Role of pumping statistics in maser and laser dynamics: Density-matrix approach

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We discuss in detail the influence that the statistical properties of the pump source have on maser and laser dynamics. We derive a general master equation for the radiation field that is valid for a wide range of different pump mechanisms. If the pump noise is eliminated, we find that the photon number noise in a micromaser and a laser can be significantly reduced below the shot-noise level. In contrast, the phase fluctuations for both maser and laser are unaffected by the noise contribution of the pump.

I. INTRODUCTION

In ordinary laser theory as well as in many maser- and laser-related problems, one commonly neglects the effects of the statistical properties of the pump mechanism. Exceptions to this attitude are some recent work on micromasers,[1–3] where the master equation for the reduced density matrix of the field is obtained after averaging over a Poissonian distribution of the incoming atoms. Furthermore, a suppression of pump noise was considered for the laser,[4] where it was shown that it can lead to squeezing of the amplitude.

In this analysis we provide a method for incorporating the statistics in maser or laser dynamics. This method allows one to go continuously from the regular injection rate of excited atoms to the Poissonian-distribution case. It also leads to a precise formulation of the conditions under which explicit consideration of the atomic distribution is irrelevant. In this paper we derive a master equation for the reduced density matrix of the field which explicitly incorporates pumping statistics. We exemplify our procedure by applying it to models for masers and lasers and calculate the photon-number variances and the phase-diffusion constant for these devices. In a planned second, forthcoming paper we will use a different approach to analyze the influence of pump fluctuations on laser and maser dynamics. We combine Langevin-operator techniques and statistical arguments to gain further insight into the role of pump noise in lasers. Although these results are derived in a completely different way than the one presented in this paper, there is a perfect agreement between the two approaches.

We find that if the transition between the two lasing atomic states becomes appreciable, i.e., the Rabi angle becomes of the order of π, there is an important dependence of the photon-number noise on the injected statistics. In the case of a complete pump-noise suppression, the photon-number fluctuations can be reduced up to 50% below the shot-noise level for the micromaser. For the laser model, considered in this paper, the reduction can be up to 25%. However, we note that the amount of noise reduction in lasers also depends on the ratio of the atomic decay constants. For example, in a semiconductor laser, noise suppression of up to 50% is possible.[4] A detailed description of this aspect will be presented in a planned second paper.[5] The phase diffusion in a maser or laser is found to be independent of pumping statistics.

Here we consider only the case in which the injected atoms are in the excited state. If they are initially in a superposition of states,[6] so that we have a nonvanishing atomic coherence, strongly-pump-dependent effects can arise, which affect the squeezing of the field.[7]

In Sec. II we present a simple combinatorial argument which explains in physical terms why the photon-number noise should depend on pumping statistics. In Sec. III we establish a general quantum-mechanical approach to this problem, applying the framework of the master equation. We obtain a generalized master equation, which is valid for a wide range of different pump mechanisms. This theory is then applied in Sec IV. to the one-photon micromaser[1,2] and to a laser oscillator, yielding the dependence of photon-number noise on the pump statistics as well as the phase-diffusion constant. Our conclusions are summarized in Sec. V.

II. INJECTION STATISTICS AND PHOTON-NUMBER NOISE: HEURISTIC DISCUSSION

In this section we extend the arguments given in Ref. 1 in order to account for a general statistical distribution of the injected atoms. We assume that a dense flux of atoms goes through an excitation region (say, one or more laser beams), and that each atom has a probability p of being excited from the ground level c to the upper level a [Fig. 1(a)]. We assume that levels a and b are involved in the lasing (or masing) transition, and that level b remains unpopulated. We further assume that the atomic beam has a regular distribution before reaching the excitation region, so that the number K of the atoms which cross that

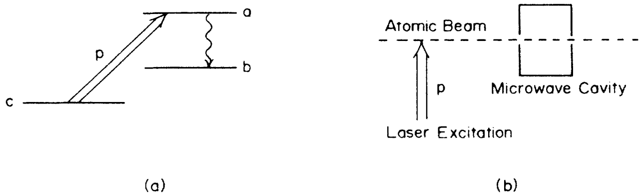

FIG. 1. (a) An atom is excited from a level c to the upper level a, assuming that the lasing occurs in the a-b transition. (b) An atomic beam, with a regular distribution, arrives at the excitation region, where the Rydberg states are generated, before entering the microwave cavity.

