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Invariant fixed point in stratified continuum percolation

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Stratified continuum percolation is a percolation model based upon self-similarity that can describe correlated transport and connectivity problems. It is demonstrated that this construction of correlated percolation retains some of the properties of random percolation. Most importantly, the critical threshold remains well defined, independent of changing correlations. These results introduce a new approach to understanding correlated flow systems.

A wide variety of patterns occurring in nature frequently have spatial correlations. Examples of spatially correlated patterns include self-similar fractal structures generated through fragmentation and aggregation processes.[1] Despite early success in applying real-space renormalization groups to correlated percolation systems,[2–4] much of recent percolation theory has continued to deal with standard uncorrelated percolation. One reason for this is that correlations can occur in arbitrary variety, and depending on the form or strength of the correlations, the system may or may not be in the same universality class. More importantly, correlations alter critical thresholds. In this paper, a correlated continuum percolation model is presented which retains many of the properties of standard uncorrelated percolation. Correlations are introduced naturally by recursively scaling up successive strata of standard percolation patterns. Using real-space renormalization, it is demonstrated that this correlated construction retains a well-defined critical threshold.

Stratified continuum percolation was initially developed to describe the patterns generated by the flow of fluid through natural fractures in granite rock.[5,6] The flow patterns were found to be highly correlated, with fractal dimensions that changed with changing stress on the fracture. This model was also applied to the problem of the dependence of the flow on deformation of the fracture as the void space is closed.[7] These applications clearly required a continuum model,[8] because of the continuous distribution of void apertures and the absence of any underlying lattice in a fracture. Furthermore, the model had to include spatial correlation to adequately describe the fractal patterns measured experimentally.

Stratified percolation patterns are constructed recursively as a hybrid between standard random continuum percolation and a random Sierpinski carpet. The Sierpinski carpet provides the self-similar skeleton upon which successively smaller tiers of standard percolation patterns are applied. The stratified percolation construction is shown graphically in Fig. 1. The construction begins by defining an initial square region, called the first tier. Within this tier, N sites are randomly selected. Around each site, a square region is defined with linear dimension reduced in scale by a factor of b from the size of the first tier. These N smaller squares constitute the second tier. The construction then continues recursively with N squares, reduced again by the scale factor b, defined within each of the squares making up the second tier. The recursive construction is terminated when the size of the square regions reaches a minimum cutoff. At this level, solid squares are plotted within the final tier, allowing overlap. The stratified percolation pattern is defined by the final set of plotted squares. On the first tier wrap-around periodic boundary conditions are applied in which points positioned beyond a boundary of the first tier are plotted within the opposing boundary. This operation specifies the first tier size as the upper cutoff of the scaling pattern. A stratified percolation pattern is shown in Fig. 2 for five tiers, $N=7$ and $b=2.37$. A total of $7^5=16807$ points have been plotted in the figure. The fractal dimension of the pattern was measured using the two-point correlation function $F(R)=\langle f(r)f(r+R)\rangle = R^{-(2-D)}$ between the upper and lower cutoffs where D is the fractal dimension. The fractal dimension in Fig. 2 was measured to be $D=1.84\pm 0.02$. The two-point correlation function operates on the entire pattern, and must be distinguished from the pair-connectedness function $\xi \approx (p-p_c)^{-\nu}$ used in the statistical analysis of percolation clusters.

The pattern in Fig. 2 is above the percolation threshold.

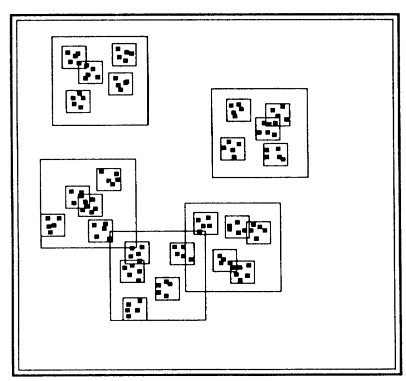

FIG. 1. Recursive construction of stratified percolation for three tiers, $N=5$ and $b=4$. The large square is the first tier. The five smaller squares located randomly within this square are the second tier, reduced by a factor of 4. Within each of these smaller tiers, five more squares are located, again reduced in scale by a factor of 4. Within the third tiers, five solid squares are finally plotted to construct the pattern. A total of $5^3=125$ squares are plotted in this figure.

