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Diffusion in a random catalytic environment, polymers in random media, and stochastically growing interfaces

Thomas Nattermann and Wolfgang Renz
Institut für Festkörperforschung der Kernforschungsanlage Jülich, Postfach 1913,
5170 Jülich, Federal Republic of Germany

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We study a family of equivalent models which includes the polymer in a random medium, the stochastically growing interface with spatially random deposition of particles, and the diffusion in a random catalytic environment $\partial_t\psi(\mathbf{x},t)=D\nabla^2\psi+U(\mathbf{x})\psi$. $U(\mathbf{x})$ denotes a frozen Gaussian random potential of strength u_0 and correlation length a. The intrinsic length scales of the problem, $l_0=(D/u_0)^{2/(4-d)}$ and a, are both assumed to be small in comparison with the diffusion length $\sqrt{Dt}$ and the system size L ($L<10^8a$). Flory-Imry-Ma–type arguments show that for dimensions $d<4$ the polymer of contour length $t\rightarrow\infty$ is both collapsed ($\nu=0$) and localized, in agreement with previous results for $a\ll l_0$ [Edwards and Muthukumar, J. Chem. Phys. **89**, 2435 (1988); and Cates and Ball, J. Phys. (Paris) **49**, 2009 (1988)]. The sample-to-sample variations of the polymer free-energy scale as t^χ with $\chi=1$. For $d>4$, a collapsed, localized or a Gaussian, delocalized polymer is found for strong or weak disorder, respectively. The disorder becomes irrelevant for self-avoiding polymers. For growing interfaces, the roughness exponent χ/ν and dynamical exponent $1/\nu$ are both equal to unity, but scaling is modified by logarithmic corrections. The results are supported by a renormalization-group (RG) approach. Vertex corrections destroy the relation $\chi=2\nu-1$. For $d>4$ we find a phase transition as a function of disorder strength. For $d<4$ there is no stable fixed point. In particular, for $d\leq 2$ the fixed point found is unstable with respect to infinitely many nonlinear terms generated under the RG. It is argued that this corresponds to the appearance of a nonanalytic elastic term in the polymer Hamiltonian. The results are substantiated by mapping the problem onto the localization of a quantum particle in a random potential. When considering activated dynamics for the polymer we find that it cannot relax to the absolutely optimal environment present in the medium during observational time scales τ but will be localized in a typical well. Due to this effect, the $\ln L$ terms occurring in the static theory have to be replaced by $\ln L_\tau$ with an additional dynamical length scale $L_\tau\ll L$.

I. INTRODUCTION

The cooperative action of *diffusion* and *amplification by a random catalytic potential* is important for a number of processes. Examples are the early stage of the order-parameter relaxation in a system with random transition temperature, autocatalytic chemical reactions in a random medium, the spreading of a migrating population with a reproduction rate which depends on local (random) conditions, the biological evolution by mutation (diffusion) and fitness-dependent reproductions in a high-dimensional phenotype space, and optimization strategies.[1-3] Recently, Ebeling *et al.*[1] and Zhang[2] have considered the dynamics of such systems described by

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t}=-\hat{H}\psi(\mathbf{x},t),\quad \hat{H}=-D\nabla^2-U(\mathbf{x})\ , \tag{1}$$

where $\psi(\mathbf{x},t)$ is proportional to the density of certain physical, chemical, or biological objects. $U(\mathbf{x})$ is a random Gaussian potential with $\langle U\rangle=0$ and

$$\langle U(\mathbf{x})U(\mathbf{x}')\rangle=u_0^2\delta_a^{(d)}(\mathbf{x}-\mathbf{x}')\ . \tag{2}$$

$\delta_a^{(d)}(\mathbf{x})$ is a d-dimensional smeared out δ function of width a. In the limit of small D, the finiteness of a will play an important role, as will be seen below. $\langle U\rangle=0$ can always be achieved by the transformation $\psi\rightarrow\psi e^{\langle U\rangle t}$. With the initial condition $\psi(\mathbf{x},0)=\delta(\mathbf{x}-\mathbf{y})$, Eq. (1) has the formal solution

$$G(\mathbf{x},t;\mathbf{y})=\int_{\mathbf{x}(0)=\mathbf{y}}^{\mathbf{x}(t)=\mathbf{x}}D\mathbf{x}(t)\exp\left[-\int_0^t dt'\left[\frac{1}{4D}\dot{\mathbf{x}}^2-U(\mathbf{x}(t'))\right]\right]\ . \tag{3}$$

Equation (3) can be interpreted as the restricted partition function of (i) a d-dimensional *Gaussian polymer* of length t in a random potential $-U(\mathbf{x})$ with $l_k=6D$ for the Kuhn's step length[4] or (ii) a $(d+1)$-dimensional *directed polymer* with a stiffness $\Gamma=T/2D$ in a random potential $-U(\mathbf{x})T$, which is correlated parallel to the t axis.[5] Here T denotes the temperature. Finally, using the transformation $h(\mathbf{x},t)=\ln\psi(\mathbf{x},t)$, $\psi(\mathbf{x},t)=\int_{\mathbf{y}}G(\mathbf{x},t;\mathbf{y})$, we end

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up with

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=D\nabla^2 h+\tilde{D}(\nabla h)^2+U(\mathbf{x}) . \tag{4}$$

Here $\int_{\mathbf{x}}=\int d^d x$ and $\tilde{D}=D$. $-Th(\mathbf{x},t)$ has the meaning of the polymer free energy. Moreover, Eq. (4) determines the height $h(\mathbf{x},t)$ of the interface of a *ballistic growth model* in a comoving frame at time t.[6] The first term on the right-hand side (rhs) describes the interface relaxation by the surface tension D. The second term considers the fact that growth occurs normally to the interface. $-U(\mathbf{x})$ denotes the stochastic part of the growth due to deposition of particles which is assumed here to be random in space, but not in time.