region during a time Δt is given by

$$K = R\,\Delta t \; . \tag{2.1}$$

Here, R is the constant injection rate and the time interval Δt was chosen to be much larger than the time interval between consecutive atoms. The probability that k atoms get excited during time Δt is given by

$$P(k,K) = \begin{bmatrix} K \\ k \end{bmatrix} p^k (1-p)^{K-k} \; . \tag{2.2}$$

From this we obtain

$$\bar{k} = pK = r\,\Delta t \; , \tag{2.3}$$

in which

$$r = pR \tag{2.4}$$

is the average injection rate of excited atoms. Furthermore, we find

$$\overline{k^2} = (1-p)\bar{k} + \bar{k}^{\,2} \; . \tag{2.5}$$

Our model is general enough to account for several limiting cases of particular interest. If, for example, $p \to 0$ and $R \to \infty$, so that the product $pR = r$ remains constant, the Bernoulli distribution goes over into a Poissonian distribution. On the other hand, if $p \to 1$ all the atoms get excited and we get a regular distribution. We see indeed that $(\Delta k)^2 = \overline{k^2} - \bar{k}^{\,2}$, as obtained from (2.5), is equal to $\bar{k}$ for $p \to 0$, and vanishes when $p \to 1$.

After crossing the pumping region, the atoms go on into the resonant cavity [Fig. 1(b)]. Then, let $P(t)$ be the probability at time t that an atom gets deexcited and emits a photon inside the cavity. We assume that the number of photons inside the cavity is large enough so that we can neglect the extra ones which are left by the atoms during the time interval Δt. In other words, we assume that the time over which the field distribution is changing is much larger than both Δt and the flight time of the atoms through the cavity. If k atoms reach the cavity during a time interval Δt, then the probability that they release $\tilde{N}$ photons into the cavity is given by

$$P(\tilde{N}/k) = \begin{bmatrix} k \\ \tilde{N} \end{bmatrix} P^{\tilde{N}} (1-P)^{k-\tilde{N}} \; . \tag{2.6}$$

Note that $P(\tilde{N}/k)$ is a conditional probability, i.e., it is the probability of having added $\tilde{N}$ photons to the cavity during Δt, provided the number of incoming excited atoms is equal to k. The total probability that $\tilde{N}$ photons are added to cavity during Δt is then

$$P(\tilde{N},\Delta t) = \sum_{k=0}^{K} P(\tilde{N}/k) P(k,K) \; , \tag{2.7}$$

with $P(k,K)$ given by (2.2). Therefore, if we are interested in the average of some function of $\tilde{N}$, say $f(\tilde{N})$, we can first calculate the conditional average using (2.6), and then average the result over the k distribution:

$$\langle f(\tilde{N}) \rangle = \sum_{k=0}^{K} \left[\sum_{\tilde{N}=0}^{k} f(\tilde{N}) P(\tilde{N}/k) \right] P(k,K) \; . \tag{2.8}$$

Thus, we find

$$\langle \tilde{N} \rangle = P\bar{k} \tag{2.9}$$

and

$$\langle (\tilde{N})^2 \rangle = (P - P^2)\bar{k} + P^2 \overline{k^2} \; , \tag{2.10}$$

where the bar indicates the average over the k distribution. We again emphasize that $\tilde{N}$ is the number of photons which are emitted into the cavity during the time interval Δt. Hence, $\tilde{N} = N(t+\Delta t) - N(t)$, where $N(t)$ is the total number of photons inside the cavity at time t. Substituting Eqs. (2.3) and (2.5) into the results above, we obtain

$$\langle [N(t+\Delta t) - N(t)] \rangle = rP\,\Delta t \tag{2.11}$$

and

$$\langle [N(t+\Delta t) - N(t)]^2 \rangle = (1-pP) rP\,\Delta t + r^2 P^2 (\Delta t)^2 \; . \tag{2.12}$$

These two relations can now be used to get the drift and the diffusion coefficient corresponding to photon number N (see, e.g., Ref. 8):

$$A_N(t) = \lim_{\Delta t \to 0} \frac{\langle [N(t+\Delta t) - N(t)] \rangle}{\Delta t} = rP(t) \tag{2.13}$$

and

$$D_N(t) = \lim_{\Delta t \to 0} \frac{\langle [N(t+\Delta t) - N(t)]^2 \rangle}{\Delta t} = (1-pP) rP \; . \tag{2.14}$$

