Squeezed-state wave functions and their relation to classical phase-space maps

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(Received 15 March 1989)

Coordinate wave functions for the one-mode squeezed states produced by the quantum analog of the general linear transformation in phase space are calculated. The probability density $[|\psi(q)|^2]$ for these states is Gaussian with center predicted by the classical transformation. The quantum image (which includes the traditional two-mode squeeze operator) of a three-parameter symplectic map in two-mode phase space equally generates squeezed states having Gaussian $|\psi(q_1,q_1)|^2$. The center of the two-mode Gaussian is again predicted by the classical mapping.

INTRODUCTION

Squeezed states of the electromagnetic field, introduced independently by Stoler[1] and Lu,[2] have in recent years received considerable attention, both theoretical[3] and experimental.[4] Yuen[5] discussed squeezed states as eigenstates of the operator

$$b=\mu a+\nu a^{\dagger} , \tag{1}$$

where a and $a^{\dagger}$ are, respectively, photon annihilation and creation operators and μ and ν are arbitrary complex numbers satisfying

$$|\mu|^2-|\nu|^2=1 . \tag{2}$$

Although Yuen derived many of the properties of these squeezed states and Schleich and Wheeler[6] give a wave function for the eigenstates of (1), most attention has been focused (partly because of their greater experimental desirability) on minimum uncertainity states[7] (MUS) whose annihilation operator has the form

$$b=a \cosh\lambda+a^{\dagger}\sinh\lambda \tag{3a}$$

or the slightly more generaly squeezed state having

$$b=a \cosh\lambda+e^{i\theta}a^{\dagger}\sinh\lambda , \tag{3b}$$

with λ and θ real.

A closer connection to experiment can be made by noting that the squeezed state is obtained by squeezing the coherent state. The squeezing performed by apparatus such as a pumped parametric oscillator is represented by a squeeze operator acting on the coherent input state. The simplest such (two-photon) squeeze operator takes the form

$$U(\lambda)=e^{\lambda[(a^{\dagger})^2-a^2]/2} \tag{4a}$$

and produces eigenstates of (3a), while the slightly more general form

$$U(\xi)=e^{[\xi(a^{\dagger})^2-\xi^* a^2]/2} \tag{4b}$$

with $\xi=\lambda e^{i\theta}$ produces eigenstates of (3b). When the result of a single squeeze is an eigenstate of (3b), sequential squeezing (two successive squeezes may be feasible) result in eigenstates of (1).

As pointed out by Han *et al.*,[8] linear transformations in phase space form a natural language for squeezed states. Indeed, operator (4a) may be recast in the intuitively appealing form[9]

$$U(\lambda)=\frac{1}{2\pi}\left[\frac{\mu+\mu^{-1}}{2}\right] \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\left|\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\ p\end{pmatrix}\right| , \tag{5}$$

where $\mu=e^{\lambda}$, $|{q \atop p}\rangle$ is the coherent state in canonical phase space representation. q and p may be identified with the amplitudes of the two quadriture components of the field or alternatively with the position and momentum of a harmonic oscillator (with units of time, length, and mass chosen to make $\hbar=m=\omega=1$). The rescaling $q\rightarrow\mu q$ and $p\rightarrow p/\mu$ is evident in this expression.

In a recent submission[10] we have investigated the more general phase space map

$$q'=Aq+Bp , \tag{6a}$$

$$p'=Cq+Dp , \tag{6b}$$

subject to $AD\text{-}BC=1$. The Hilbert space image U, of (6), takes the form

$$U=\frac{1}{2\pi}s^{1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\ p\end{pmatrix}\right| , \tag{7}$$

and was shown to transform the annihilation operator according to

$$U(r,s)aU^{\dagger}(r,s)=sa+ra^{\dagger} , \tag{8}$$

where

$$s=\tfrac{1}{2}[(A+D)+i(B-C)] , \tag{9a}$$

$$r=\tfrac{1}{2}[(D-A)-i(B+C)] , \tag{9b}$$

and $|s|^2-|r|^2=1$. (8) is in fact precisely the form of the general squeezed-state annihilation operator (1).

