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Quantum tunneling and thermal activation in the parametric oscillator

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The mean time taken for the degenerate parametric oscillator to quantum tunnel between its two above-threshold stable states is calculated in the large photon-number limit. The result is compared with an earlier approximate result using a truncated Fokker-Planck equation in the Wigner representation. This earlier theory is equivalent to stochastic electrodynamics. Our quantum theory results indicate orders of magnitude faster tunneling far above threshold when compared to the truncated Wigner theory.

The problem of quantum fluctuations in nonthermal equilibrium systems is an open area of modern physics. Recently, the discovery of squeezed states[1] has demonstrated the existence of new, phase-dependent statistical properties in laser-driven devices. These types of statistical properties are not obtainable in thermal-equilibrium Gibbs ensembles. In fact, in most cases the relevant density matrix for nonthermal equilibrium systems is only available through direct solutions of their Heisenberg or master equations.[2,3] In this Rapid Communication, we present a treatment of quantum tunneling in a degenerate parametric oscillator. This provides a new and sensitive testing ground for theories of nonlinear quantum fluctuations.

An early treatment of the degenerate parametric oscillator was developed by Graham[2] using a Wigner[4] operator representation. This involved an approximation in which third-order derivative terms were dropped from the time-evolution equation. The resulting truncated Wigner theory is identical to stochastic electrodynamics[5] (SED). This treatment of quantum fluctuations is essentially a hidden variable theory[6] with a classical phase space. Surprisingly, the truncated Wigner, or SED, treatment is sufficient to reproduce the results of squeezing experiments in the below-threshold parametric amplifiers.[7] However, use of the truncation approximation is questionable in a tunneling calculation.[8] Graham also omits pump quantum noise, which has a large effect above threshold.

We present a new calculation of the mean tunneling time using a coherent-state positive-P representation[9] which allows the derivation of exact stochastic equations, since the positive-P representation gives no third-order terms. Instead, it has extra phase-space dimensions that allow a stochastic description of the quantum features, even when coherent-state superpositions occur. Our results for tunneling times are completely different from those of the truncated Wigner theory. The tunneling rates agree in the near threshold limit, but our results give orders of magnitude faster tunneling for driving fields well above threshold. Of most interest is the fact that the difference between the quantum mechanical and the truncated Wigner calculation is greatly augmented in the limit of macroscopic systems.

The behavior of the degenerate parametric oscillator may be formulated in terms of a simple model consisting of an optical cavity and two quantized electromagnetic field modes with frequencies ω and 2ω which interact via a $\chi^{(2)}$ susceptibility. Both modes are taken to be resonant with the cavity and losses at the cavity mirrors are included. The input pump field has frequency 2ω and is treated classically. This gives rise to the interaction Hamiltonian for the system

$$\hat{H} = i\hbar\bar{g}(\hat{a}^{\dagger 2}\hat{b} - \hat{a}^{2}\hat{b}^{\dagger})/2 + i\hbar E(\hat{b}^{\dagger} - \hat{b}) + \hat{H}_{L} . \quad (1)$$

Here $\hat{a},\hat{a}^{\dagger}$ and $\hat{b},\hat{b}^{\dagger}$ are the annihilation and creation operators for the two modes with frequencies ω and 2ω, respectively, and $\bar{g}$ is the intermode coupling constant. Thus the first term in the Hamiltonian describes the interaction between the two modes. E is the classical pump amplitude injected into the pump mode $\hat{b}$ at frequency 2ω. The third term, $\hat{H}_{L}$, describes the losses for the two modes at the cavity mirrors, with decay rates γ_a and γ_b for modes a and b, respectively.