Therefore the change of the mean photon number due to gain is given at all times by the expression

$$\frac{d}{dt} \langle N(t) \rangle = r \langle P(t) \rangle \; . \tag{2.15}$$

On the other hand, applying the generalized Einstein relation[8] to the variable $\Delta N = N - \langle N \rangle$, we find the equation for the variance:

$$\frac{dv}{dt} = 2\langle A_N \Delta N \rangle + 2\langle D_N \rangle \; , \tag{2.16}$$

where

$$v = \langle (\Delta N)^2 \rangle \; . \tag{2.17}$$

Equation (2.16) can be used to get the steady-state photon-number variance, when dissipation is also taken into account. This will be done in the following section. However, from Eq. (2.14) alone we already see that the statistical parameter p plays an important role in the photon-number–diffusion constant, provided the deexcitation probability P becomes appreciable. On the other hand, if only small Rabi angles are involved for each atom inside the cavity ($P \ll 1$), the dependence on the pumping statistics can be neglected.

We will now see how a precise quantum theory can be formulated, which explicitly includes the effects of pumping statistics.

III. MASTER EQUATION FOR GENERALIZED PUMPING STATISTICS

Let us first neglect dissipation effects. Let τ be the time spent by each atom inside the cavity, and t_j the arrival time of atom j. The change of the radiation field due to the interaction with atom j can be written as[8]

$$\rho(t_j+\tau)=M(\tau)\rho(t_j) , \tag{3.1}$$

where ρ is the reduced density matrix of the field, and M is an operator whose explicit form depends on the model under consideration. For example, for a one-photon micromaser, in which nondecaying atoms pass through the cavity during a time interval τ, the operator M is given by[8]

$$M(\tau)\rho=\cos(\lambda\tau)\rho\cos(\lambda\tau)+g^2a^\dagger\frac{\sin(\lambda\tau)}{\lambda}\rho\frac{\sin(\lambda\tau)}{\lambda}a . \tag{3.2}$$

Here we assumed that the atoms are initially prepared in their upper lasing level. The operator λ is defined by

$$\lambda=g(a^\dagger a+1)^{1/2} , \tag{3.3}$$

and g is the electric dipole coupling constant, while a and $a^\dagger$ are the usual annihilation and creation operators for the resonant mode. On the other hand, for a laser the atomic lifetime is much smaller than τ, so we must replace (3.2) by its average over the distribution $\Gamma\exp(-\Gamma\tau)$, where Γ^{-1} is the atomic lifetime.[8]

We now assume that each atom contributes independently of the others to the field (this is certainly the case if there is at most one atom in the cavity at a time; however, this restriction is not necessary, and our treatment remains valid as long as the active atoms constitute a dilute gas[9]). Therefore, if k atoms are passed through the cavity from 0 to t, the density matrix at time t is determined by

$$\rho^{(k)}(t)=M^k\rho(0) . \tag{3.4}$$

If the distribution of incoming atoms is now given by Eq. (2.2), we find the density matrix at time t, averaged over that distribution, to be

$$\rho(t)=\sum_{k=0}^{K}\begin{bmatrix}K\\k\end{bmatrix}p^k(1-p)^{K-k}M^k\rho(0)$$
$$=[1+p(M-1)]^K\rho(0) , \tag{3.5}$$

where, according to (2.1), $K=Rt$.

In writing (3.5), we have made a continuous approximation of the steplike time evolution of the system. This is valid on a time scale which is large compared to the average time between two atoms, i.e., $t\gg 1/r$, with r given by (2.4).

Differentiating (3.5) with respect to t, we get

$$\dot\rho(t)=\frac{r}{p}\ln[1+p(M-1)]\rho(t) . \tag{3.6}$$

This equation constitutes our generalized master equation. We immediately see that when $p\to 0$, while at the same time r is kept constant, we get

$$\dot\rho(t)=r(M-1)\rho(t) \quad (p\to 0,\ r\ \text{constant}) . \tag{3.7}$$

This is the usual Scully-Lamb master equation,[8] which therefore corresponds to a Poissonian pumping (see Ref. 3 for an alternative derivation). The same result is obtained from (3.6) for any p, if $M-1$ is small in some sense. However, this is certainly not the case if the atomic Rabi angle becomes of the order of π as the atoms go through the cavity. Such large Rabi angles arise, for example, in the usual experiments, involving micromasers,[1–3,10,11] or in laser operation far above threshold.