A central quantity for the diffusion problem is

$$R_A^2[t,U]=\int_{\mathbf{x}}\int_{\mathbf{y}}(\mathbf{x}-\mathbf{y})^2 G(\mathbf{x},t;\mathbf{y})\left[\int_{\mathbf{x}}\int_{\mathbf{y}} G(\mathbf{x},t;\mathbf{y})\right]^{-1} \tag{5}$$

and its average value $R_A^2(t)=\langle R_A^2[t,U]\rangle$, which describes the *local spreading* of an *initially homogeneous* density (problem A). For polymers, $R_A(t)$ denotes the rms end-to-end distance which is expected to scale with the chain length as $R_A(t)\simeq\mathcal{A}_A t^{\nu_A}$. For directed polymers $R_A(t)$ describes their transverse wandering. A second exponent χ_A describes the fluctuations of

$$h(t)=\ln\int_x \psi(\mathbf{x},t) ,$$

$$\Delta h(t)=[\langle h^2(t)\rangle-\langle h(t)\rangle^2]^{1/2}\sim t^{\chi_A} .$$

A different quantity $R_B[t,U;\mathbf{y}]$ is connected with the spreading of an *initially localized* distribution $\psi(\mathbf{x},0)=\delta(\mathbf{x}-\mathbf{y})$ (problem B). $R_B[t,U;\mathbf{y}]$ follows from (5) if we omit the integrations over $\mathbf{y}$. In the polymer context $R_B(t)=\langle R_B^2[t,U;\mathbf{y}]\rangle^{1/2}$ denotes the mean polymer end-to-end distance (or transverse wandering) if one end of the polymer is kept fixed. Note that, because of the strong correlations of $U(\mathbf{x})$ parallel to the t axis, $R_A(t)$ and $R_B(t)$ are in general different. The fluctuations of $h(\mathbf{y},t)$ behave as

$$\langle |h(\mathbf{y},t)-h(\mathbf{y}',t')^2\rangle^{1/2}\sim|\mathbf{y}-\mathbf{y}'|^{\chi_B/\nu_B} f\left[\frac{|t-t'|^{\nu_B}}{|\mathbf{y}-\mathbf{y}'|}\right] ,$$

where χ_B/ν_B is the roughening exponent of the growing interface.

The problems formulated at the beginning involve the *four characteristic length scales*: a, l_0, $R_D(t)$, and L. a is a small scale cutoff, e,g., the scale over which the random potential is correlated. A second length scale $l_0=(D/u_0)^{2/(4-d)}$ follows simply from dimensional analysis. The third scale $R_D(t)\simeq(Dt)^{1/2}$ connects t with $\mathbf{x}$ and enters via the Einstein diffusion length. Finally, we have the linear extension L of the whole system. In this paper we will restrict ourselves to the case

$$a,l_0\ll R_D(t)\ll L\le 10^8 a$$

[for $R_D(t)\ll l_0$, the influence of the disorder can be neglected and we have $\nu=\frac{1}{2}$ and $\chi=0$].

For a wave packet of extension R, the two terms on the rhs of (1) scale as DR^{-2} and $u_0 R^{-d/2}$ (provided $R>a$), respectively, from which we can form a *dimensionless* bare *coupling constant* $g_0(R)=(u_0^2/D^2)R^{4-d}=(R/l_0)^{4-d}$. In the *strong-coupling* (*or classical*) case $g_0(R)\gg 1$ the disorder wins over the diffusion. This occurs always for $d<4$ if $R\to\infty$. On the contrary, the *weak coupling (quantum)* case is given by $g_0(R)\ll 1$ which appears for $R\to\infty$ if $d>4$. Clearly, on small scales $R\approx a$ the situation may be different. In particular, the white-noise limit $a\to 0$ (or $a\ll l_0$) corresponds to $g_0(a)\ll 1$ for $d<4$ and $g_0(a)\gg 1$ for $d>4$. To our knowledge, the distinction between the classical and the quantum case goes back to Lifshitz.

If, for $d<4$, we are in the strong-coupling case, the polymers configurations will be dominated, as shown below, by the statistics of the potential wells only. So far we estimated the depth of the wells on scale R by $u_0 R^{-d/2}$. However, in a system of size L we have $(L/R)^d$ wells, the deepest of which scales as $\langle U_{\min}(R)\rangle\simeq u_L R^{-d/2}$ and

$$\Delta U_{\min}^2(R)=\langle U_{\min}^2(R)\rangle-\langle U_{\min}(R)\rangle^2\approx 2u_L^2 R^{-d}$$

with $u_L\simeq u_0[d\ln(L/R)]^{1/2}$ (see the Appendix). Similarly, the mean energy difference between the two lowest wells is of the order

$$\langle\delta U(R)\rangle\approx 2u_0 R^{-d/2}[d\ln(L/R)]^{-1/2} .$$

(Here we differ from Cates and Ball, who state that the energy difference between the two lowest wells remains finite for $L\to\infty$. Our formula is derived in the Appendix.) Note that $[d\ln(L/a)]^{1/2}\le 8$ and 4 for real 3D systems and computer simulations, respectively, although it goes to infinity at the thermodynamic limit $L\to\infty$ or $a\to 0$. When talking about potential wells and minima we mean those of $-U(\mathbf{x})$, the polymer potential, which correspond to maxima of the catalytic potential $U(\mathbf{x})$.