This Hamiltonian can be used to obtain an operator master equation. A Fokker-Planck equation is then derived using the positive-P representation to give positive definite diffusion.[9] This is equivalent to four Ito stochastic differential equations. These four equations describe the degenerate parametric oscillator in terms of the vari-

ables α, $\alpha^\dagger$, β, and $\beta^\dagger$. They are simplified by adiabatically eliminating the pump mode $\hat{b}$. The two remaining equations for the subharmonic mode $\hat{a}$ in the zero-temperature limit are as follows:[3]

$$d\alpha = [-\alpha + \alpha^\dagger(\lambda - g^2\alpha^2)]d\tau + (\lambda - g^2\alpha^2)^{1/2}dW_1 , \quad (2a)$$

$$d\alpha^\dagger = [-\alpha^\dagger + \alpha(\lambda - g^2\alpha^{\dagger 2})]d\tau + (\lambda - g^2\alpha^{\dagger 2})^{1/2}dW_2 , \quad (2b)$$

where

$$\langle dW_i dW_j \rangle = \delta_{ij}d\tau .$$

In (2), $g = \bar{g}/(2\gamma_a\gamma_b)^{1/2}$ is the coupling scaled by the geometric mean of the mode decay rates. In the following calculations we take $g^2 \ll 1$, corresponding to the current experimental limits of large threshold photon numbers. The time τ is measured in cavity lifetimes γ_a^{-1}. Also, $\lambda = |\bar{g}E|/(\gamma_a\gamma_b)$ is the pump amplitude scaled so that the threshold condition of parametric oscillation is at $\lambda = 1$. The terms dW_1 and dW_2 are independent, δ-correlated, real noise variables which have a Gaussian probability distribution.[9] The complex variables α and $\alpha^\dagger$ are stochastically independent in the positive P representation, although they represent the Hermitian conjugate operators $\hat{a}$ and $\hat{a}^\dagger$ and have conjugate expectation values and moments.

The steady-state limit is well known from the complex-P representation solution[3] for the equivalent Fokker-Planck equation. However, Wolinsky and Carmichael have recently shown that the dynamics of the quantum stochastic equations are simplified by the existence of a real, bounded subspace in which any stochastic trajectory can be trapped.[10] Given that a trajectory starts in this real subspace of $g^2\alpha^2 < \lambda$ and $g^2\alpha^{\dagger 2} < \lambda$, inspection of the terms in Eq. (2) shows that both the drift and noise terms will remain real, so that the trajectory must stay on the real plane. The trajectory is prevented from crossing the boundaries at $g^2\alpha^2 = \lambda$ and $g^2\alpha^{\dagger 2} = \lambda$, because there the quantum noise transverse to it vanishes. Thus the trajectory must follow the deterministic flow inwards or be driven along the boundary by the other noise component. On this manifold, the complex and positive P representations are identical.

Although the manifold is bounded, it has regions in which the stochastic equations become unstable to complex fluctuations. These could be caused by phase fluctuations in the pump or by thermal noise. The eigenvalues controlling the stability can be obtained by expanding the stochastic equations (2) to first order in their imaginary parts about the real manifold. They are

$$\Lambda_\pm = -(1 + 2g^2\alpha\alpha^\dagger) \pm [(\lambda - g^2\alpha^2)(\lambda - g^2\alpha^{\dagger 2})]^{1/2} . \quad (3)$$

Above threshold the unstable regions stretch from the $\alpha = -\alpha^\dagger$ corners of the manifold across the saddle point at $\alpha = \alpha^\dagger = 0$. This saddle point is the region through which a trajectory is most likely to travel while tunneling from one state to the other. While in our model this does not cause problems, it suggests that for very small g^2, corrections due to phase fluctuations or nonzero temperatures could have relatively large effects on tunneling times.