In Sec. I of this Brief Report we derive the coordinate representation wave function of the squeezed state produced by the action of $U(r,s)$ on the coherent state. We show that the displacement of the coherent state in phase space caused by squeezing is exactly that predicted by the classical mapping giving rise to U. The width of the resulting Gaussian is also easily related to te classical map. In consequence, if the effect of squeezing on the quadriture amplitudes of the field are known, the uncertainties follow trivially. In Sec. II the wave function of the states implied by the generalization of Ref. 10 for the two-mode squeezed state is obtained. The two-mode density leads again to remarkably simple results when expressed in terms of the classical mapping parameters.

I. ONE-MODE SQUEEZED-STATE WAVE FUNCTION

In Ref. 10 the operator (7) was integrated to obtain

$$U(r,s)=s^{-1/2}\exp\left[\frac{r}{2s}(a^\dagger)^2\right]\exp\left[a^\dagger a\ln\frac{1}{s}\right]\times\exp\left[\frac{r^*}{2s}a^2\right]. \tag{10}$$

The squeezed state $|\alpha;r,s\rangle$ produced by the action of this operator on the coherent state $|\alpha\rangle$ is

$$\begin{aligned}|\alpha;r,s\rangle=U(r,s)|\alpha\rangle&=s^{-1/2}\exp\left[-\frac{ra^{\dagger2}}{2s}\right]\exp\left[a^\dagger a\ln\frac{1}{s}\right]\exp\left[\frac{r^*a^2}{2s}\right]|\alpha\rangle\\&=s^{-1/2}\exp\left[\frac{r^*\alpha^2}{2s}-\frac{|\alpha|^2}{2}\right]\exp\left[\frac{ra^{\dagger2}}{2s}\right]\exp\left[\frac{\alpha a^\dagger}{s}\right]|0\rangle\ .\end{aligned}\tag{11}$$

To obtain the wave function in coordinate representation, we calculate $\langle q|\alpha;r,s\rangle$, where $|q\rangle$ is the coordinate eigenstate with Fock space representation

$$|q\rangle=\pi^{-1/4}\exp[-\tfrac{1}{2}q^2+\sqrt{2}qa^\dagger-\tfrac{1}{2}(a^\dagger)^2]|0\rangle\ . \tag{12}$$

Thus

$$\langle q|\alpha;r,s\rangle=s^{-1/2}\pi^{-1/4}\exp\left[-1/2\left[q^2+|\alpha|^2-\frac{r^*}{s}\alpha^2\right]\right]\left\langle 0\left|\exp(\sqrt{2}qa-\tfrac{1}{2}a^2)\exp\left[-\frac{r}{2s}(a^\dagger)^2+\frac{\alpha}{s}a^\dagger\right]\right|0\right\rangle\ . \tag{13}$$

Inserting the overcompleteness relation for the coherent state $|z\rangle$, $\int d^2z\,\pi^{-1}|z\rangle\langle z|=1$, into (13) and using the identity $|0\rangle\langle 0|=:e^{-a^\dagger a}:$ (where : : denotes normal ordering), we integrate by means of the integration-within-ordered-products[11] (IWOP) technique to obtain for the matrix element in (13):

$$\begin{aligned}&\langle 0|\exp(\sqrt{2}qa-\tfrac{1}{2}a^2)\exp\left[\frac{r}{2s}(a^\dagger)^2+\frac{\alpha}{s}a^\dagger\right]|0\rangle\\&\quad=\langle 0|\exp(\sqrt{2}qa-\tfrac{1}{2}a^2)\int d^2z\pi^{-1}|z\rangle\langle z|\exp\left[-\frac{r}{2s}(a^\dagger)^2+\frac{\alpha}{s}a^\dagger\right]|0\rangle\\&\quad=\langle 0|\int d^2z\pi^{-1}:\exp\left[-|z|^2+\sqrt{2}qz-\tfrac{1}{2}z^2+za^\dagger-\frac{r}{2s}(z^*)^2+\frac{\alpha}{s}z^*+z^*a-a^\dagger a\right]:|0\rangle\\&\quad=(1-r/s)^{-1/2}\exp\left[\frac{1}{s-r}\left[\sqrt{2}q\alpha-\frac{\alpha^2}{2s}-rq^2\right]\right]\ .\end{aligned}\tag{14}$$