Previous work on the problems mentioned are restricted to an approximate treatment of several limiting cases, e.g., $g_0(a)\gg 1$, $d<4$ for problem B in Ref. 1, $g_0(a)\ll 1$ and $d<4$ in Refs. 4 and 7. It is the aim of the present paper to consider the systems described for all values of $g_0(a)$ and d, using Flory arguments, a renormalization-group treatment, and the connection to the localization of a quantum particle. The results will be discussed mainly in terms of the polymer picture; we come back to the problem of a spreading population and the growing interface at the end of this paper.

II. FLORY-IMRY-MA-TYPE ARGUMENTS

We start with problem A. First we present a Flory-Imry-Ma-type estimate for the free energy $-Th(R,t)$ of a polymer with free ends, confined in a region of linear extension R. From (2) and (3) we obtain (ignoring prefactors)

$$-h(R,T)\approx\frac{R^2}{Dt}+\frac{Dt}{R^2}-t\frac{u_L}{R^{d/2}}\quad\text{if } R\ge a , \tag{6}$$

where the first two terms describe the entropy of a free polymer for $R^2\gg Dt$ and $Dt\gg R^2$, respectively.[8] The second term is simply the loss in entropy t/t_{enc} of a polymer of length t, confined into a well of size R, but free inside. $t_{\text{enc}}\approx R^2/D$ is the mean contour length between two encounters of the polymer with the walls of the well. The third term on the rhs of (6) estimates the disorder-averaged maximal free-energy gain at a given system size L (see the Introduction and Appendix). Equation (6) has been analyzed for $g_0(a) \ll 1$, $d < 4$ in Ref. 7.

Minimizing (6) for $d < 4$, we get a *single* minimum at $R \approx R_c[\ll R_D(t)]$

$$R_c = \max(l_c, a), \quad l_c = (D/u_L)^{2/(4-d)} ,$$
$$u_L = u_0[d \ln(L/R_c)]^{1/2} . \tag{7}$$

Hence, the Gaussian polymer subjected to a random potential $U(\mathbf{x})$ is *collapsed* to a size R_c; the directed polymer makes only finite transverse excursions. For $a \ll l_c$ this result agrees with that of Edwards and Muthukumar,[4] Cates and Ball[7] and earlier numerical work by Baumgärtner and Muthukumar.[9] Since the Flory argument corresponds to a saddle-point approximation applied on (3), we get with the same accuracy from (5), $R_A(t) \simeq R_c$, i.e., $\nu_A \equiv 0$.

So far we have neglected the translational entropy $d \ln(L/R_c)$, which is smaller than $h(R_c,t) \approx t u_L / R_c^{d/2}$ if

$$t \frac{u_0}{R_c^{d/2}} [d \ln(L/R_c)]^{-1/2} \approx \tfrac{1}{2} t \langle \delta U(R_c) \rangle \gg 1 . \tag{8}$$

In this case only a single potential minimum contributes to (3) and (5), i.e., the polymer is *collapsed and localized.* The fluctuations $\Delta U_{\min}(R_c)$ determine the fluctuations in $h(t)$, which yields $\chi_A = 1$, i.e., χ_A and ν_A do not obey the scaling relation $\chi = 2\nu - 1$.[5] The lhs of Eq. (8) can be considered as the energy difference between two polymer configurations in the two lowest potential wells. For $l_c \ll a$, $R_c \approx a$, and

$$t \langle \delta U(a) \rangle \simeq \left[\frac{R_D(t)}{a} \right]^2 [g_0(a)]^{1/2} [d \ln(L/a)]^{-1/2} ,$$

which is always assumed to be large since $[d \ln(L/a)]^{-1/2} \geq 0.1$ even for $L \simeq 10^8 a$. For $a \ll l_c$, $R_c \approx l_c$, and

$$\langle t \delta U(R_c) \rangle \simeq \left[\frac{R_D(t)}{l_0} \right]^2 [d \ln(L/a)]^{(d-2)/(4-d)}$$

which is also always larger than unity, since for $d = 1$, $L \approx 10^8 a$, $[\ln L/a]^{-1/3} \approx 0.4$, and $R_D(t) \gg l_0$. Thus, in realistic situations $L < 10^8 a$, the translational entropy can be ignored. For completeness we mention that, in the true thermodynamic limit $L \to \infty$ or for short polymers, inequality (8) is no longer fulfilled for $d < 4$ if $l_c \ll a$ and $d < 2$ if $a \ll l_c$, respectively. In this case many potential valleys contribute to (5), the polymer is *delocalized,* and both the nominator and the denominator are *self-averaging.* Hence for $L \to \infty$ the quenched and the annealed average agree with each other [in (5) the integral $\int_{\mathbf{y}}$ becomes a disorder average times L^d] and, consequently, $-t u_L / R^{d/2}$ in (6) has to be replaced by $-t^2 u_0^2 / R^d$ which acts as a self-attraction and yields $R_A^{2-d} \simeq \max(D/t u_0^2, a^{2-d})$ in $d < 2$ and $R_A \simeq a$ in $d \geq 2$.