FIG. 2. Stratified percolation pattern for 5 tiers, $N=7$ and $b=2.37$. The total area fraction occupied (colored black in the figure) is 0.64. The fractal dimension is $D=1.84$. This pattern is slightly above the percolation threshold.

The area fraction occupied (black) is 0.64, already reduced from the critical area fraction $A_c=0.7$ of standard continuum percolation. The spatial correlation introduced by the recursive construction has lowered the critical threshold. This lowering of the critical threshold can be understood qualitatively through the tendency of the correlation to "clump" the percolation clusters. Connected clusters therefore develop for relatively small occupancies. The stronger the correlation, the more the sites clump into connected clusters, and the lower the percolation threshold becomes. The strongly correlated structure of Fig. 2 can be seen more easily by considering the density of sites, plotted in Fig. 3. White denotes the absence of sites. Black denotes the highest density of points.

It is easy to see that, by construction, standard continuum percolation patterns arise within every tier. This presents the possibility of defining an area fraction per tier which is calculated in the identical manner as for standard continuum percolation. For N squares of reduced size b plotted within a square region of unit area, the result from standard continuum percolation is

$$a(N,b)=1-(1-1/b^2)^N, \tag{1}$$

FIG. 3. Density of sites for the pattern in Fig. 2. White denotes unoccupied sites. Black denotes the highest density of sites.

where a denotes the area fraction per tier. A recursive expression for the total area fraction of the stratified percolation pattern can be defined by applying Eq. (1) for the changing area fraction of each successive tier. For n tiers this recursive expression is

$$\begin{aligned} A(n,N)&=1-[1-A(n-1,N)/b^2]^N, \\ A(1,N)&=a(N,b). \end{aligned} \tag{2}$$

A simpler, nonrecursive approximate expression is obtained by expanding Eq. (2) as

$$A(n,N)=a[a+(1-a)a^2]^{n-1}. \tag{3}$$

The approximate expression for the fractal dimension is derived in a similar fashion as

$$D=2+\frac{\ln[a+(1-a)a^2]}{\ln b}. \tag{4}$$

The critical percolation threshold for stratified percolation was determined through Monte Carlo simulations using the real-space continuum renormalization group.[9] The probability that the patterns with N points per tier percolated within test squares of linear size s is given by the percolation probability $R(N;s)$. The fixed point in the percolation probability defines the percolation threshold N^* such that $R(N^*;s)=R(N^*;s')$ is invariant to changes in the scale $s\to s'$ of the test regions. The percolation probability is conventionally written as a function of the occupied area fraction $R(A;s)$. The critical area fraction is then given by $A_c=A(n,N^*)$. To determine if a specific pattern percolated, the continuum pattern was first "digitized" into a 300×300 array. The digitizing resolution was set so that the smallest black square consisted of 16 occupied sites. This fine resolution ensures that the continuum properties are maintained at the smallest level, yet at the same time allowing the use of site percolation algorithms. A pattern was deemed to percolate within a test square with linear size s if a connected path existed top to bottom or side to side in the test square.[10] The connected path was identified using a cluster numbering algorithm.[11]

Percolation probabilities $R(A;s)$ for stratified percolation were tabulated by varying the number of points per tier N. The results for 100 simulations of three tiers with $b=4.22$ are shown in Fig. 4(a), plotted for $s=1$, $s=\frac{1}{2}$, $s=\frac{1}{3}$, and $s=\frac{1}{6}$ as functions of total area fraction. The fixed point in the percolation probability occurs at $A_c=0.53\pm0.02$, reduced significantly from $A_c=0.7$ for standard continuum percolation. By including correlation into the percolation problem, a well-defined fixed point has been lost. This is a general feature of all correlated percolation problems. Stratified percolation, on the other hand, incorporates correlations while keeping the essential features of standard continuum percolation. In particular, the area per tier, $a(N,b)$, plays a key role in the stratified percolation construction. The percolation probability of Fig. 4(a) is replotted in Fig. 4(b) as $R(a;s)$, a function of the area per tier. The striking result of the plot is that the critical threshold occurs at $a_c=0.71\pm0.02$, which is the canonical value from standard percolation to within the

