Scaling in multifractals: Discretization of an eigenvalue problem

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A matrix approximant to the generalized Frobenius-Perron equation is presented, the largest eigenvalues of which approach the most important eigenvalues of the operator (e.g., the so-called free energy). It is pointed out that also certain eigenfunctions not accessible to iteration of the continuous problem can easily be obtained in the discretized formalism. For general one-dimensional maps the spectrum of generalized entropies is shown to appear as the largest eigenvalue of a truncated version of the matrix. The statistical interpretation of the eigenfunction associated with the free energy is given in terms of configuration probabilities of spin chains.

I. INTRODUCTION

Certain multifractal[1] or, more generally, thermodynamic[2–19] properties of dynamical systems have recently been shown to appear as eigenvalues of certain operators. For one-dimensional maps $x'=f(x)$ this eigenvalue problem can be written in the form of a generalized Frobenius-Perron equation

$$\lambda(\beta)Q(x')=\sum_{x\in f^{-1}(x')}\frac{Q(x)}{|f'(x)|^{\beta}} . \tag{1}$$

Here β is a parameter ($-\infty<\beta<\infty$), f' and f^{-1} denote the derivative and the inverse of f, respectively, $\lambda(\beta)$ is an eigenvalue of the generalized Frobenius-Perron (GFP) operator, defined by the right-hand side of (1), and $Q(x)$ is the corresponding eigenfunction.

Generalized versions of the Frobenius-Perron equation have long been discussed in mathematics.[2,3] In the physical literature, they appeared first in connection with the transient chaotic dynamics of one-dimensional maps[8] and were then extended to study permanent chaotic behavior.[9–13] Recent results indicate the applications of the GFP equation for describing thermodynamic properties of nonnatural measures,[14–16] and, moreover, also of general fractals showing up beyond the scope of dynamical systems.[17,12]

For concreteness, in the main part of the paper we restrict our attention to fully developed chaotic cases (complete maps) when a chaotic attractor exists and is mapped by $f(x)$ 2 to 1 onto itself.[20] Functions with a maximum of order z are considered. As a prototype for such maps we take the family

$$x'=1-2|x|^{z} , \tag{2}$$

with $z>\frac{1}{2}$ where the attractor is the interval $[-1,+1]$.

First, we shortly summarize some properties of the GFP operator for later purposes. The eigenvalues that can be obtained in an iterative procedure are of special importance. By this we mean eigenvalues whose eigenfunctions can be constructed by an iterative solution of (1), starting with some Q_0. $Q(x)$ appears then as the $k\to\infty$ limit of a function series $Q_0,\dots Q_k,\dots$ and is independent of the choice of the initial Q_0, in some function space. The eigenvalues themselves are the particular values of $\lambda(\beta)$ which ensure the convergence of the series $\{Q_k\}$.[9]

In the space of functions having a singularity of order $(1-1/z)\beta$ at both ends of the attractor [i.e., $Q(x)\sim(1\pm x)^{-(1-1/z)\beta}$], an eigenvalue $\lambda_1(\beta)$ (the largest one in this space) is associated with the aforementioned iteration procedure. This $\lambda_1(\beta)$ was proven[11,12] to be connected with the generalized entropies K_q (Ref. 21) with respect to the natural measure on the attractor as

$$\ln\lambda_1(q)=(1-q)K_q . \tag{3}$$

By using smooth initial functions, another eigenvalue $\lambda_*(\beta)\equiv\exp[-\beta F(\beta)]$ ensures the convergence of the series $\{Q_k\}$. It was shown[12,17] that

$$-\ln\lambda_*(\beta)\equiv\beta F(\beta)=G(\beta) , \tag{4}$$

where $G(\beta)$ is defined by the relation $\sum_{i=1}^{2^{n-1}}\Delta_i^{\beta}\sim e^{-G(\beta)n}$, in the large-$n$ limit. The length scales $\Delta_i=x_{2i}-x_{2i-1}$ appearing in the sum are obtained from the nth preimages $x_j (j=1,2,3,\dots,2^n)$ of some seed point x^*, ordered along the x axis. Note that the Δ_i's provide a partial coverage of the attractor only, since the intervals (x_{2i},x_{2i+1}) are not taken into account.