For $d > 4$, Eq. (6) has *two* minima $R_1(t) \simeq R_D(t)$ and $R_2 \approx a$. Substituting these into (6) we see that the Gaussian polymer solution $R_D(t)$ ($\nu_A = \frac{1}{2}$) is stable for $a > l_c$, i.e., for weak disorder. Clearly, the polymer is also not localized in this case. For *strong disorder* $a \ll l_c$ the polymer *is collapsed* to a size a ($\nu_A = 0$) and *localized,* since condition (8) agrees with that for $l_c \ll a$ and $d < 4$. For $d = 4$, $R_A \simeq R_D(t)$ if $D > u$ and $R_A \simeq a$ if $D < u$. The exponent χ is $\chi_A = 1$ for strong disorder (as for $d < 4$) and $\chi_A = 0$ for weak disorder. The scaling relation $\chi_A = 2\nu_A - 1$ is only fulfilled for weak disorder and $d > 4$.

So far, our polymer was not self-avoiding. Self-avoidance (SA) is described by an additional term $w t^2 / R^d$ in the free energy (6). This introduces a new length scale $l_{\mathrm{SA}} = (D^2/w)^{1/(4-d)}$. In a pure system the SA can be neglected if $l_{\mathrm{SA}} \gg R_D(t)$ and $d < 4$. In impure systems this is only the case if $(R_D/l_{\mathrm{SA}})^2 (l_{\mathrm{SA}}/l_c)^{d-2} \ll 1$. In the opposite case the free energy reads

$$-h(R,t) = \frac{R^2}{R_D^2} + \frac{R_D^2}{(R^{2+d} l_{\mathrm{SA}}^{4-d})^{1/3}} - \frac{R_D^2 l_c^{(d-4)/2}}{R^{d/2}}$$
$$+ \frac{R_D^4 l_{\mathrm{SA}}^{d-4}}{R^d} , \tag{9}$$

where $R_D^2 = Dt$ carries the t dependence. The second term in (9) is modified with respect to (6) since the polymer behaves self-avoiding between two encounters with the boundaries of the well of size R. Minimization of (9) yields the Flory result for a SA walk $R_{\mathrm{SA}}(t) = l_{\mathrm{SA}} [R_D(t)/l_{\mathrm{SA}}]^{6/(2+d)}$, i.e., the disorder is irrelevant. These results supplement those of the authors of Refs. 4 and 7, who considered the case $l_c \gg a$, $d < 4$, and $l_{\mathrm{SA}} \to \infty$ only. In an annealed situation, the SA and disorder terms add, $(w - u_0^2) t^2 / R^d$, and consequently a transition occurs at the Θ point $w = u_0^2$ from a SA to a compact polymer, in agreement with Ref. 10. We will not consider this possibility any further, since the annealed situation is not realistic [see argument given above after Eq. (8)].

Next we consider problem B. For $D \to 0$ but $u_0 D$ finite, the main contribution to $G(\mathbf{x},t;\mathbf{y}=\mathbf{0})$ comes from the saddle-point configuration $\mathbf{x}_s(t')$. In order to estimate this, we note that the maximum value $U_{\max} = U(\mathbf{x}_{\max})$ of $U(\mathbf{x})$ in a region of size $|\mathbf{x}|^d$ scales as $U_{\max}^2 \simeq (u_0^2/a^d) d \ln(|x|/a)$. With $\mathbf{y} = \mathbf{0}$ and t fixed, $\mathbf{x}_s(t')$ can be shown by elementary means to consist of an essentially straight initial part, which makes an angle α with the t axis, until it reaches $\mathbf{x} \approx \mathbf{x}_{\max}$ at a time t_1.[11] For $t_1 < t' < t$, $\mathbf{x}_s(t') \simeq \mathbf{x}_{\max}$ [here we assume $g_0(a) \gg 1$]. Minimization of the energy of the path ($|\mathbf{x}_{\max}| = R$)

$$-h(R,\alpha) \simeq \frac{1}{4D} R \tan\alpha - u_0 a^{-d/2} (t - R/\tan\alpha)$$
$$\times (d \ln R/a)^{1/2} \tag{10}$$

yields $\tan\alpha \approx (2D^2 \tilde{u}^2 / a^d)^{1/4}$ and

$$R = R_B \simeq (Dt/a)[g_0(a)(\ln t)^{-3}]^{1/4} ,$$

where $\tilde{u} = u_0 (d \ln t)^{1/2}$. Thus the behavior for strong disorder is *weakly subballistic,* $R_B \propto t^{\nu_B} \ln^{\tilde{\nu}_B} t$, with $\nu_B = 1$ and a logarithmic correction with an exponent $\tilde{\nu}_B = -\frac{3}{4}$ in all dimensions d. Clearly, Eq. (10) makes sense only as long as $R_B(t) \ll L$.