On the other hand, for $p=1$ (regular distribution), we get

$$\dot\rho(t)=r(\ln M)\rho(t) \quad (p=1) . \tag{3.8}$$

In order to apply Eq. (3.6) to a specific problem, we may consider p an expansion parameter, and write

$$\frac{1}{p}\ln[1+p(M-1)]=(M-1)-\frac{p}{2}(M-1)^2+\cdots . \tag{3.9}$$

We will follow this procedure in the next section. We will show that, surprisingly enough, only the first two terms in the expansion (3.9) need to be considered, if the pumping rate is sufficiently high.

IV. PHOTON-NUMBER NOISE AND PHASE DIFFUSION IN MASERS AND LASERS

We first exemplify our procedure with a one-photon micromaser,[1,2,10] assuming that a monokinetic atomic beam is injected into the cavity, with practically no population in the lowest lasing state. The operator $M(\tau)$, defined in Eq. (3.1), is then given by Eq. (3.2). Later in this section we will consider the laser case.

We now replace expansion (3.9), with M given by (3.2), in Eq. (3.6), and keep only terms up to first order in p. We then get

$$\dot{\rho}=r(1+p)\left[\cos(\lambda\tau)\rho\cos(\lambda\tau)+g^2a^\dagger\frac{\sin(\lambda\tau)}{\lambda}\rho\frac{\sin(\lambda\tau)}{\lambda}a\right]$$
$$-r\left[1+\frac{p}{2}\right]\rho-\frac{rp}{2}\left[\cos^2(\lambda\tau)\rho\cos^2(\lambda\tau)+g^2\cos(\lambda\tau)a^\dagger\frac{\sin(\lambda\tau)}{\lambda}\rho\frac{\sin(\lambda\tau)}{\lambda}a\cos(\lambda\tau)\right.$$
$$\left.+g^2a^\dagger\frac{\sin(\lambda\tau)}{\lambda}\cos(\lambda\tau)\rho\cos(\lambda\tau)\frac{\sin(\lambda\tau)}{\lambda}a+g^4a^\dagger\frac{\sin(\lambda\tau)}{\lambda}a^\dagger\frac{\sin(\lambda\tau)}{\lambda}\rho\frac{\sin(\lambda\tau)}{\lambda}a\frac{\sin(\lambda\tau)}{\lambda}a\right]. \tag{4.1}$$

From this expression we now obtain the equation of motion for the matrix elements of ρ in the photon-number representation. In particular, the equations for $\rho_{N,N}$ and $\rho_{N,N+1}$ will allow us to calculate the photon-number noise and the phase-diffusion constant, respectively.

A. Reduction of photon-number noise

In the photon-number representation, the diagonal elements satisfy the equation

$$\dot{\rho}_{N,N}=r[-\sin^2(g\sqrt{N+1}\tau)\rho_{N,N}+\sin^2(g\sqrt{N}\tau)\rho_{N-1,N-1}]$$
$$+\frac{rp}{2}\{-\sin^4(g\sqrt{N+1}\tau)\rho_{N,N}+[\sin^4(g\sqrt{N}\tau)+\sin^2(g\sqrt{N}\tau)\sin^2(g\sqrt{N+1}\tau)]\rho_{N-1,N-1}$$
$$-\sin^2(g\sqrt{N}\tau)\sin^2(g\sqrt{N-1}\tau)\rho_{N-2,N-2}\}. \tag{4.2}$$

We can now calculate the increase of the mean number of photons due to the atomic gain by

$$\langle\dot{N}\rangle=\sum_{N=0}^{\infty}N\dot{\rho}_{N,N}=r\sum_{N=0}^{\infty}\alpha_N\rho_{N,N}=r\langle\alpha_N\rangle, \tag{4.3}$$

in which

$$\alpha_N=\sin^2(g\sqrt{N+1}\tau)\{1+(p/2)[\sin^2(g\sqrt{N+1}\tau)-\sin^2(g\sqrt{N+2}\tau)]\}. \tag{4.4}$$

We notice that, in the semiclassical limit, $N\gg1$ and $N\gg g\tau$. Then the p-dependent terms in (4.4) cancel, so that

$$\alpha_N\simeq\sin^2(g\sqrt{N}\tau)\quad(N\gg1,g\tau). \tag{4.5}$$

In this limit α_N coincides with the semiclassical transition probability from the upper to the lower masing state, which we denoted by P in Sec. II. Equation (4.3) then becomes identical to Eq. (2.15).