The wave function then becomes

$$\langle q|\alpha;r,s\rangle=\pi^{-1/4}(s-r)^{-1/2}\exp\left[\frac{-1}{2(s-r)}[(s+r)q^2-2\sqrt{2}q\alpha-\alpha^2(r^*-s^*)]-\left|\frac{\alpha}{2}\right|^2\right]\ . \tag{15}$$

Calculating the probability density $\rho=|\langle q|\alpha;r,s\rangle|^2$ with the aid of (7), we obtain the simple result

$$\rho(q)=\frac{1}{[\pi(A^2+B^2)]^{1/2}}\exp\left[\frac{[q-\sqrt{2}(\alpha_1A+\alpha_2B)]^2}{(A^2+B^2)}\right]\ , \tag{16}$$

where α_1 and α_2 are, respectively, the real and imaginary parts of α. If we denote the expectation value $\langle\alpha;r,s|\hat{Q}|\alpha;r,s\rangle$ of the position operator $\hat{Q}=[(a^\dagger+a)/\sqrt{2}]$ in the squeezed state as Q' and expectation values of $\hat{Q}$ and $\hat{P}=[(a-a^\dagger)/i\sqrt{2}]$ in the coherent state by $Q=\sqrt{2}\alpha_1$ and $P=\sqrt{2}\alpha_2$, the argument of the exponential in (16) may be written as $-(q-Q')^2/(A^2+B^2)$, where $Q'=AQ+BP$. The classical transformation (6a) is apparent in the displacement of the probability density. The wave function in momentum representation is similarly given by

$$\langle p|\alpha;r,s\rangle=\pi^{-1/4}(s+r)^{-1/2}\times\exp\left[\frac{-1}{2(r+s)}[(s-r)p^2+2\sqrt{2}ip\alpha-\alpha^2(r^*+s^*)]-\frac{|\alpha|^2}{2}\right] \tag{17}$$

to give the probability density

$$\rho(p)=\frac{1}{[\pi(C^2+D^2)]^{1/2}}\times\exp\left[\frac{[p-\sqrt{2}(\alpha_1 C+\alpha_2 D)]^2}{(C^2+D^2)}\right]. \tag{18}$$

Its mean mimics the classical map (6b).

For any state $|\psi\rangle$, $\langle\psi|\hat{Q}|\psi\rangle=\int q|\langle q|\psi\rangle|^2dq$ and $\langle\psi|\hat{Q}^2|\psi\rangle=\int q^2|\langle q|\psi\rangle|^2dq$, whence, from (16) we obtain the variance for the squeezed state

$$(\Delta q)^2=\tfrac{1}{2}(A^2+B^2) \tag{19}$$

and from (18) the analogous result

$$(\Delta p)^2=\tfrac{1}{2}(C^2+D^2) \tag{20}$$

giving the uncertainty product

$$(\Delta q)(\Delta p)=\tfrac{1}{2}[(A^2+B^2)(C^2+D^2)]^{1/2}. \tag{21}$$

In retrospect, we note that these results for the expectation values, expressed in terms of μ and ν, of (1) were obtained from the commutator relation $[b,b^\dagger]=1$ in Ref. 5. The traditional squeezed state resulting from squeezing the coherent state $|\alpha\rangle$ by the two parameter squeeze operator (4b) is recovered by setting

$$A=\cosh\lambda-\sinh\lambda\cos\theta,\quad D=\cosh\lambda+\sinh\lambda\cos\theta,$$
$$B=C=-\sinh\lambda\sin\theta. \tag{22}$$

II. TWO-MODE SQUEEZED-STATE WAVE FUNCTION

It was shown in Ref. 10 that the two-mode squeeze operators (26) form a subset of the unitary operators engendered by the symplectic two-mode phase-space mapping

$$\begin{pmatrix} q_1' \\ q_2' \\ p_1' \\ p_2' \end{pmatrix}=\frac{1}{2}\begin{pmatrix} A+D & A-D & B-C & B+C \\ A-D & A+D & B+C & B-C \\ C-B & B+C & A+D & D-A \\ B+C & C-B & D-A & A+D \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}. \tag{23}$$