For weak disorder and $d<4$, the part of the path with $\mathbf{x} \approx \mathbf{x}_{\max}$ will have an intrinsic width l_c instead of a. The above result can easily be generalized to this case and we obtain

$$R_B(t) \simeq \frac{Dt}{l_0}(\ln t)^{(d-3)/(4-d)} . \tag{11}$$

Note that for weak disorder $\nu_B=1$, but $\tilde{\nu}_B=(d-3)/(4-d)$, in agreement with Ref. 7. For completeness we note, that the *end point* $\mathbf{x}_t$ of the extremal path considered as a function of t performs random jumps of increasing height $|\Delta \mathbf{x}_t| \simeq R_B(t)$ followed by parts $\mathbf{x}_t=\text{const}$ of increasing length $\Delta t \simeq t$. The relevance of this fact for the diffusion problem is discussed in Sec. V.

III. RENORMALIZATION-GROUP APPROACH

So far, we have considered the influence of disorder (via Flory or saddle-point arguments) on a single scale only. This will be valid if $g_0(a) \gg 1$ for $d<4$ and $g_0(a) \ll 1$ for $d>4$. In the remaining cases, fluctuations on intermediate scales can modify the large-scale behavior. Therefore we study the flow of the coefficients D, $\tilde{D}$, and u under a RG transformation. Our procedure follows closely that described in Ref. 6. A RG treatment of Eq. (4) with long-range correlated disorder $U(\mathbf{x},t)$ has recently been studied in detail by Medina *et al.*[12] We note, however, that their model *does not* cover our case of *t-independent* $U(\mathbf{x})$. The Fourier transform of Eq. (4) is solved perturbatively in the vertex using the free propagator $(-i\omega+D\mathbf{k}^2)^{-1}$. The first terms in the perturbation expansions for the full propagator, the correlation function, and the full vertex diverge for $d<4$. The perturbation expansion is reorganized into a RG calculation by integrating out the modes with $e^{-\nu l}\Lambda < |\mathbf{k}| < \Lambda$. The parameters are then rescaled as $\mathbf{k}'=e^{\nu l}\mathbf{k}$, $\omega'=e^{l}\omega$ and $h=e^{\chi l}h'$.[13] In a one-loop calculation the resulting flow equations are

$$\begin{aligned}
\frac{dD}{dl} &= D\left[1-2\nu-2\nu K_d \frac{d-2}{d}\left[\frac{\tilde{D}u}{D^2}\right]^2\right] , \\
\frac{d\tilde{D}}{dl} &= \tilde{D}\left[\chi+1-2\nu-\nu K_d \frac{4}{d}\left[\frac{\tilde{D}u}{D^2}\right]^2\right] , \\
\frac{d(u^2)}{dl} &= u^2\left[2-2\chi-\nu d+2\nu K_d\left[\frac{\tilde{D}u}{D^2}\right]^2\right] .
\end{aligned} \tag{12}$$

Note that because of the violation of Galilean invariance of Eq. (4) there is now a nontrivial vertex renormalization which leads to a violation of the relation $\chi=2\nu-1$.[12] Equation (12) can be converted into an equation for the coupling constant $g=(\tilde{D}u/D^2)^2$

$$\frac{1}{\nu}\frac{dg}{dl} = g\left[4-d+\frac{2}{d}(5d-12)K_d g\right] , \tag{12'}$$

with $g(l=0)=g_0(a)$. χ and ν follow from the three fixed points (FP) of (12) and (12'): (i) The Gaussian FP $u^*=0$ is stable only for $d>4$ and corresponds to $\nu=\frac{1}{2}$, $\chi=0$, i.e., we have simple diffusion. (ii) The Edwards-Wilkinson FP $\tilde{D}^*=0$ is also stable only for $d>4$ and gives $\nu=\frac{1}{2}$ and $\chi=(4-d)/4$.[14] Since for $\chi<0$ the large-scale behavior is dominated by scale-independent contributions we have $\chi_{\text{eff}}=0$. (iii) The third FP is given by $K_d g^*=[(4-d)d]/[2(12-5d)]$ and is unstable for $d>\frac{12}{5}$. For $d<\frac{12}{5}$, we obtain from g^*, $\chi=\nu[(4-d)^2/(12-5d)]$ and $1/\nu=2+(d-2)(4-d)/(12-5d)$. Unfortunately, this FP is unstable with respect to infinitely many perturbations of the form $\tilde{D}_{2n}(\nabla h)^{2n}$, $n>1$, in (4) since $\chi/\nu>1$. Such terms are generated under the RG transformation and give rise to diagrams which involve additional loops consisting of two propagators and one u^2 insertion, which diverge as $t^{2(\chi-\nu)}$ (i.e., at the Edwards-Wilkinson FP for $d<2$). We were not able to sum up these divergences. Since, according to (12), the stiffness $1/\tilde{D}$ flows to large values it is tempting to presume that the large-scale behavior of $G(\mathbf{x},t;\mathbf{y})$ is determined by an *effective action* (3) where $(1/4D)\dot{\mathbf{x}}^2$ is replaced by $J|\dot{\mathbf{x}}|$. In this case, the corresponding Eq. (1) includes infinitely many even-power gradient terms with coefficients related to J. However, this change in the action appears only for $g(e^{\nu l}) \geq 1$, i.e., for $R>R_c$. From this we conclude that our Flory results for case A remain essentially unchanged. But also for case B the change in the action only affects the exponent $\tilde{\nu}_B$ which is, e.g., now $\tilde{\nu}_B=-\frac{1}{2}$ if $g_0(a) \gg 1$. For $\frac{12}{5}<d<4$ or for $d>4$ and $g_0(a)>g^*$, g flows to a strong coupling FP which is not attainable by perturbation theory.[15] Since large values of g correspond to a polymer collapse we expect $\nu=0$ in this case. Only for $d>4$ and $g_0(a)<g^*$ the flow is towards the Gaussian FP. (i) The unstable fix point g^* corresponds for $d>4$ to a phase transition from a collapsed to an extended polymer as a function of the disorder strength, in agreement with our Flory arguments.