On the other hand, for the variance $v=\langle N^2\rangle-\langle N\rangle^2$, we get

$$\dot{v}=2r\langle\alpha_N\Delta N\rangle$$
$$+r\langle\alpha_N-p\sin^2(g\sqrt{N+1}\tau)\sin^2(g\sqrt{N+2}\tau)\rangle, \tag{4.6}$$

with $\Delta N=N-\langle N\rangle$.

Again, in the semiclassical limit this equation becomes identical to Eq. (2.16), since $r\alpha_N$ then coincides with the drift coefficient A_N defined by (2.13).

In order to analyze the steady-state value of v, one must necessarily include dissipation so that the system can actually have a stable steady state. This is done in the usual way, in which one assumes that the damping time of the cavity (γ^{-1}) is much larger than the transit time τ. In this limiting case, we may neglect dissipation when considering the interaction of each atom with the field in the cavity, and add the usual loss contribution to the master equation (3.4),

$$(\dot{\rho})_{\text{loss}}=\frac{\gamma}{2}(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a). \tag{4.7}$$

Here we have assumed that the damping heat reservoir for the field is at zero temperatures. Adding the gain and loss contributions for the field, we get

$$\langle\dot{N}\rangle=r\langle\alpha_N\rangle-\gamma\langle N\rangle \tag{4.8}$$

and

$$\dot{v}=2r\langle\alpha_N\Delta N\rangle+r\langle\alpha_N-p\alpha_N^2\rangle-2\gamma v+\gamma\langle N\rangle. \tag{4.9}$$

In the last equation we have again assumed the semiclassical limit, in which the photon number is much larger than gr. Then the squares of the sine terms in Eq. (4.6) can be replaced by α_N as given by Eq. (4.5). The steady state for the mean photon number, denoted N_S, is now obtained as

$$r\langle\alpha_N\rangle=\gamma N_S. \tag{4.10}$$

The solution to this equation can be found graphically as the intersections of the gain curve (left-hand side of the above equation) with the loss curve (right-hand side), as shown in Fig. 2. It is easy to see that the steady state N_S is stable if and only if the slope of the gain curve is small-

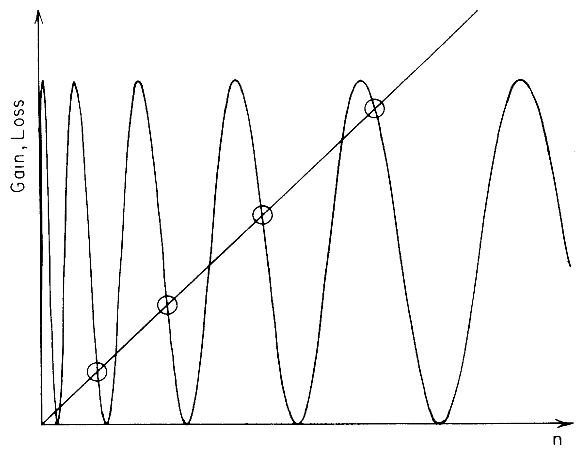

FIG. 2. The steady-state photon number is found by intersecting the loss straight line with the gain curve.

er than the slope of the loss curve, i.e., $r(d\alpha_N/dN)|_{N=N_S}<\gamma$. We next define the normalized photon number $n=N/N_{\rm ex}$ in which $N_{\rm ex}=r/\gamma$. Physically, the parameter $N_{\rm ex}$ is the number of excited atoms which enter the cavity during the cavity damping time. It is easy to see from Eq. (4.10) that $N_{\rm ex}$ also specifies the maximum number of photons inside the cavity so that n fulfills the relation $0\le n\le 1$. Furthermore, we denote the normalized derivative of α_N at the steady-state point by α'_S, i.e., $\alpha'_S=N_{\rm ex}(d\alpha_N/dN)_{N=N_S}$. The stability condition can then be written as

$$\alpha'_S<1 \; . \tag{4.11}$$

We now assume that the photon distribution is concentrated around a stable steady state N_S (in case of a multiple-peaked distribution, which is characteristic for a multistable system,[1-3] one must assume that the peak around N_S is much higher than the other peaks). We can then expand α_N in (4.9) around N_S, and obtain, up to second order in ΔN,

$$\langle \alpha_N \Delta N\rangle=\alpha'_S v/N_{\rm ex} \; . \tag{4.12}$$

Substituting this result into (4.9) allows us to calculate the steady-state variance:

$$v\simeq\frac{1}{1-\alpha'_S}\left\langle N-\frac{pN_{\rm ex}}{2}\alpha_N^2\right\rangle \; . \tag{4.13}$$

This expression explicitly exhibits the role of pumping statistics on the photon-number noise. In the micromaser case, one can get a sub-Poissonian distribution even if $p\to 0$ (provided $\alpha'_S<0$, which is depicted in Fig. 2), thus confirming the results of Ref. 2. As p increases, the sub-Poissonian character is enhanced even further.