The eigenvalues $\lambda_1(\beta)$ and $\lambda_*(\beta)$ are found[12] to agree in the region $\beta/\beta_c<1$ where β_c is a critical point ($\beta_c\gtrless 0$ for $z\lessgtr 1$). Furthermore, $\lambda_1(\beta)$ is a smooth function of β, but there is a break in the free energy $F(\beta)$ at β_c, interpreted as a phase transition (Refs. 22,5,9,7). The condensed phase $\beta/\beta_c>1$ is characterized by the inequality $\lambda_*(\beta)>\lambda_1(\beta)$. In fact, as pointed out in Ref. 12 (see also Ref. 13), the phase transition is caused by a crossing of two eigenvalue branches

$$\lambda_*(\beta)=e^{-\beta F(\beta)}=\max\{\lambda_0(\beta),\lambda_1(\beta)\} , \tag{5}$$

where $\lambda_0(\beta)$ is an eigenvalue existing for $\beta/\beta_c>0$ only. The branch $\lambda_0(\beta)$ is determined by the slope c, supposed to be finite, of $f(x)$ at the left end of the attractor: $\lambda_0(\beta)=c^{-\beta}$.[12]

The GFP operator is non-Hermitian; its spectrum possesses both discrete and continuous components.[12] The aim of this paper is to find a matrix approximant which is much easier to handle than the original operator. This matrix is presented in Sec. II. We then study how the eigenvalues λ_* and λ_1 appear in the discretized version and point out that also certain eigenfunctions not accessible by iterating (1) can easily be obtained in the matrix formalism (Sec. III). The largest eigenvalue of a simple truncated version of the matrix is found to approach $\lambda_1(\beta)$, i.e., to be related to the generalized entropies of the dynamical system, as discussed in Sec. IV. It will be shown that $\beta F(\beta)$ coincides with the growth rate $\overline{G}(\beta)$ of a sum containing the length scales raised to a power β in a *full* coverage of the attractor. Consequently, $-\beta F(\beta)$ agrees with the quantity called pressure.[2,5] Finally the statistical interpretation of the eigenfunctions associated with $\lambda_*(\beta)$ is given in terms of probabilities of spin-chain configurations in an equilibrium ensemble at temperature $T=1/\beta$ (Sec. V). Possible further applications are discussed in Sec. VI.

II. MATRIX APPROXIMATION

First, we define the coverage of the attractor we are going to use in what follows. This is the so-called generating partition[23] obtained by considering the nth preimages of the whole attractor for $n\to\infty$. Let $I_j^{(n)}$ $(j=1,2,3,\ldots,2^n)$ denote the preimage intervals (cylinders) at a fixed value of n where the subscripts j are chosen in such a way that the $I_j^{(n)}$'s are ordered along the x axis. They completely cover the attractor. Each cylinder can be uniquely associated with a symbol sequence $\epsilon_1,\epsilon_2,\ldots,\epsilon_n$, where ϵ_i is 0 or 1 depending on whether trajectories starting from $I_j^{(n)}$ lie, after $i-1$ steps, left or right from the point x_c belonging to the maximum of $f(x)$.

Notice that the Frobenius-Perron equation[24] is recovered from (1) for $\beta=1$. In this important case the maximal eigenvalue in the class of smooth initial functions is 1 corresponding to the existence of a stable stationary distribution $\rho(x)$ on the attractor. It was Ulam[25] who conjectured the existence of a matrix approximant to the Frobenius-Perron operator.[24] For hyperbolic maps $[1<|f'(x)|<\infty]$ the conjecture was proven,[26] while for nonhyperbolic cases numerical evidence supports its validity.[25] In our notation the conjecture says that the matrix

$$a_{ij}=\frac{l(I_j^{(n)}\cap f^{-1}(I_i^{(n)}))}{l(I_j^{(n)})},\quad i,j=1,2,3,\ldots,2^n \tag{6}$$

is an nth-order approximant to the operator where $l(I)$ is the length (Lebesgue measure) of an interval I. The matrix element a_{ij} is just that portion of the length of $I_j^{(n)}$ which is mapped under f into $I_i^{(n)}$. (Similar approximations to time-evolution operators of probability distributions have been used also in other contexts.[27] The largest eigenvalue of the matrix a_{ij} is unity. The corresponding right eigenvector $(c_1,c_2,\ldots,c_{2^n})$ has the following property:[25] the piecewise constant function

$$\rho(n,x)=\frac{c_j}{l_j},\quad x\in I_j^{(n)} \tag{7}$$

approaches the stationary distribution as $n\to\infty$. Here the shorthand notation $l_j\equiv l(I_j^{(n)})$ has been applied.

