Mean first-passage time for random-walk span: Comparison between theory and numerical experiment

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The predictions of a recent theoretical analysis developed by Weiss, DiMarzio, and Gaylord [J. Stat. Phys. **42**, 567 (1986)] on the mean first-passage time for a random-walk span in a "free" system to reach a given target span S are compared with the results of numerical simulation. Although the results are in good agreement qualitatively, a systematic quantitative discrepancy is found between theoretical predictions and experimental results in the "large but finite" S region, thus indicating the need for further theoretical analysis of the problem. Numerical results are also obtained in the case of a random-walk process driven by Gaussian colored noise.

I. INTRODUCTION

A number of papers in recent years have been devoted to the theoretical[1–9] and experimental[10–15] studies of first-passage-time (FPT) problems. Although remarkable progress has recently been made towards a more complete understanding of these problems, a full and rigorous description of first-passage-time processes has been obtained only in a few particular cases. Moreover, to the best of our knowledge, all the existing analytic approaches refer to the case of very simple non-Gaussian or white Gaussian driving noises, so that even in the simplest cases of "free" or linearly bounded systems the interplay between Gaussian-noise statistics and color is not yet fully understood. The problems related to the extension of the well-known Kramer's theory to the case of colored Gaussian noise have been evidenced in a recent paper.[14] Although the joint use of the time-dependent Fokker-Planck equation approach and local linearization techniques (LLT's) has been shown to furnish reliable predictions about the dependence of the mean first-passage time (MFPT) on the color of the Gaussian driving noise, the quantitative agreement between theoretical predictions and experimental results is still poor.

In recent years this situation has motivated an intensive research of theoretical tools other than the Fokker-Planck equation approach to FPT problems. Statistical methods based on the explicit construction of stochastic trajectories have been proposed[2,3,5–7] and applied successfully to interpret the experimental results[7,11,15] in the case of non-Gaussian colored driving noises (namely, dichotomic, shot, or Poisson noises).

A similar approach has recently been followed by Weiss, DiMarzio, and Gaylord[16] for deriving the MFPT for a random-walk span in the "free" system, driven by the flux

$$\dot{x}(t)=\xi(t)\ , \tag{1}$$

in which $\xi(t)$ represents a driving noise described by a given statistics.

The span of this process can be defined as

$$s(t)=x_{\max}(t)-x_{\min}(t)\ , \tag{2}$$

where $x_{\max}(t)$ $[x_{\min}(t)]$ represents the maximum (minimum) value assumed by the x coordinate during the time interval t. $s(t)$ is clearly a nondecreasing function of the time; in Ref. 16 this property has been used to derive approximate formulas related to the earliest time (FPT) at which $s(t)$ reaches a given target level S.

Our main interest in this problem is related to the fact that the theoretical approach of Weiss, DiMarzio, and Gaylord,[16] applied to the case of impulsive driving noise, predicts the MFPT for $s(t)$ to reach the target S ($\langle\Theta(S)\rangle$) to depend (in the limit $S \gg \sigma^2$) only on the amplitude σ^2 of the driving noise and on the mean time T between two successive "shots," in the form

$$\langle\Theta(S)\rangle=T\left[1+\frac{S^2}{2\sigma^2}\right]\ . \tag{3}$$

Moreover, the MFPT is predicted to be independent of the particular statistics governing the amplitude of the noise. In order to test this prediction, it is appropriate to perform a numerical experiment, comparing the results obtained using impulsive driving noises characterized by different statistical distributions of the shot amplitudes; moreover, we also studied the case of "continuous" noise (colored-Gaussian-noise case), using a fast and accurate computer algorithm described in detail in Ref. 17.

II. COMPUTER EXPERIMENT

The determination of the MFPT for the span of the x coordinate in the random-walk process was obtained by computer simulation of Eq. (1); the details of the numerical procedure and the algorithm used have been described in detail elsewhere.[11,15,17] The system was prepared at time $t=0$ in the state $x=0$ $[s(t=0)=0]$. Keeping track of the time needed for the stochastic quantity $s(t)$ [Eq. (2)] to reach a given target span S (FPT), the evolution of the system has been followed. Then the system was brought back to the initial conditions and the whole process was repeated, averaging the FPT over a number of realizations in order to have a good statistical sample (in the present experiment, typical samples of 1000 independent realizations have been used). The former procedure allowed us to obtain the experimental counterpart of the $\langle \Theta(S) \rangle$ function.