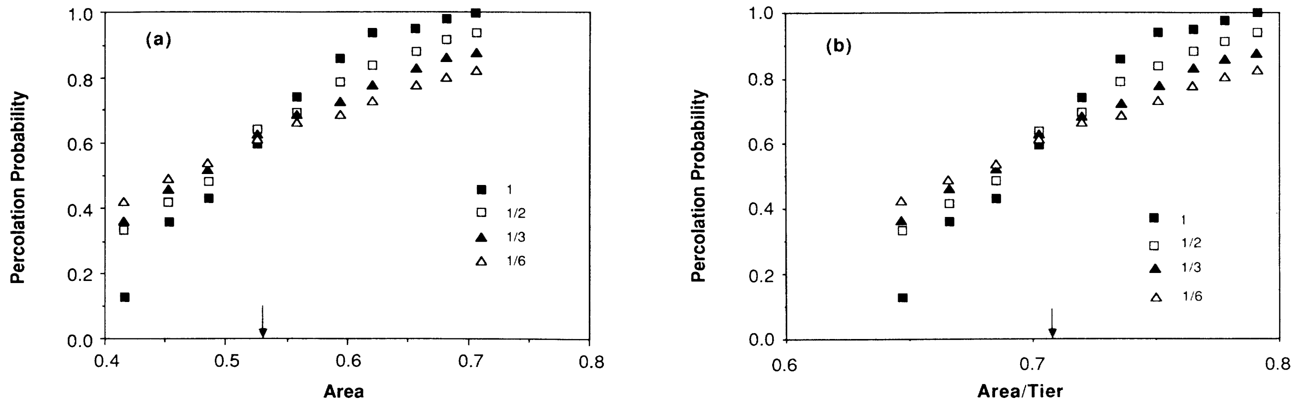

FIG. 4. Percolation probability of three-tier stratified percolation patterns calculated using real-space renormalization for four grid sizes: $s=1$, $s=\frac{1}{2}$, $s=\frac{1}{3}$, $s=\frac{1}{6}$. The fixed point defines the percolation threshold. This is shown in (a) as a function of occupied area fraction. The threshold for three tiers is $A_c=0.53$. The percolation probability is plotted as a function of area fraction per tier in (b). Plotted in this way, the percolation threshold has remained invariant at the canonical value of 0.7.

accuracy of the simulations. Monte Carlo results for 350 simulations of five tiers with $b=2.37$ also indicate $a_c=0.71\pm0.01$. *Therefore, the stratified percolation construction retains the well-defined fixed point.* As correlations are changed within the constructs of stratified percolation, the critical threshold (stated in terms of the area fraction per tier) remains invariant within the computational uncertainty.

Stratified percolation may be expected to be in the same universality class as standard percolation because it already retains many of the standard features. Evidence for universality class is obtained by considering the critical exponent ν of the pair-connectedness length. This exponent is derived from $R(a;s)$ by the expression[9,12]

$$\nu=\frac{\ln(s)-\ln(s')}{\ln[dR(a;s)/da]_{a_c}-\ln[dR(a;s')/da]_{a_c}},$$

where the derivatives are evaluated at the critical threshold. Evaluating the Monte Carlo $R(a;s)$ at a_c leads to a value of $\nu=1.7\pm0.4$. The magnitude of the computational uncertainty makes it impossible at this time to draw any conclusions about the universality class of stratified percolation. Further simulations are necessary to reduce the uncertainty in this value.

In conclusion, stratified continuum percolation is a new model that can describe transport and connectivity problems that possess self-similarity, such as flow-through fractures. Most importantly, this construction of correlated percolation retains many of the results from standard percolation. In particular, the critical threshold of stratified percolation remains well defined, independent of changing correlations. This correlated percolation model was not originally motivated by magnetic problems, or by dynamic aggregation processes such as cluster-cluster aggregation or surface island formation. However, work is in progress to determine if this recursive construction of percolation networks has a wider applicability to general self-similar percolation processes. If so, then the ease with which the critical threshold can be recognized in stratified patterns promises a powerful tool for many correlated percolation problems.

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