An alternative RG approach for higher-dimensional manifolds with a partition function given by Eq. (3), but t replaced by $\underline{t}=(t_1,\ldots,t_{d'})$ is considered in Ref. 16. For a potential $U(\mathbf{x},t)$ which depends randomly both on $\mathbf{x}$ and on $\underline{t}$, this leads to an expansion in $\epsilon=4-d'$. In the present case $U(\mathbf{x},t) \equiv U(\mathbf{x})$, however, there is no such upper critical dimensionality (it is shifted to infinity) and hence also this approach does not work.

The model considered here has a Fourier-transformed disorder correlator

$$\langle \hat{U}(\omega,k)\hat{U}(\omega',k') \rangle = c(\omega)\delta(\omega+\omega')\delta(k+k')$$

with $c(\omega) \sim \delta(\omega)$ [see Eq. (2)]. This scales in the same way as the special case $\theta=\frac{1}{2}$ in Medina *et al.*[12] but $c(\omega) \sim 1/\omega$ in their case, leading to additional infrared divergencies. This prevents them from treating the case $\theta=\frac{1}{2}$. In our model these divergencies do not occur so that we are able to derive Eq. (12) and (12').

IV. RELATION TO LOCALIZATION OF A QUANTUM PARTICLE

In the last part of this paper we discuss the relation of the present problem to the *localization of a quantum particle in a random potential*.[17,18] Indeed, we may expand

$G(\mathbf{x},t;\mathbf{y})$ with respect to eigenfunctions $\phi_m(\mathbf{x})$ of $\hat{H}$, $\hat{H}\phi_m = E_m \phi_m$ (Ref. 1)

$$G(\mathbf{x},t;\mathbf{y}) = \sum_m \phi_m^*(\mathbf{y})\phi_m(\mathbf{x})e^{-E_m t} . \tag{13}$$

It is well known that for the lattice (tight-binding) version of $\hat{H}$, the low-lying states are localized in any dimension.[19] We assume that this is valid also for our continuum version and that the functions

$$\phi_m(\mathbf{x}) \approx L_c^{-d/2} \exp[-|\mathbf{x}-\mathbf{x}_m|/L_c(E_m)]$$

are localized around a center $\mathbf{x}_m$ and $L_c(E) = (D/E)^{1/2}$. It is clear from (13) that it is essential whether the energy difference between the lowest energy levels $\Delta E_n = E_{n+1} - E_n$ (n small, $E_1 < E_2 < \cdots$) fulfills the inequality $\Delta E_n t \gg 1$ or not.

If $\Delta E_n t \gg 1$, $R_A^2[t,U]$ is dominated by the ground state and the nominator and denominator do not self-average separately. Then in the classical case $g_0(a) \gg 1$, E_0 is essentially given by the energy of the deepest potential well $E_0 \approx -u_L a^{-d/2}$. The penetration depth of the wave function is roughly given by $L_c(E_0) = a g_0^{1/4}(a) \ll a$, i.e., the wave function extends essentially over a length $L_c \approx a$. With these findings, we get from (5) and (13) $R_A(t) \approx a$, i.e., the polymer is collapsed and localized, in agreement with our Flory arguments. In the quantum case, $g_0(a) \ll 1$, the potential energy acts on the scale a as a small perturbation to the kinetic energy. Bound states exist for an isolated shallow well in one and two dimensions but, since the penetration depth $\lambda \approx a g_0^{-1/2(2-d)} \gg a$, the wave functions of different wells now overlap strongly. For low enough energies, however, the states are still localized[19] and will dominate (13) and (15). Since, for $g_0(a) \ll 1$ and $d < 4$, l_c is the only length scale in the problem we conclude $R_A(t) \approx l_c$ again in agreement with our Flory argument.

If, on the other hand, $\Delta E_n t \ll 1$,

$$\frac{1}{L^d}\int_{\mathbf{x}}\int_{\mathbf{y}} (\mathbf{x}-\mathbf{y})^{2n} G(\mathbf{x},t;\mathbf{y}) = (-1)^n \int_{-\infty}^{\infty} dE\, e^{-Et} \left[\frac{\partial^{2n}}{\partial \mathbf{k}^{2n}} A(\mathbf{k},E) \right]_{\mathbf{k}=0} \tag{14}$$

is self-averaging. On the rhs of Eq. (14) we have introduced the spectral density[20]

$$A(\mathbf{k},E) = \frac{1}{L^d} \sum_m \left\langle \left| \int_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \phi_m(\mathbf{x}) \right|^2 \delta(E - E_m) \right\rangle .$$

Its asymptotic form for small $\mathbf{k}$ and $E \to -\infty$ is proportional to the density of state $\rho(E)$.