The numerical simulation of Eq. (1) was performed in two different cases, characterized by the different statistics which the noise $\xi(t)$ obeys.

A. Case (a): Shot noise

In this case the stochastic process $\xi(t)$ in Eq. (1) describes an impulsive noise (shot noise) such as

$$\xi(t)=\sum_i \gamma_i \delta(t-t_i) \ . \tag{4}$$

The random variables γ_i $(i=1,2,\ldots)$ are independent and identically distributed. An exponential form is assumed for the probability distribution $\phi(t-t_i)$ of the time intervals $(t-t_i)$ between two successive shots, i.e.,

$$\phi(t-t_i)=\lambda^{-1}\exp[-\lambda(t-t_i)] \ . \tag{5}$$

The parameter λ represents the reciprocal of the mean interval between two pulses.

The noise characterized by Eqs. (4) and (5) corresponds to a Poisson sequence of random impulses; because of its impulsive nature, the shot-noise process is always uncorrelated. However, it is still possible to define a characteristic time of the process as

$$T=\int_0^\infty t\phi(t)dt \ . \tag{6}$$

Having fixed the probability distribution between two successive shots, the shot-noise process itself can be unequivocally determined defining the probability distribution $\psi(\gamma_i)$ for the shot amplitudes. Earlier theoretical approaches[5] adopted the choice of positive amplitude shots characterized by either an exponential probability distribution

$$\begin{aligned} &\psi(\gamma_i)=0 \quad \text{if } \gamma_i<0 \ , \\ &\psi(\gamma_i)=\gamma \exp(-\gamma_i/\gamma) \quad \text{if } \gamma_i>0 \ , \end{aligned} \tag{7}$$

in which γ represents the mean amplitude of the shots, or a δ-like distribution

$$\psi(\gamma_i)=\delta(\gamma_i-\gamma), \quad \gamma>0 \tag{8}$$

(i.e., the γ_i are deterministic quantities).

It is a simple exercise to demonstrate that the MFPT for the "free" process to reach a given target span S is[5]

$$\langle \Theta(S) \rangle = T\left[1+\frac{S}{\gamma}\right] \tag{9}$$

in the case of exponential distribution of the shot amplitudes [Eq. (7)] and

$$\langle \Theta(S) \rangle = T\left[1+\sum_{n=1}^{\infty}\Theta(S-n\gamma)\right] , \tag{10}$$

where $\Theta(x)$ represents the Heaviside function, in the case of a δ-like distribution [Eq. (8)].

The results of digital simulation referring to the former cases are reported in Fig. 1, compared with the theoretical expressions [Eqs. (9) and (10)] (for details of the numerical simulation method, see Ref. 17). Remarkably good agreement is found between these results, instilling confidence in the simulation program.

However, more interesting results can be obtained by relaxing the constraint of having positively defined amplitude shots, according to the theoretical treatment of Ref. 16. The numerical results shown in Fig. 2 are obtained adopting an exponential distribution function for the absolute value of the amplitude of the shots

$$\psi(\gamma_i)=\frac{\gamma}{2}\exp(-\sqrt{2}|\gamma_i|/\gamma) \ , \tag{11}$$

a "heads or tails" distribution

$$\psi(\gamma_i)=\tfrac{1}{2}[\delta(\gamma_i-\gamma)+\delta(\gamma_i+\gamma)] \ , \tag{12}$$

and a "Gaussian" distribution

$$\psi(\gamma_i)=\exp\left[-\frac{\gamma_i^2}{2\gamma^2}\right] , \tag{13}$$

respectively. In all the former cases the distribution function of the shot noise is zero centered and its variance σ^2 coincides with γ^2.

