|^2=25$.

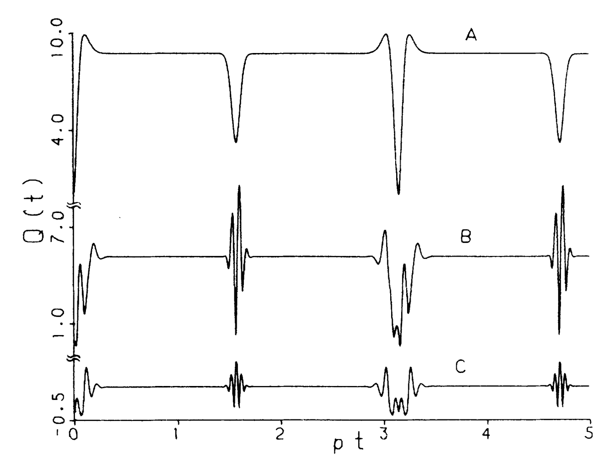

FIG. 4. $Q(t)$ as a function of pt for the two modes initially in the coherent state $|\alpha,\beta\rangle$ with $|\alpha|^2=10$, $|\beta|^2=25$, and $\theta=0$ (A), $\pi/4$(B), and $\pi/2$(C).

cally evaluated $I(t)$, which is found to remain positive at least in the range of times shown in Fig. 4.

In the above discussion we have taken the initial state to be a coherent state $|\alpha,\beta\rangle$. The results for other states can be obtained by averaging over the initial P distribution of the field. For the initial Fock states the mean value can be obtained from the relation

$$\langle n_a,n_b|G|n_a,n_b\rangle=\frac{e^{-(|\alpha|^2+|\beta|^2)}}{n_a!n_b!}\frac{\partial^{2n_a+2n_b}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}\partial\beta^{n_b}\partial\beta^{*n_b}}e^{(|\alpha|^2+|\beta|^2)}\langle\alpha,\beta|G|\alpha,\beta\rangle\Big|_{\alpha=\alpha^*=\beta=\beta^*=0}, \tag{5.17}$$

where G is some operator like number operator. If G is chosen as $a^\dagger a$ then, as shown in the Appendix, the mean number of photons in mode a is

$$\langle a^\dagger(t)a(t)\rangle=\frac{1}{2}\left[n_a+n_b+(n_a-n_b)\sum_{l=0}^{n_a}\frac{n_a!n_b!(-1)^l\sin^{2l}(2pt)[\cos(2pt)]^{n_a+n_b-2l-1}}{(n_a-l)!(n_b-l)!(l!)^2}\right],\quad n_b>n_a\ . \tag{5.18}$$

The evolution of $\langle a^\dagger(t)a(t)\rangle$ as a function of time is exhibited in Fig. 5 for $n_b=25$ and for different values of n_a. The photon number oscillates about the average of the number of photons in the two modes. For $n_a=0$, there are only "pulses" of photons centered at $pt=n\pi/2$. For $n_a\neq0$, the oscillations develop. These oscillations are, however, qualitatively different from the ones observed in a field, initially in coherent state.

VI. QUANTUM-MECHANICAL CHANGES IN THE DEGREE OF POLARIZATION

In this section we show that the quantum nature of the field changes the degree of polarization of the field. This is in contrast to the result in the semiclassical theory. The degree of polarization P can be defined[15] in terms of the elements of the coherence matrix J

$$J=\begin{bmatrix}\langle a^\dagger a\rangle & \langle a^\dagger b\rangle\\ \langle b^\dagger a\rangle & \langle b^\dagger b\rangle\end{bmatrix},\quad P^2=1-\frac{4\det J}{(\mathrm{Tr}J)^2}\ . \tag{6.1}$$

Let us assume that the initial state of the field is a coherent state $|\alpha,\beta\rangle$, then

$$J=\begin{bmatrix}\alpha^*\alpha & \alpha^*\beta\\ \alpha\beta^* & \beta^*\beta\end{bmatrix},\quad P=1\ . \tag{6.2}$$

The time dependence of the coherence matrix can be obtained from (5.7) and (5.8). Calculations show that

$$P^2(t)=1-\frac{4|\alpha_+|^2|\alpha_-|^2}{(|\alpha_+|^2+|\alpha_-|^2)^2}\times(1-\exp\{-2(|\alpha_+|^2+|\alpha_-|^2)\times[1-\cos(2pt)]\})\ . \tag{6.3}$$

The degree of polarization is unity when

$$pt=n\pi\ , \tag{6.4}$$

otherwise $|P(t)|<1$. Thus, the quantum nature of the field converts a fully polarized field into a partially polarized field. In general, the degree of polarization remains unity only if α_+ or $\alpha_-=0$, i.e., for input fields which are circularly polarized one way or the other.

Note added in proof. We have learned that Chandra and Prakash have also studied the squeezing characteristics of light propagating through the Kerr media.

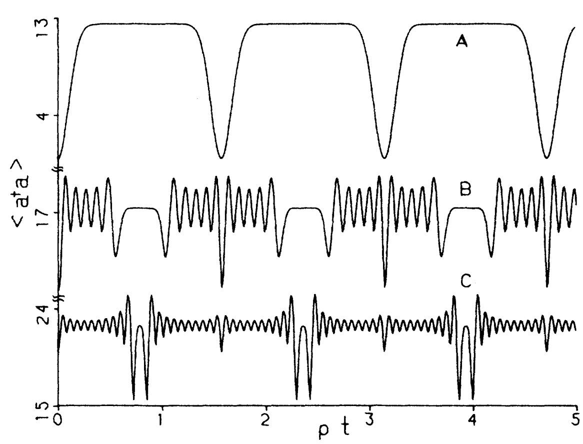

FIG. 5. $\langle a^\dagger(t)a(t)\rangle$ as a function of pt for the two modes initially in the Fock state $|n_a,n_b\rangle$ with $n_b=25$ and $n_a=0$ (curve *A*), $n_a=10$ (curve *B*), and $n_a=20$ (curve *C*).