Based on exactly tractable piecewise linear examples, we extend Ulam's conjecture by considering the matrix $P(n,\beta)$ with elements

$$P_{ij}=\begin{cases}(a_{ij})^\beta & \text{if } a_{ij}\neq 0\ ,\\ 0 & \text{otherwise}\end{cases} \tag{8}$$

to be the nth-order approximant to the GFP operator. Certain eigenvalues $\lambda(n,\beta)$ of the matrix tend for $n\to\infty$ to $\lambda(\beta)$, an eigenvalue of the operator. The piecewise constant function

$$Q(n,x)=\frac{s_j}{(l_j)^\beta},\quad x\in I_j^{(n)} \tag{9}$$

approaches then $Q(x)$, the eigenfunction belonging to $\lambda(\beta)$, where s_j is the jth element of the right eigenvector belonging to $\lambda(n,\beta)$.

It is worth noting that for $\beta\neq 1$ $Q(x)$ cannot be interpreted as the density of a usual measure as one sees by comparing Eqs. (7) and (9). For $\beta=1$ the function $\rho(x)$ is a probability distribution on the attractor, and that is why Ulam's original conjecture could be formulated for arbitrary partitions of the attractor. However, the extensions (8) and (9) give meaningful results only with the generator.

In what follows we support the conjecture by numerical evidence. In particular, we have numerically investigated the largest eigenvalue $\lambda_1(n,\beta)$ of the matrix $P(n,\beta)$ for maps of type (2). By applying $P(n,\beta)$ several times on any unit vector with positive elements and renormalizing in each step, the normalization constant and the vector generated tend to $\lambda_1(n,\beta)$ and to the corresponding right eigenvector, respectively. The convergence is, in general, quite rapid: applying $P(n,\beta)$ $m=20-30$ times an accuracy of 0.1% is observed.

We have found that $\lambda_1(n,\beta)$ obtained in this way approaches, for large n, the eigenvalue $\lambda_*(\beta)$ of the GFP operator:

$$\lambda_1(n,\beta)\xrightarrow[n\to\infty]{}\lambda_*(\beta)\ . \tag{10}$$

This implies that in the condensed phase $\lambda_1(n,\beta)\to\lambda_0(\beta)$, while $\lambda_1(n,\beta)\to\lambda_1(\beta)$ otherwise [see Eq. (5)]. For β values of order of unity, $\lambda_*(\beta)$ is approximated within an accuracy of 0.2% already at $n=10$. At the critical point β_c a sharper and sharper bend develops in the plot $\lambda_1(n,\beta)$ vs β as $n\to\infty$. According to the existence of a phase transition, the convergence slows down here in both m and n.

We have checked the validity of Eq. (9), too. The analytic form of the eigenfunctions $Q^{(\beta)}$ associated with $\lambda_*(\beta)$ are known[13] for the logistic map [$z=2$ in (2)]. Figure 1 illustrates in which sense $Q(n,x)$ approaches the singular eigenfunction of the GFP operator. $Q(6,x)$ and $Q(8,x)$ were calculated from the right eigenvectors belonging to $\lambda_1(6,1.2)$ and $\lambda_1(8,1.2)$, respectively. By com-

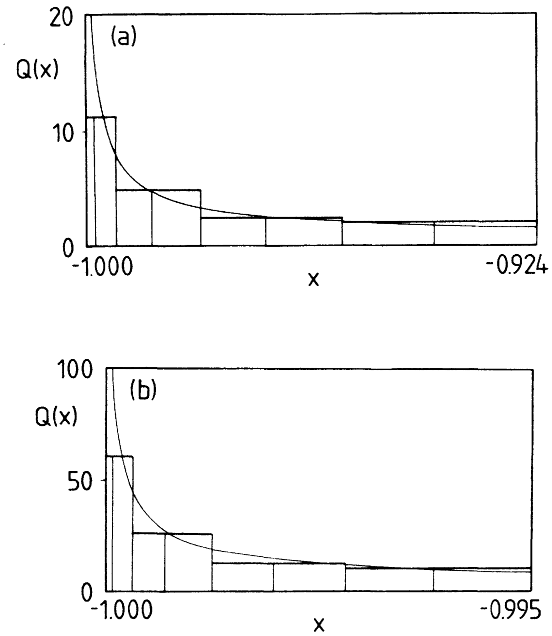

FIG. 1. The approximant (9) and the exact eigenfunction (solid line) associated with the eigenvalue $\lambda_1(n,\beta)$ of the GFP operator for $x'=1-2x^2$, $\beta=1.2$. The approximants $n=6$, $m=20$ (a) and $n=8$, $m=20$ (b) are plotted. The range shown in (b) corresponds to the first two cylinders of (a). Note the change of scale on the vertical axis (arbitrary units).