A comparison of the experimental results with the theoretical predictions of Eq. (2) shows that the MFPT in order to reach a given target span S, does not depend ap-

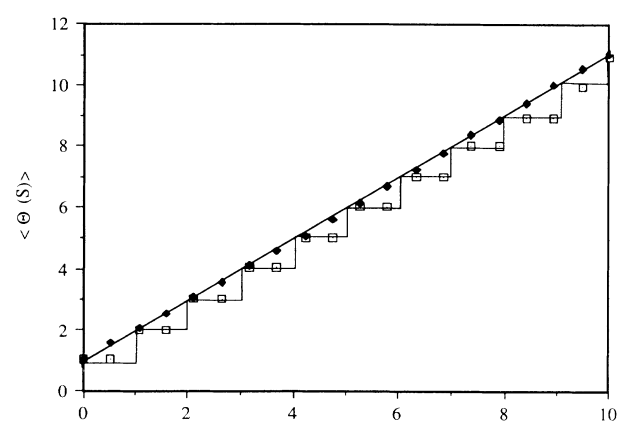

FIG. 1. Mean-first-passage time for the span $s(t)$ ($\langle \Theta \rangle$) vs S in the case of shot noise described by exponential (♦) and δ-like (□) amplitude probability distributions. Continuous lines refer to the theoretical predictions of Eqs. (9) and (10). For both the curves $\gamma=1$ and $T=1$.

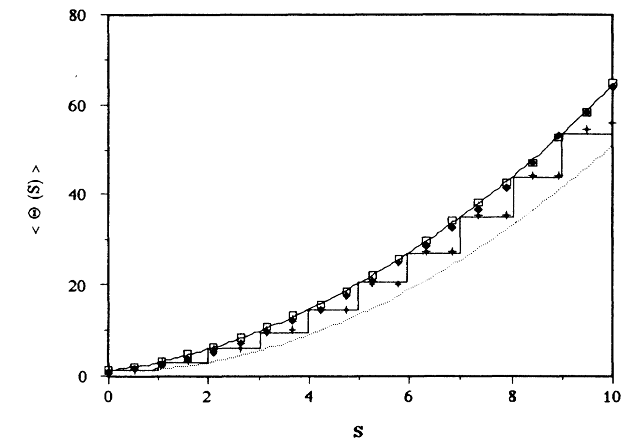

FIG. 2. $\langle \Theta \rangle$ vs S in the case of zero-centered shot noise with exponential (□), Gaussian (♦), and "heads or tails" (+) amplitude probability distributions. For all the curves $\gamma=1$ and $T=1$. The solid smooth line corresponds to the best fit with Eq. (15) in the case of the numerical data obtained using the first two distributions, while the "stair-case-like" curve connects the data obtained using the heads and tails amplitude probability distribution for the shot noise. The dotted curve refers to the predictions of Eq. (3).

preciably on the particular statistics chosen for the shot-noise amplitudes, according to the theoretical findings of Ref. 16. However, from the best fit of the experimental results with the polynomial function

$$\langle \Theta(S) \rangle = a + bS + cS^2 , \tag{14}$$

it is clear that the MFPT dependence of S shows a nonvanishing linear component b which is not reproduced by the theoretical analysis [see Eq. (3)]. The dependence of the three fit coefficients a, b, and c on the parameters T and γ characterizing the shot-noise process can be determined analyzing the $\langle \Theta(S) \rangle$ behavior while varying the same parameters. A set of numerical data obtained using an exponential distribution of the shots and varying their mean amplitude γ is shown in Fig. 3. The best-fit procedure of these numerical results is consistent with the phenomenological expression

$$\langle \Theta(S) \rangle = T \left[1 + \frac{\sqrt{2}}{\gamma} S + \frac{1}{2\gamma^2} S^2 \right] , \tag{15}$$

as shown in Fig. 4, where the numerically determined dependence of the two parameters b and c on γ is compared with the predictions of Eq. (15). Similar results have been obtained using a Gaussian probability distribution of the shot amplitude.

The above-mentioned results allow us to derive three main conclusions about the behavior of the $\langle \Theta(S) \rangle$ function.

(i) The numerical results for $\langle \Theta(S) \rangle$ are well reproduced by a second-order polynomial function. There is no evidence for the presence of higher-order terms (S^3, S^4, . . .).