We now transform the variables α and $\alpha^\dagger$ to give constant diffusion, or additive stochastic noise

$$u = \sin^{-1}\left(\frac{g\alpha}{\sqrt{\lambda}}\right) + \sin^{-1}\left(\frac{g\alpha^\dagger}{\sqrt{\lambda}}\right) , \quad (4a)$$

$$v = \sin^{-1}\left(\frac{g\alpha}{\sqrt{\lambda}}\right) - \sin^{-1}\left(\frac{g\alpha^\dagger}{\sqrt{\lambda}}\right) . \quad (4b)$$

These new variables are constrained to have a range such that $|u| - |v| \leq \pi/2$. Referring back to the variables α and $\alpha^\dagger$ it can be seen that the u axis represents the classical subspace of the phase space, where $\alpha = \alpha^\dagger$. Thus the variable v is a nonclassical dimension which allows the creation of quantum features. The stochastic equations corresponding to these variables are

$$du = \left\{\lambda\sin(u) - \sigma\left[\tan\left(\frac{u+v}{2}\right) + \tan\left(\frac{u-v}{2}\right)\right]\right\}d\tau + \sqrt{2}g dW_u , \quad (5a)$$

$$dv = \left\{-\lambda\sin(v) - \sigma\left[\tan\left(\frac{u+v}{2}\right) - \tan\left(\frac{u-v}{2}\right)\right]\right\}d\tau + \sqrt{2}g dW_v . \quad (5b)$$

Here $\sigma = 1 - g^2/2$. These Ito equations have a corresponding Fokker-Planck equation and a probability distribution in the limit as $\tau \to \infty$ of

$$P(u,v) = N\exp[-V(u,v)/g^2] , \quad (6)$$

where the potential $V(u,v)$ is

$$V(u,v) = -2\sigma\ln|\cos(u) + \cos(v)| + \lambda\cos(u) - \lambda\cos(v) . \quad (7)$$

Above threshold the potential has two minima corresponding to the stable states of the oscillator. These minima have equal intensities and amplitudes of opposite sign, and are at classical locations with $\alpha = \alpha^\dagger$:

$$(u_0, v_0) = (\pm 2\sin^{-1}[(\lambda - \sigma)^{1/2}/\sqrt{\lambda}], 0) , \quad (8a)$$

or

$$g\alpha_0 = \pm(\lambda - 1 + g^2)^{1/2} . \quad (8b)$$

There is also a saddle point at $(u_s, v_s) = (0,0)$. An important feature to note is that along the u axis the second derivative of the potential in the v direction is always positive. The classical subspace ($v = 0$) is therefore at a minimum of the potential with respect to variations in the nonclassical variable v. This valley along the u axis between the two potential wells is the most probable path for a stochastic trajectory in switching from one well to the other. Since there is only one such path on the manifold, the switching rate between them will be dominated by the rate due to trajectories along this route. Using the method of Landauer and Swanson[11] the mean time taken for the oscillator to switch from one state to the other in the limit of $g^2 \ll 1$ is[12]

$$T_P = \frac{\pi}{\gamma_a}\left[\frac{\lambda + \sigma}{\lambda(\lambda - \sigma)^2}\right]^{1/2}\exp\left[\frac{2}{g^2}[\lambda - \sigma - \sigma\ln(\lambda/\sigma)]\right] . \quad (9)$$

We note that increasing the pump amplitude λ increases the switching time, as does reducing g^2. Small g^2 corresponds to large threshold photon numbers.

The switching time calculated by us for the degenerate parametric oscillator is markedly different to a previously calculated result[2] which used the Wigner representation. This had only a classical phase space, with coordinates x and p, and could therefore be written as a function of a single variable $\alpha = x + ip$. In general, an exact Wigner distribution can develop negative values. In the truncation approximation it is positive, with

$$W(p,x) = \exp[-\phi(x+ip)], \tag{10}$$

where

$$\phi(x+ip) = x^2+p^2+1\tfrac{1}{2}g^2(x^2+p^2)^2-\lambda(x^2-p^2). \tag{11}$$

Above threshold this potential has two minima, at $\alpha = \pm(\lambda-1)^{1/2}$. In the limit of large threshold photon numbers, these minima are very close to those obtained in (8). After taking the different operator correspondences into account, the Graham theory[2] and the positive-P theory[3] predict identical results for small linear fluctuations near threshold. However, the result for the tunneling time using this approximate Wigner distribution is

$$T_W = \frac{1}{4\pi\gamma_a}\left[\frac{\lambda+1}{\lambda(\lambda-1)^2}\right]^{1/2}\exp\left[\frac{1}{g^2}(\lambda-1)^2\right]. \tag{12}$$

Comparing this to (9), we see that the terms in the exponential are very different in character. The result of (12) predicts slower switching above threshold. Since the differences can be many orders of magnitude (see Fig. 1) it should be easy to experimentally observe this difference.