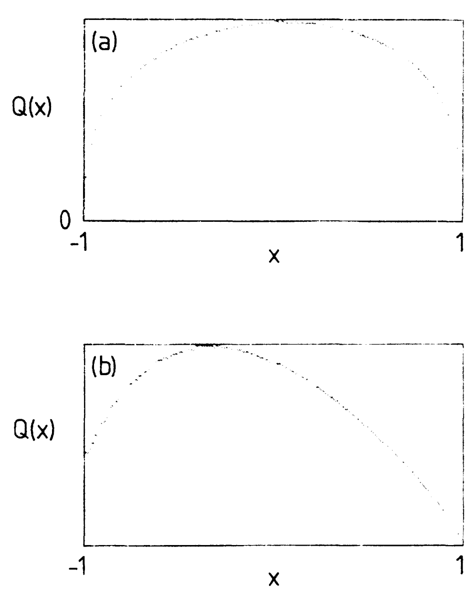

FIG. 2. Approximate eigenfunctions (9) associated with the eigenvalue $\lambda_1(n,\beta)$ at critical points, which cannot be found by iterating (1). (a) $z=4$, $\beta_c=-0.504$; (b) $z=0.8$, $\beta_c=3.174$. In both cases the approximates $n=10$, $m=200$ are shown, and maps of type (2) are taken. [In (b) the results obtained for $x\rightarrow -1$ are not yet reliable due to the slow convergence in that region.]

paring them with the exact result $(1-x^2)^{-0.6}$ one sees that Eq. (9) provides, with increasing n, a refining approximation for this singular function. Similar agreement has been found also at other values of β for both above and below the critical point $\beta_c=-1$.[28] At the critical point itself the iteration of (1) starting with a smooth initial function diverges.[13] Nevertheless, $(1-x^2)^{0.5}$ is an eigenfunction associated with $\lambda_*(\beta_c)=4$ but cannot be reached by iterating Eq. (1). We found, however, that this function can directly be obtained via Eq. (9).

For other members of the family (2) no analytic results are known. We have calculated, for different values of z, the eigenfunctions $Q^{(\beta)}(x)$ associated with the eigenvalue $\lambda_*(\beta)$ of (1) and their discretized versions (9), and an agreement has been found. The situation is somewhat different at critical points. Here we used Eq. (9) to find the approximate eigenfunctions since they cannot be reached by iterating (1). The results obtained in this way are exhibited in Fig. 2. For $\beta\rightarrow\beta_c$, the eigenfunction $Q^{(\beta)}$ obtained by iterating (1) from smooth initial functions can be shown[28] to behave as $Q^{(\beta)}(x) = A(x)|\beta-\beta_c|^{-1}$, where $A(x)$ is a finite positive function. In fact, the approximant (9) at β_c yields just $A(x)$. The prefactor $|\beta-\beta_c|^{-1}$ does not appear owing to the normalization of the eigenvectors.

III. THE SECOND EIGENVALUE AND ITS EIGENFUNCTION

The second largest eigenvalue $\lambda_2(n,\beta)$ of the matrix $P(n,\beta)$ can be obtained by applying the matrix several times on a unit vector orthogonal to the *left* eigenvector belonging to $\lambda_1(n,\beta)$. The factors used to normalize the vector in each step tend, for $n\rightarrow\infty$, to $\lambda_2(n,\beta)$, while the vector obtained in this way is the associated right eigenvector of $P(n,\beta)$.