(ii) The linear component of the $\langle \Theta(S) \rangle$ function, which is not reproduced by the theoretical analysis of Ref. 16, increases while decreasing the mean amplitude γ of the shots.

(iii) The MFPT behavior is correctly described by the theoretical analysis of Ref. 16 only in the limit $S \to \infty$; a different approach is probably needed to reproduce the "large but finite" S region.

B. Case (b): Gaussian noise

In order to study the MFPT dependence of the noise parameters in the finite S region, we also studied the case in which the driving noise $\xi(t)$ in the random-walk pro-

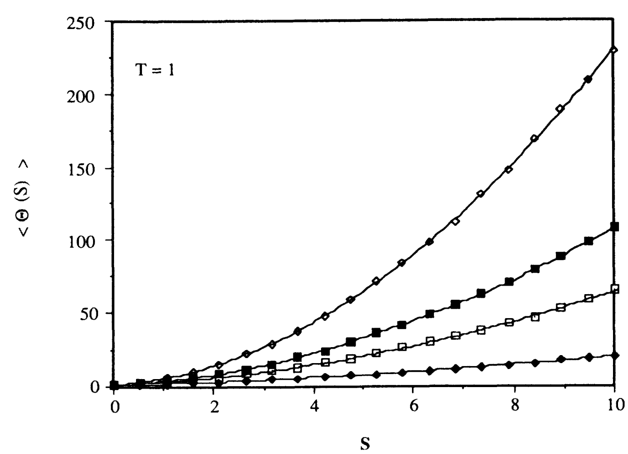

FIG. 3. $\langle \Theta \rangle$ vs S in the case of zero-centered shot noise with exponential amplitude probability distributions with $\gamma=0.5$ (◇), $\gamma=0.75$ (■), $\gamma=1$ (□), and $\gamma=2$ (♦). For all the curves $T=1$. The solid lines correspond to the predictions of Eq. (15).

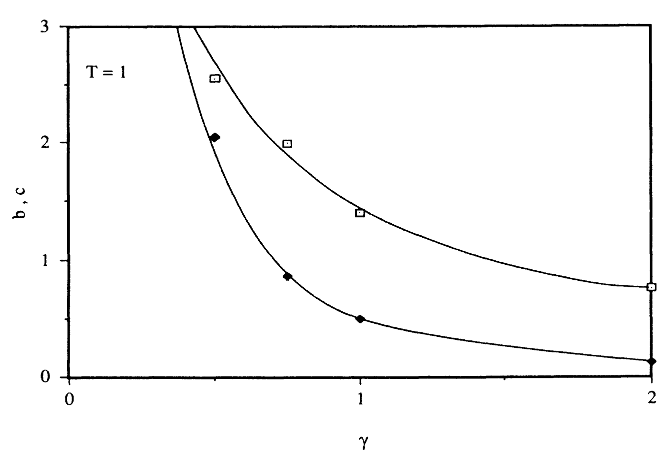

FIG. 4. Fitting parameters b (□) and c (♦) as a function of the mean shot amplitude γ. The data have been obtained from the best fit of the experimental data shown in Fig. 3 with Eq. (14) ($T=1$). The solid lines represent the predictions of Eq. (15).

cess [Eq. (1)] represents a Gaussian stochastic process *with finite correlation time* τ, described by the relations

$$P(\xi)=\exp\left[\frac{\xi^2}{2\langle\xi^2\rangle}\right],$$
$$\langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp\left[-\frac{(t-t')}{\tau}\right], \quad (16)$$

where $D=\langle\xi^2\rangle\tau$ is the intensity of the noise.

It should be stressed that the original theoretical analysis of Weiss, DiMarzio, and Gaylord[16] cannot be directly applied to the case of Gaussian driving noise, so that the comparison of the simulation results with the predictions of Ref. 16 is, in this case, only qualitative.