A simpler expression for the variance can be obtained if $N_S\gg 1$. The average of α_N^2 in Eq. (4.13) can then be approximated by $\langle \alpha_N\rangle^2$, neglecting terms of order $1/N_S$. Using the steady-state equation (4.10), we get

$$v\simeq\frac{N_S}{1-\alpha'_S}\left[1-\frac{pn_S}{2}\right] , \tag{4.14}$$

in which the index S again denotes the steady-state value.

Equation (4.14) suggests that v can be reduced up to 50% by choosing $n_S=1$. This corresponds to an intersection of the loss curve with the gain curve at one of its maxima (see Fig. 3). However, it follows from the work in Refs. 2 and 3 that the probability peak, which corresponds to such a steady state, is usually not the highest one. Therefore, the approximation which led to Eq. (4.13) would not be valid. On the other hand, if N_S corresponds to the first peak of the gain curve (see Fig. 3), then the conditions for the validity of (4.13) are fulfilled, and we can indeed get a 50% reduction due to the effect of the regular atomic injection as compared to the Poissonian case.

In the laser case, Eq. (4.14) can also be applied, provided we replace N_S and α'_S by expressions which take the atomic decay into account. Averaging over the atomic lifetime, we obtain from Eq. (4.5), in the semiclassical limit,

$$\alpha_N=\int_0^\infty d\tau\,\Gamma e^{-\Gamma\tau}\sin^2(g\sqrt{N}\,\tau)=\frac{2g^2N}{\Gamma^2+4g^2N} \; . \tag{4.15}$$

In doing so, we have assumed that all atomic decay rates are equal to Γ. If we again assume a peaked photon distribution, we find from Eq. (4.10) the following expression for the steady state of the normalized photon number in a laser,

$$n_S=\frac{2g^2N_S}{\Gamma^2+4g^2N_S} \; . \tag{4.16}$$

Thus, in a laser, n_S is always smaller than $\frac{1}{2}$, approaching its upper bound in the high-intensity limit. Furthermore, it is easy to see from Eq. (4.15) that α'_S is always positive:

$$\alpha'_S=1-2n_s\ge 0 \; . \tag{4.17}$$

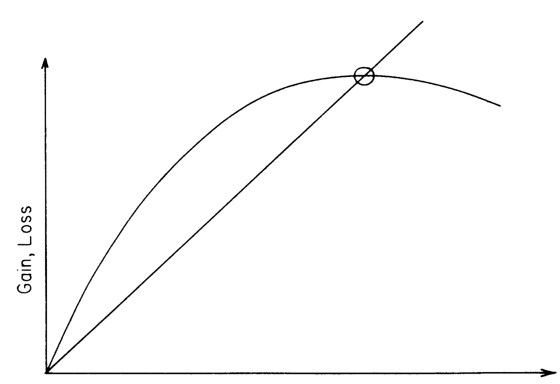

FIG. 3. Steady-state photon number in the particular case in which the loss straight line intersects the gain curve in the first maximum.

Substituting Eq. (4.17) into (4.14), we find the simple expression

$$v=\frac{1}{2n_S}\left[1-\frac{pn_S}{2}\right]N_S \ . \tag{4.18}$$

We can now easily discuss the influence of pump fluctuations on the photon-number variance in a laser. In the case of a Poissonian distribution over the incoming atoms ($p=0$), the variance is always larger than the mean number of photons, as we expect from the usual laser theory. In the high-intensity limit this variance has its smallest value, approaching a Poisson-like distribution of the photon number. In the case of a regular distribution, in which $p=1$, we can obtain a significant noise reduction far above threshold. The normalized photon number n_S then approaches the value $\frac{1}{2}$, so that one can obtain 25% reduction of the photon-number fluctuations below the shot-noise level.