In the quantum case $g_0(a) \ll 1$ and $d < 4$,

$$\rho(E) \simeq \exp(-c|E/E_0|^{(4-d)/2})$$

(Ref. 20) with $E_0 = D l_0^{-2}$, the integral (14) is dominated by the saddle point $E_s \simeq \{[t/(4-d)]E_0^{(4-d)/2}\}^{2/(2-d)}$ which gives

$$R_A \simeq L_c(E_s) \simeq [(4-d)D/t u_0^2]^{1/(2-d)}$$

for $d < 2$, in agreement with our result from Flory arguments for this case.

In the classical case $g_0(a) \gg 1$,

$$\rho(E) \simeq \exp[-c(E^2/u_0^2)a^d] ,$$

an analogous estimation yields

$$R_A \approx [a^2/R_D(t)] g_0^{-1/2}(a) \ll a .$$

Since R_A is a least of the order a, this indicates again $R_A \approx a$ in agreement with the results of Sec. II.

V. CONCLUDING REMARKS

We first translate some of our results already discussed for a random-walk polymer into the language of the other applications mentioned in the Introduction.

For the problem of the random autocatalytic diffusion Eq. (1) we have shown that an originally localized population $\psi(\mathbf{x},0) = \delta(\mathbf{x}-\mathbf{y})$ spreads over the whole space. Nevertheless, it is mainly localized at a single attractive place for a fixed time. The further the attractive places $\mathbf{x}$ are away from the starting point $\mathbf{y}$, the bigger the local potential $U(\mathbf{x})$ has to be in order to grow a peak faster there, which finally will overcome the peaks growing nearer to $\mathbf{y}$. The quantity $R_B(t)$, which is a measure of this spreading, behaves for strong disorder as $R_B(t) \sim t \ln^{-3/4} t$. The exponent of the logarithm $\tilde{\nu}_B = -\frac{3}{4}$ is different from the value $\tilde{\nu}_B = -\frac{1}{2}$ found in Ref. 1. However, we note that Ebeling *et al.*[1] use, in their derivation, the *average* density of states, which is only correct, if the nominator and the denominator in the definition of $R_B^2[t,U;y]$ are self-averaging, or if $\langle \psi(\mathbf{x},t) \rangle$ is considered. Neither is the case (for realistic, finite systems). Clearly, the results of Ref. 1 would also be changed if the density of states in the quantum case were considered, i.e., if $a \to 0$. The finiteness of a is essential for $D < D^*$ in $d < 4$ and $D > D^*$ in $d > 4$ with

$$D^* = d u_0 a^{(4-d)/2} \ln^{1/2}(L/R_c) .$$

A similar behavior was obtained earlier for a directed polymer in a random medium.[21] We also mention that $\langle \psi^m(\mathbf{x},t) \rangle$, m integer, has been studied earlier.[22,23] In particular, Zeldovich *et al.*[22] argued for a lattice version of our path integral (3) that the optimal paths contain ballistic parts at their ends and a dominant stationary part at the high potential wells, which is in agreement with our considerations around Eq. (10). But they consider only self-averaging quantities.

The growing interface is characterized by a roughness exponent $\chi_B/\nu_B = 1$ and a dynamical exponent $1/\nu_B = 1$, but the scaling is modified by logarithmic corrections. Since the case of weak disorder has been considered in some detail in Ref. 7, we discuss here only the case of strong disorder. The structure of the interface consists of conical mountains of mean spacing $R_B(t) \sim t(\ln t)^{-3/4}$ and peak-to-valley separation $\Delta h \approx t(\ln t)^{1/2}$. Thus, the average slope of the hills $\Delta h/R_B(t)$ increases slowly with time as $(\ln t)^{5/4}$ for $R < R_c$ and as $\ln t$ for $R > R_c$ with R_c defined in (7).

Finally, let us again consider the random-walk polymer with free ends (problem A). In Sec. II this is shown to be collapsed and localized around the lowest potential well in the system, from which effect the $\ln L$ terms in (6)–(8) arise. The $\ln L$ only makes sense if the system is in equilibrium, i.e., for real time $\tau \to \infty$. When the polymer is given an activated dynamics it will not be able, when starting at an arbitrary position, to find the lowest well in the system during experimentally accessible observational time scales τ, but it can diffusively explore only typical wells in some finite volume $L_\tau^d \ll L^d$. Thus, the static factors $\ln L$ in (6)–(8) have to be replaced by the dynamic factor $\ln L_\tau$ with

$$L_\tau = R_0(\tau/\tau_0)^{1/2}\exp(\delta h) \ . \tag{15}$$

Here $R_0 = \max[a, l_0 = (D/u_0)^{2/(4-d)}]$ is the distance between neighboring typical potential wells, $\tau_0 = \tau_0(t)$ is an elementary jumping time, and δh accounts for the barrier height between the wells.

When assuming the collapsed polymer chain as a whole to perform jumps we get $\tau_0 \simeq t^z$ with $z=1$ and, from (6), $\delta h \simeq -tu_0/R_0^{d/2}$ and L_τ easily becomes very small. On the other hand, one should take into account tunnel states in (3), i.e., two tails of the chain stay in different wells simultaneously, whereas the middle part of contour length t_m extends over the barrier. Minimizing the free energy yields t_m and δh independent of t. Thus for long chains the latter case will happen and $L_\tau \propto \tau_0^{-1/2} \propto t^{-z/2}$ with $z \ge 1$ expected. Further dynamical considerations are skipped here.[24]

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APPENDIX

Here we present some results of the order statistics of the potential U, which are given in the introduction and are mainly used throughout Sec. II.