The scientific debate about the MFPT determination for stochastic processes driven by colored Gaussian noise is still open: for instance, in a recent paper Doering, Hagan, and Levermore[8] proposed an approximate method of analysis which, in the case of the "free" process described by Eq. (1), successfully reproduced the experimentally determined MFPT dependence of the noise correlation time τ.[11] Unfortunately, this theoretical treatment cannot be used to derive the same analytical information about the MFPT for the span $s(t)$; however, from the best fit of our numerical results for the function $\langle\Theta(S)\rangle$, shown in Figs. 5 and 6, it is possible to derive useful information about the functional dependence of $\langle\Theta(S)\rangle$ on the noise correlation time τ and the noise intensity D. Adopting the fitting formula described by Eq. (14), we obtained the a, b, and c parameters as a function of D and τ: the parameter a was always found to be vanishingly small, and it does not show any particular dependence on D and τ; the corresponding results for the b and c coefficients are shown in Figs. 7 and 8. It is clear that the quadratic coefficient c is almost independent (within experimental error) of the noise color τ, while its dependence on D is of the form

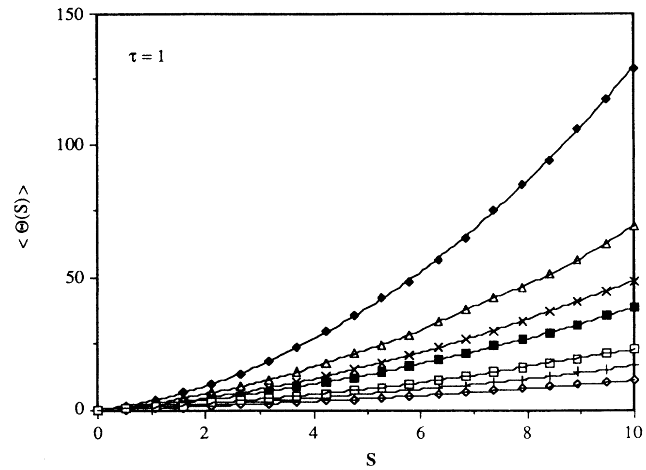

FIG. 6. $\langle\Theta\rangle$ vs S in the case of colored Gaussian noise. □, $D=5$; ■, $D=3$; ◆, $D=2$; □, $D=1$; +, $D=0.75$; ■, $D=0.5$; ◆, $D=0.25$. For all the curves $\tau=1$. The continuous lines have been calculated according to the predictions of Eq. (19).

$$c=1/D\ . \quad (17)$$

On the contrary, the linear coefficient b shows a dependence on both D and τ which is well reproduced by the relation

$$b=\lambda_M\sqrt{\tau/D}\ . \quad (18)$$

λ_M represents the characteristic Milne extrapolation length given in terms of the Riemann ζ function by $\lambda_M=-\zeta(\frac{1}{2})=1.46\ldots$ The choice of this particular constant might appear arbitrary; however, this choice is suggested by the fact that the corresponding MFPT for the x coordinate shows the same square-root dependence on

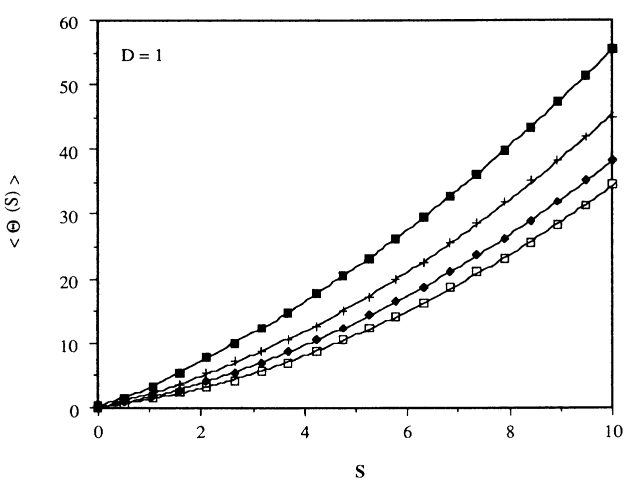

FIG. 5. $\langle\Theta\rangle$ vs S in the case of colored Gaussian noise. □, $\tau=0.5$; ◆, $\tau=1$; +, $\tau=2$; ■, $\tau=5$. For all the curves $D=1$. The continuous lines have been calculated according to the predictions of Eq. (19).