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APPENDIX: DERIVATION OF EQ. (5.18)

In this appendix we give a derivation of Eq. (5.18), i.e., the expression for the mean number of photons in a Cartesian mode when both the modes are initially in a Fock state.

Let the initial state of the field be $|n_a,n_b\rangle$, where $n_a(n_b)$ is the number of photons in mode $a\,(b)$. The average

$$\langle n_a,n_b|a^\dagger(t)a(t)|n_a,n_b\rangle$$

is then given by Eq. (5.17) with $G\equiv a^\dagger(t)a(t)$. Now, using Eqs. (5.7) and (5.2), we have

$$\langle \alpha,\beta|a^\dagger(t)a(t)|\alpha,\beta\rangle = \tfrac{1}{2}(|\alpha|^2+|\beta|^2) + \tfrac{1}{4}\{[(|\alpha|^2-|\beta|^2)-i(\alpha\beta^*+\alpha^*\beta)]\exp[(\alpha^*\beta-\alpha\beta^*)\sin(pt)]+\text{c.c.}\}\exp[-2(|\alpha|^2+|\beta|^2)\sin^2(qt)] \ . \tag{A1}$$

Next we substitute Eq. (A1) in Eq. (5.11) and evaluate $\langle n_a,n_b|a^\dagger(t)a(t)|n_a,n_b\rangle$ by using the following results:

$$\begin{aligned} I &= \frac{\partial^{2(n_a+n_b)}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}\partial\beta^{n_b}\partial\beta^{*n_b}}[\exp(x_1|\alpha|^2+x_2|\beta|^2+x_3\alpha^*\beta+x_4\alpha\beta^*)]\Big|_{\alpha=\alpha^*=\beta=\beta^*=0} \\ &= \frac{\partial^{2n_a}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}}\exp(x_1|\alpha|^2)\frac{\partial^{n_b}}{\partial\beta^{*n_b}}\exp(x_4\alpha\beta^*)(x_2\beta^*+x_3\alpha^*)^{n_b}\Big|_{\alpha=\alpha^*=\beta^*=0} \\ &= \frac{\partial^{2n_a}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}}\exp(x_1|\alpha|^2)\sum_{l=0}^{n_b}\frac{(n_b!)^2(x_3x_4)^l|\alpha|^{2l}x_2^{n_b-l}}{(n_b-l)!(l!)^2}\Bigg|_{\alpha=\alpha^*=0} \\ &= \sum_{l=0}^{n_b}\frac{(n_a!)^2(n_b!)^2(x_3x_4)^l|\alpha|^{2l}x_2^{n_b-l}}{(n_a-l)!(n_b-l)!l!^2} \ . \end{aligned} \tag{A2}$$

It then follows that

$$\frac{\partial^{2(n_a+n_b)}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}\partial\beta^{n_b}\partial\beta^{*n_b}}(|\alpha|^2\pm|\beta|^2)\exp(x_1|\alpha|^2+x_2|\beta|^2+x_3\alpha^*\beta+x_4\alpha\beta^*)|_{\alpha=\alpha^*=\beta=\beta^*=0} = \left[\frac{\partial I}{\partial x_1}\pm\frac{\partial I}{\partial x_2}\right] = \sum_{l=0}^{n_b}\frac{(n_a!)^2(n_b!)^2[(n_a-l)x_2\pm(n_b-l)x_1]x_1^{n_a-l-1}x_2^{n_b-l-1}(x_3-x_4)^l}{(n_a-l)!(n_b-l)!l!^2} \ , \tag{A3}$$

$$\frac{\partial^{2(n_a+n_b)}}{\partial\alpha^{n_a}\partial\alpha^{*n_a}\partial\beta^{n_b}\partial\beta^{*n_b}}(\alpha^*\beta+\alpha\beta^*)\exp(x_1|\alpha|^2+x_2|\beta|^2+x_3\alpha^*\beta+x_4\alpha\beta^*)|_{\alpha=\alpha^*=\beta=\beta^*=0} = \frac{\partial I}{\partial x_3}+\frac{\partial I}{\partial x_4} = \sum_{l=0}^{n_b}\frac{l(n_a!)^2(n_b!)^2(x_3+x_4)(x_3-x_4)^{l-1}x_1^{n_a-l}x_2^{n_b-l}}{(n_a-l)!(n_b-l)!l!^2} \ . \tag{A4}$$

Substituting Eqs. (A1), (A3), and (A4) with appropriate values of x_i's in Eq. (5.17) we obtain

$$\langle n_an_b|a^\dagger(t)a(t)|n_a,n_b\rangle = \frac{1}{2}\left[n_a+n_b+(n_a-n_b)\sum_{l=0}^{n_a}\frac{n_a!n_b!(-1)^l\sin^{2l}(2pt)[\cos(2pt)]^{n_a+n_b-2l-1}}{(n_a-l)!(n_b-l)!(l!)^2}\right] \tag{A5}$$

which is Eq. (5.18).

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[11]Macroscopic superpositions of coherent states have been discussed by several authors Yurke and Stoler (Ref. 2), Milburn and Holmes (Ref. 2).

[12]For a derivation of this relation, see, for example, Ref. 6.

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