Assume a set of random variables U_j, $j=1,\dots,N$ equally distributed with a probability density (PD) $f(U)$. Then the PD of the ordered set $U_1 \le U_2 \le \cdots \le U_n$ is evidently given by

$$g_N(U_1,\dots,U_N) \approx \begin{cases} N!\prod_{j=1}^{N} f(U_j) & \text{if } U_1 \le U_2 \cdots \le U_N \\ 0 & \text{otherwise,} \end{cases} \tag{A1}$$

where the combinatorial factor accounts for normalization. From this expression, any quantity of the ordered set can be derived. The PD of the mth and $(m+1)$th ordered values $U_1 \le \cdots \le U_m \le U_{m+1} \le U_N$ is given by integration (see, e.g., Ref. 25)

$$\begin{aligned} g_{2,m}(U_m,U_{m+1}) &= \int dU_1 \cdots dU_{m-1} dU_{m+2} \cdots dU_N g_N(U_1,\dots,U_N) \\ &= \begin{cases} 0 & \text{if } U_{m+1} < U_m \ , \\ \dfrac{N!}{(m-1)!(N-m-1)!} F^{m-1}(U_m) f(U_m) f(U_{m+1})[1-F(U_{m+1})]^{N-m-1} & \text{otherwise} \ , \end{cases} \end{aligned} \tag{A2}$$

where

$$F(U) = \int_{-\infty}^{U} f(U')dU' \ . \tag{A3}$$

From (A2) the average of the distance between two successive values yields

$$\begin{aligned} \langle \delta U_m \rangle &= \langle U_{m+1} - U_m \rangle \\ &= \int_{-\infty}^{\infty} dU_{m+1} \int_{-\infty}^{U_{m+1}} dU_m (U_{m+1}-U_m) g_{2,m}(U_m,U_{m+1}) \\ &= \frac{N!}{m!(N-m)!} \int_{-\infty}^{\infty} dU [F(U)]^m [1-F(U)]^{N-m} \ . \end{aligned} \tag{A4}$$

Furthermore, the PD of the mth-ordered value U_m follows from (A2) by integrating U_{m+1}

$$\begin{aligned} g_{1,m}(U_m) = {} & \frac{N!}{(m-1)!(N-m)!} F^{m-1}(U_m) \\ & \times f(U_m)[1-F(U_m)]^{N-m} \end{aligned} \tag{A5}$$

and the averages

$$\langle U_m^k \rangle = \int_{-\infty}^{\infty} dU\, U^k g_{1,m}(U), \quad k = 1,2,\dots \ . \tag{A6}$$

Now we calculate, by saddle-point evaluation, the integrals for the case considered here, i,e., the Gaussian distribution

$$f(U)=\frac{1}{\sqrt{2\pi}U_0}\exp(-U^2/2U_0^2)$$

$$F(U)=\begin{cases} -U_0^2 f(U)/U & \text{if } U\rightarrow -\infty \\ \frac{1}{2}+\frac{1}{\sqrt{2\pi}}U/U_0 & \text{if } |U| \ll U_0 \\ 1-U_0^2 f(U)/U & \text{if } U\rightarrow \infty \ . \end{cases} \tag{A7}$$

The saddle-point approximation

$$\int_{-\infty}^{\infty}\exp[S(U)]dU \simeq \sqrt{2\pi}\exp[S(U_s)] \times |\partial_{UU}^2 S(U_s)|^{-1/2}$$

with $\partial_U S(U_S)=0$ becomes exact in the limit $N\rightarrow\infty$. From (A4) we have $\langle \delta U_m \rangle \propto U_0[Nf(U_S)]^{-1}$ with the saddle point U_s, where $F(U_s)=m/N$ and thus, using (A7),

$$\langle \delta U_m \rangle \propto U_0 \begin{cases} \frac{1}{m\ \ln^{1/2}N} & \text{if } |U| \gg U_0 \\ \sqrt{2\pi}/N & \text{if } |U| \ll U_0 \ . \end{cases} \tag{A8}$$

Consequently, the values of U_m near the mean value 0 have an average density $\propto N$ whereas the density in the tails scales like $\propto \ln^{1/2}N$. The first expression of (A8) is used in Eq. (8) of the text. Similarly, the saddle point of (A6) is U_s with

$$(m-1)/F(U_s)=N-1+(U_s/U_0^2-k/U_s)/f(U_s) \ ,$$

which, in the case $1=m \ll N$ and $k=1,2$, simplifies to $(N-1)f(U_s)\simeq U_s/U_0^2$ and

$$U_s \simeq \sqrt{2}U_0 \ln^{1/2}N - o(\ln(\ln N)/\ln N) \ .$$

With this we obtain

$$\langle U_1 \rangle \simeq \frac{2\sqrt{\pi}}{e}U_0 \ln^{1/2}N \tag{A9}$$

and $\langle U_1^2 \rangle \simeq e/\sqrt{2\pi}\langle U_1 \rangle^2$ so that $\langle \Delta U_1^2 \rangle \simeq 0.2U_0 \ln N$. Equation (A9) is quite obvious and widely used but fluctuations are of the same size as the average.

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