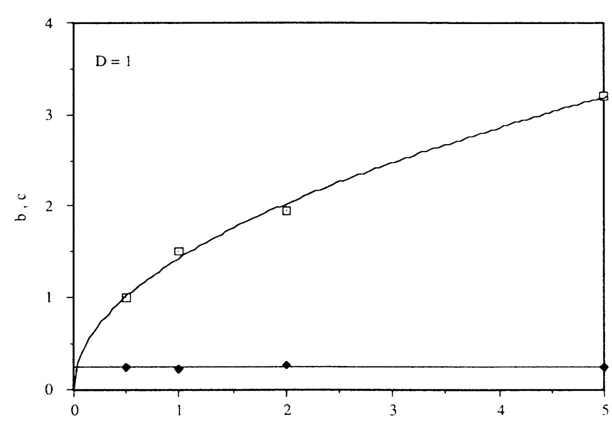

FIG. 7. The fitting parameters b (□) and c (◆) as a function of the Gaussian-noise correlation time τ. The data have been obtained from the best fit of the experimental data shown in Fig. 5 with Eq. (14) ($D=1$). The solid lines have been calculated according to Eqs. (17) and (18).

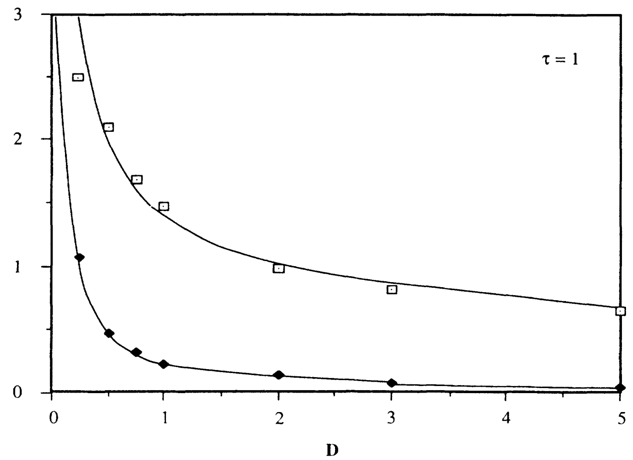

FIG. 8. The fitting parameters b (□) and c(♦) as a function of the Gaussian-noise intensity D. The data have been obtained from the best fit of the experimental data shown in Fig. 6 with Eq. (14) ($\tau=1$). The solid lines have been calculated according to Eqs. (17) and (18).

the ratio (τ/D) and it can be expressed in terms of the same characteristic extrapolation length λ_M.[8,11]

In conclusion, our experimental results for the MFPT for the span $s(t)$ to reach a given target level S are reproduced well by the empirical formula

$$\langle \Theta(S) \rangle = \lambda_M \sqrt{\tau/D}\, S + \frac{1}{D} S^2 \; . \tag{19}$$

It is worthwhile to note that the linear component of $\langle \Theta(S) \rangle$ exactly vanishes only in the $\tau \to 0$ limit (white-noise limit). Moreover, the ratio R of the linear term and the quadratic one in Eq. (19), which is a measure of the relative deviation with respect to the pure quadratic behavior, is

$$R = \frac{\lambda_M (D\tau)^{1/2}}{S} = \frac{\lambda_M (\langle \xi^2 \rangle)^{1/2}}{S} \tau \; . \tag{20}$$

The relative deviation is thus a linear function of the noise correlation time τ. This means that in the case of strongly colored Gaussian noise the *quantitative* error which is done assuming a pure quadratic behavior of the function $\langle \Theta(S) \rangle$ can be quite large even in the region $S \gg \langle \xi^2 \rangle^{1/2}$.

III. CONCLUSIONS

The comparison between the experimental results obtained for the MFPT of a random-walk span, driven by either shot or Gaussian noises, and the analytical approach of Weiss, DiMarzio, and Gaylord,[16] clearly show that the former can correctly reproduce the qualitative behavior of the latter results (especially in the large-S–small-τ region). However, a *quantitative* description of the process probably needs a more detailed theoretical analysis. This work could contribute to stimulate further efforts and studies towards a more complete understanding of the problems still open in this field.

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