The approximate Wigner calculation involves the dropping of third-order derivative terms in order to get a second-order Fokker-Planck equation. As noted by one of us,[8] the effect of the dropping of higher-order derivative terms in Fokker-Planck equations is highly representation dependent. Although it is often a good approximation for linear fluctuations, it can change switching times by many orders of magnitude. Our calculation in the positive-P representation involves no such truncation, and gives an exact second-order Fokker-Planck equation. There is however an enlargement of the classical phase space. The exact Wigner function solution can be obtained by convolving the exact P function[8] with a complex Gaussian. The equation of motion of this exact solution therefore has identical eigenvalues and tunneling rates to our result.

An interesting feature of the truncated Wigner Fokker-Planck equation is that the diffusion term is identical to that for classical thermal noise, with an occupation number of half a photon per mode. The quantum noise appears to be caused only by losses at the cavity mirrors which couple to vacuum fluctuations. In this sense, the truncated Wigner theory is identical to stochastic electrodynamics. However, the higher-order derivatives in the Wigner Fokker-Planck equation are dependent on the interaction strength g, so that the true nonlinear character of the quantum noise is removed when these are neglected. This does not occur with the exact positive P Fokker-Planck equation. In fact, in normally ordered P representations the quantum noise is *solely* due to the interaction, because normally ordered vacuum fluctuations vanish identically at zero temperature.

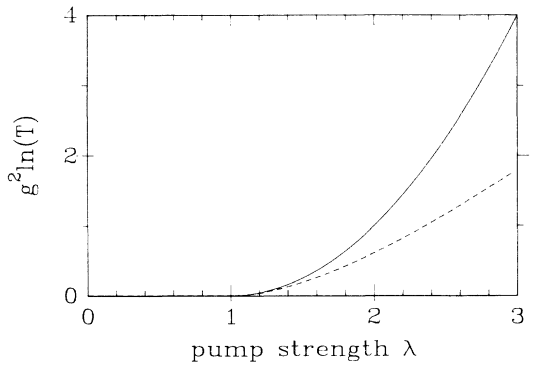

FIG. 1. This figure shows the variation in the logarithm of the tunneling time with the pump amplitude λ. The solid line is the truncated Wigner result and the dotted line is our positive-P result. With $g^2 = 0.1$, a unit difference on the vertical log scale is equivalent to a difference in switching times of a factor of e^{10}.

Thus, the neglect of third-order derivatives implies that the resulting SED-type theory is not equivalent to quantum mechanics for this nonlinear problem. With pump noise included the Wigner theory should be accurate for linear fluctuations. Above threshold these additional pump noise terms are very large and cannot be neglected, even for linearized calculations. However, the inclusion of pump quantum noise violates the potential conditions on the Wigner Fokker-Planck equation. They are omitted in Graham's treatment. We will compare our result with the full SED theory elsewhere.

These two results indicate the difference between the rates of classical thermal activation and true quantum tunneling. Classical thermal activation rates are slower than quantum tunneling rates far above threshold, because for large barriers the thermal trajectory must go over the barrier. However, a quantum process can shortcut this by tunneling. Our calculations indicate that care must be taken with a stochastic electrodynamics approach to vacuum fluctuations. An experimental measurement of the tunneling rate would provide direct evidence to distinguish quantum mechanics from the earlier theory, in a situation with a relatively large or macroscopic number of photons present in the quantum system.[13] This would be observable as a random telegraph signal in the output detected with a local oscillator.[14]

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