At this point we should pay closer attention to the expansion (3.9), since otherwise it should be doubtful whether our results are valid in the limit $p=1$. In order to discuss this problem, it is convenient to introduce the high-N expansion of the master equation, as presented in Ref. 3.

Defining $t'=rt$, $\delta=1/N_{\rm ex}$, $n=N/N_{\rm ex}$, and $\phi=g\sqrt{N_{\rm ex}}\tau$, and converting $\rho_{N,N}$ into a continuous variable function $\rho(n,t')$, one can rewrite Eq. (4.2) as follows:

$$\begin{aligned}\frac{d\rho(n,t')}{dt'}=&-\sin^2\phi\sqrt{n+\delta}\rho(n,t')+\sin^2\phi\sqrt{n}\,\delta(n-\delta,\,t')\\&+\frac{rp}{2}\{-\sin^4\phi\sqrt{n+\delta}\rho(n,t')+[\sin^4\phi\sqrt{n}+\sin^2(\phi\sqrt{n}\,)\sin^2\phi\sqrt{n+\delta}]\rho(n-\delta,\,t')\\&-\sin^2\phi\sqrt{u}\,\sin^2\phi\sqrt{n-\delta}\rho(n-2\delta,\,t')\}\ .\end{aligned}\tag{4.19}$$

Now, we expand Eq. (4.19) in powers of δ, which corresponds to a high-N expansion, getting

$$\frac{d\rho(n,t')}{dt'}=-\delta\frac{\partial}{\partial n}[a_1(n)\rho(n,t')]+\frac{\delta^2}{2}\frac{\partial^2}{\partial n^2}[a_2(n)\rho(n,t')]+O(\delta^3,(\delta\phi)^3)\ , \tag{4.20}$$

where

$$\begin{aligned}a_1(n)&=\sin^2\phi\sqrt{n}+\frac{\delta}{2n}\sin2\phi\sqrt{n}-\frac{3rp}{4}\frac{\delta}{n}\sin^3\phi\sqrt{n}\,\cos\phi\sqrt{n}\ ,\\a_2(n)&=\sin^2\phi\sqrt{n}+\frac{\delta}{2n}\sin2\phi\sqrt{n}\ .\end{aligned}\tag{4.21}$$

The above equation corresponds to the usual Poissonian distribution for the incoming atoms, and therefore one can make the identification $[(M-1)\rho]_{N,N}$ with the right-hand side of (4.19). This implies that

$$\begin{aligned}[(M-1)^2\rho]_{n,n}&=-\delta\frac{\partial}{\partial n}\{a_1(n)[(M-1)\rho]_{n,n}\}+\frac{\delta^2}{2}\frac{\partial^2}{\partial n^2}\{a_2(n)[(M-1)\rho]_{n,n}\}+O(\delta^3,(\delta\phi)^3)\\&=\delta^2\frac{\partial}{\partial n}\left[a_1(n)\frac{\partial}{\partial n}[a_1(n)\rho(n,t')]\right]+O(\delta^3,(\delta\phi)^3)\\&=\delta^2\frac{\partial^2}{\partial n^2}[a_1^2(n)\rho(n,t')]-\delta^2\frac{\partial}{\partial n}\left[\frac{\partial a_1(n)}{\partial n}a_1(n)\rho(n,t')\right]\ .\end{aligned}\tag{4.22}$$

Therefore, the first two terms of expansion (3.9) yield, up to second order in δ and $\delta\phi$,

$$\frac{d}{dt'}\rho(n,t')=-\delta\frac{\partial}{\partial n}\left[\left[a_1(n)+\frac{p\delta}{2}\frac{\partial a_1(n)}{\partial n}a_1(n)\right]\rho(n,t')\right]+\frac{\delta^2}{2}\frac{\partial^2}{\partial n^2}\{[a_2(n)-pa_1^2(n)]\rho(n,t')\}+O(\delta^3,(\delta\phi)^3)\ . \tag{4.23}$$

We again obtain the pumping-statistics-dependent correction to the diffusion coefficient, plus a correction of order δ to the drift coefficient [the existence of this correction is already evident in (4.4)]. In comparing (4.23) with the high-N expansion of (4.5), one should keep in mind that $a_1(n)$, as defined in Eq. (4.21), already contains terms of order δ. When these contributions are taken into account, one gets the same result for the drift coefficient from both equations.