The unitary operator image $U^{(2)}(r,s)$,

$$U^{(2)}(r,s)=\exp\left[-\frac{r}{s}a_1^\dagger a_2^\dagger\right]\exp\left[(a_1^\dagger a_1+a_2^\dagger a_2+1)\ln\frac{1}{s}\right]\times\exp\left[\frac{r^*}{s}a_1a_2\right], \tag{24}$$

of this mapping (23), generalizes the squeezed annihilation operators as $b_1=U^{(2)}a_1(U^{(2)})^\dagger=sa_1+ra_2^\dagger$ and

$$b_2=U^{(2)}a_2(U^{(2)})^\dagger=sa_2+ra_1^\dagger, \tag{25}$$

where the subscripts indicate which mode they act on. (The parameters A, B, C, and D, have been redefined from Ref. 10 in order that the traditional (two-parameter) two-mode squeeze operator

$$U^{(2)}=e^{(\xi a_1^\dagger a_2^\dagger-\xi^* a_1 a_2)},\quad \text{with } \xi=\lambda e^{i\theta} \tag{26}$$

may be recovered when the expressions (22) are substituted in (9). This change replaces s^* and r of Ref. 10 by s and $-r$.)

Denoting the two-mode coherent state by $|\alpha,\beta\rangle$ we obtain, as in the one-mode case, the wave function of the two-mode squeezed state $U^{(2)}(r,s)|\alpha,\beta\rangle$ by inserting the completeness relation, and integrate using the IWOP technique,

$$\begin{aligned}\langle q_1,q_2|U^{(2)}(r,s)|\alpha,\beta\rangle &= s^{-1}\langle q_1,q_2|\exp\left[-\frac{r}{s}a_1^\dagger a_2^\dagger\right]\exp\left[(a_1^\dagger a_1+a_2^\dagger a_2)\ln\frac{1}{s}\right]\exp\left[\frac{r^*}{s}\alpha\beta\right]|\alpha,\beta\rangle \\ &= s^{-1}\pi^{-1/2}\exp[-\tfrac{1}{2}(q_1^2+q_2^2+|\alpha|^2+|\beta|^2]\langle 0,0|\exp(\sqrt{2}q_1a_1+\sqrt{2}q_2a_2-\tfrac{1}{2}a_1^2-\tfrac{1}{2}a_2^2) \\ &\quad\times\int d^2z_1d^2z_2\pi^{-2}|z_1,z_2\rangle\langle z_1,z_2|\exp\left[-\frac{r}{s}a_1^\dagger a_2^\dagger+\frac{\alpha}{2}a_1^\dagger+\frac{\beta}{s}a_2^\dagger+\frac{r^*}{s}\alpha\beta\right]|0,0\rangle \\ &= s^{-1}\pi^{-1/2}\exp[-\tfrac{1}{2}(q_1^2+q_2^2+|\alpha|^2+|\beta|^2)]\langle 0,0|\int d^2z_1d^2z_2\pi^{-2}\exp(-|z_1|^2-|z_2|^2-\tfrac{1}{2}z_1^2-\tfrac{1}{2}z_2^2) \\ &\quad\times\exp\left[\sqrt{2}q_1z_1+\sqrt{2}q_2z_2-\frac{r}{s}z_1^*z_2^*+\frac{\alpha}{s}z_1^*+\frac{\beta}{s}z_2^*-a_1^\dagger a_1-a_2^\dagger a_2+\frac{r^*}{s}\alpha\beta\right]:|0,0\rangle \\ &= [\pi(s^2-r^2)]^{-1/2}\exp\left[-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\frac{1}{(s^2-r^2)}\left[-\tfrac{1}{2}(s^2+r^2)(q_1^2+q_2^2)-2q_1q_2rs\right.\right. \\ &\quad -\frac{\alpha^2}{2}-\frac{\beta^2}{2}+\sqrt{2}\beta(q_2s+q_1r) \\ &\quad \left.\left.+\sqrt{2}\alpha(q_1s+q_2r)+\alpha\beta(r^*s-rs^*)\right]\right].\end{aligned} \tag{27}$$