It is clear that, since $(M-1)\rho$ is of order δ, $(M-1)^2\rho$ will bring contributions up to order δ^2, and analogously $(M-1)^n\rho$ will bring corrections of order δ^n. This proves that, as far as the drift and the diffusion coefficients are concerned, one does not need to beyond the second term in expansion (3.9), as long as $\delta,\delta\phi \ll 1$.

B. Phase diffusion

Following a procedure similar to Eqs. (4.19)–(4.23), one can now write a differential equation for the off-diagonal elements of ρ. Defining $g(n)\equiv\rho_{n,\,N+1}$, and with Eqs. (4.1) and (4.7), we get

$$\begin{aligned}\dot g(N)=r\{&[\cos(g\sqrt{N+1}\tau)\cos(g\sqrt{N+2}\tau)-1]g(N)+\sin(g\sqrt{N}\,\tau)\sin(g\sqrt{N+1}\tau)g(N-1)\}\\&+\frac{rp}{2}\{[2\cos(g\sqrt{N+1}\tau)\cos(g\sqrt{N+2}\tau)-1-\cos^2(g\sqrt{N+1}\tau)\cos^2(g\sqrt{N+2}\tau)]g(N)\\&\quad+[2\sin(g\sqrt{N}\,\tau)\sin(g\sqrt{N+1}\tau)-\cos(g\sqrt{N+1}\tau)\cos(g\sqrt{N+2}\tau)\sin(g\sqrt{N}\,\tau)\sin(g\sqrt{N+1}\tau)\\&\quad-\sin(g\sqrt{N}\,\tau)\cos(g\sqrt{N}\,\tau)\sin(g\sqrt{N+1}\tau)\cos(g\sqrt{N+1}\tau)]g(N-1)\\&\quad-\sin^2(g\sqrt{N}\,\tau)\sin(g\sqrt{N-1}\tau)\sin(g\sqrt{N+1}\tau)g(N-2)\}-\frac{\gamma}{2}(2N+1)g(N)+\gamma\sqrt{(N+1)(N+2)}g(N+1)\;.\end{aligned}\tag{4.24}$$

In order to get the phase-diffusion constant, we use the method presented in Ref. 3. We express this equation in terms of the normalized variable $n=N/N_{\rm ex}$, and expand it in powers of $\delta=1/N_{\rm ex}$. This yields an equation of the form

$$\frac{d}{dt'}g(n,t')=-\mu(n)g(n,t')-\delta\frac{\partial}{\partial n}[f_1(n)g(n,t')]+\frac{\delta^2}{2}\frac{\partial^2}{\partial n^2}[f_2(n)g(n,t')]+O(\delta^3,(\delta\phi)^3)\;. \tag{4.25}$$

The coefficient $\mu(n)$, evaluated at the steady-state point n_S, is the phase diffusion constant (divided by r, since $t'=rt$). This interpretation is valid only if the distribution is sufficiently concentrated around n_S (see, e.g., Refs. 3 and 8).

Following this procedure, we find, for the phase-diffusion constant,

$$\mu=\frac{(1+\phi^2)\gamma}{8N_S}\;, \tag{4.26}$$

as in usual laser theory.[8]

This result is independent of the parameter p: phase diffusion, at least up to order δ^2, does not depend on pumping statistics. It will be shown in a planned, forthcoming paper that this result remains valid up to all orders of δ.

We notice again that, for large enough pumping, only the first two terms in expansion (3.9) need to be considered; the remaining terms yield contributions of order δ^3 or $(\delta\phi)^3$ to the diffusion constant.

V. CONCLUSION

The fact that usual laser theory implicitly assumes a Poissonian statistic for the injected atoms has been pointed out recently.[1–3] We have been able to derive a master equation which is valid not only in the Poissonian case, but also for any general Bernoulli-type distribution of the incoming excited atoms. The regular distribution, which corresponds to equal-time intervals between consecutive atoms, is especially interesting, since it leads to the maximum possible reduction of the photon-number noise. This reduction can be up to 50% for the internal field of a micromaser and up to 25% for a laser with equal atomic decay rates and far above threshold.

In this paper we have based our treatment on the master-equation approach. Further insight into the influence of the statistical properties of the pump on the laser dynamics can be obtained through a very different analysis, involving Langevin-operator techniques. This is planned to be the subject of a separate paper.

Note added. After finishing this work, we learned that M. J. Collet, F. Haake, and D. Walls have independently arrived at conclusions similar to those presented here.[12]

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