In an attempt to gain some feeling for this rather formidable-looking expression, the squared modulus of (27) was plotted using a personal computer for various values of the complex parameters s, r, α, and β. On inspection of a number of these graphs it was soon evident that the squeeze operator mapped the centers of the probability densities in accord with the classical mapping (23), i.e., for $Q_2=\sqrt{2}\alpha_1$, $Q_1=\sqrt{2}\beta_1$ and $P_1=\sqrt{2}\alpha_2$, $P_2=\sqrt{2}\beta_2$ one finds empirically that

$$\begin{aligned} Q_1' &= \tfrac{1}{2}[(A+D)Q_1+(A-D)Q_2 \\ &\quad +(B-C)P_1+(B+C)P_2] , \\ Q_2' &= \tfrac{1}{2}[(A-D)Q_1+(A+D)Q_2 \\ &\quad +(B+C)P_1+(B-C)P_2] . \end{aligned} \tag{28}$$

Encouraged by this result, we attempted, as in the one mode case, to express the probability density in terms of the classical map parameters A, B, C, and D. The observation that *every* two-mode squeezed state has its major axis at 45° to the q_1 and q_2 axes led us to try expressing the argument of the exponent of the density $\rho(q_1,q_2)$ in terms of the rotated coordinates $x=2^{-1/2}(q_1+q_2)$ and $y=2^{-1/2}(q_1-q_2)$ to obtain, after some lengthy but straightforward algebra, the seductively simple expression for $\rho(q_1,q_2)=|\langle q_1,q_2|\alpha,\beta;r,s\rangle|^2$:

$$\rho(q_1,q_2)=\frac{1}{\pi[(A^2+B^2)(C^2+D^2)]^{1/2}} \times\exp\left[\frac{(x-X)^2}{(A^2+B^2)}-\frac{(y-Y)^2}{(C^2+D^2)}\right] , \tag{29}$$

with $X\equiv[A(\alpha_1+\beta_1)+B(\alpha_2+\beta_2)]$ and $Y\equiv[D(\alpha_1-\beta_1)-C(\alpha_2-\beta_2)]$. The mapping (28) in terms of Q_1 and Q_2 now follows immediately; the means are mapped directly by the first two lines of (23). The variances of the distribution along the principal axes can now be immediately read from (29). It is to be noted that as the traditional two-mode squeeze operator has only two free parameters (it requires $B=C$ in addition to $AD-BC=1$), a single variance, $(\Delta x)^2$, of the two-mode quadrature operator[12] $\hat{X}=(a_1+a_1^\dagger+a_2+a_2^\dagger)/2\sqrt{2}$, is sufficient to describe the coordinate uncertainity of the state. For our more general state, two uncertainties, $(\Delta x)^2=(A^2+B^2)/2$ and $(\Delta y)^2=(C^2+D^2)/2$, are required. Defining $u=2^{-1/2}(p_1+p_2)$ and $v=2^{-1/2}(p_1-p_2)$, the probability density in momentum space is similarly given by

$$\rho(p_1,p_2)=\frac{1}{\pi[(A^2+B^2)(C^2+D^2)]^{1/2}} \times\exp\left[\frac{(u-U)^2}{(C^2+D^2)}-\frac{(v-V)^2}{(A^2+B^2)}\right] , \tag{30}$$

with $U=[C(\alpha_1+\beta_1)+D(\alpha_2+\beta_2)]$ and $V=[-B(\alpha_1-\beta_1)+A(\alpha_2-\beta_2)]$. Again we find the means mapped by (23), this time by rows 3 and 4. The uncertainity products are now easily obtained,

$$\Delta\left[\frac{\hat{Q}_1\pm\hat{Q}_2}{\sqrt{2}}\right]\Delta\left[\frac{\hat{P}_1\pm\hat{P}_2}{\sqrt{2}}\right]=\frac{[(A^2+B^2)(C^2+D^2)]^{1/2}}{2} . \tag{31}$$

CONCLUSION

In this Brief Report we have shown that the squeeze operators engendered by the classical linear phase-space transformation maps the coherent state to a squeezed state with Gaussian probability density whose mean, in position as well as in momentum representation, is given precisely by the action of the classical linear transformation on the coherent state's mean position and momentum. This relationship holds for both single- and two-mode squeezed states. In the two-mode case, the mode mixing produced by squeezing is clearly evident in the classical image of the quantum-mechanical squeeze operator.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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