Our numerical investigations show that in the condensed phase $\beta/\beta_c>1$

$$\lambda_2(n,\beta)\xrightarrow[n\rightarrow\infty]{}\lambda_1(\beta)\ , \tag{11}$$

while in the region $0<\beta/\beta_c<1$

$$\lambda_2(n,\beta)\xrightarrow[n\rightarrow\infty]{}\lambda_0(\beta)\ . \tag{12}$$

Consequently, for large n $\lambda_2(n,\beta)$ approaches $\lambda_1(n,\beta)$ at β_c, and a critical slowing down shows up in n when calculating $\lambda_2(n,\beta)$. At any finite value of n the two branches $\lambda_1(n,\beta)$ and $\lambda_2(n,\beta)$ exhibit an "avoided crossing" approaching, for $n\rightarrow\infty$, the crossing of $\lambda_0(\beta)$ and $\lambda_1(\beta)$ (Fig. 3). It is to be noted that, in numerical studies, $\lambda_2(n,\beta)$ turns out[28] to be complex in the range $\beta/\beta_c<0$ and cannot be used to approach the real eigenvalue of the GFP operator there.

Equation (9) is again a useful approximant to the eigenfunction (for $\beta/\beta_c>0$). In the condensed phase an agreement is found with the solutions of (1) belonging to $\lambda_1(\beta)$. In the region $0<\beta/\beta_c<1$ the eigenfunctions associated with $\lambda_0(\beta)$ have not yet been known since they also cannot be reached by iterating (1), due to the fact that $\lambda_0(\beta)$ is not the largest eigenvalue here. However, Eq. (9) pro-

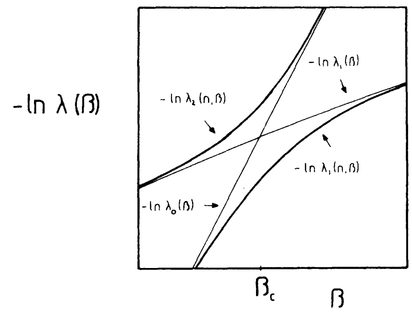

FIG. 3. The avoided crossing of the branches $\lambda_1(n,\beta)$, $\lambda_2(n,\beta)$ around the critical point β_c. (Schematic drawing for a case $z>1$.)

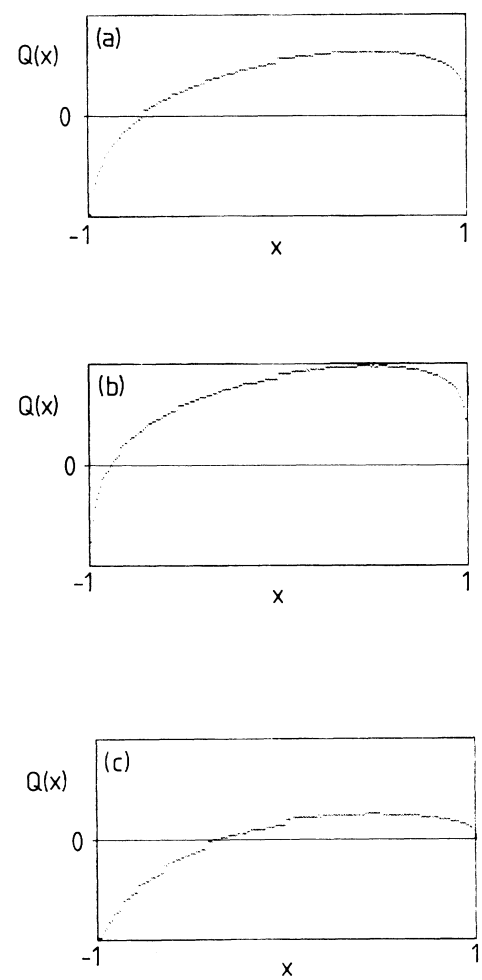

FIG. 4. Approximate eigenfunctions (9) associated with the second largest eigenvalue $\lambda_2(n,\beta)$ in the range $0<\beta/\beta_c<1$, which cannot be obtained by iterating (1). (a) $z=2$, $\beta=-0.5$; (b) $z=4$, $\beta=-0.3$; (c) $z=0.8$, $\beta=1$. In all cases $n=10$, $m=40$ and maps of type (2) are taken.

vides a tool for finding these eigenfunctions. The results depicted in Fig. 4 show that all these functions possess a node.

IV. THE TRUNCATED MATRIX—RELATION TO ENTROPIES

The phase transition occurring in the free energy, $\ln 1/\lambda_*(\beta)$, of (1) was shown[12] to be a consequence of the anomalous scaling of the leftmost cylinder. This is reflected also in the fact that the branch $\lambda_0(\beta)$ dominating the condensed phase is specified by the slope of $f(x)$ at the left end of the attractor. It seems, therefore, evident to consider a truncated matrix $C(n,\beta)$ obtained by cutting the outermost rows and columns of $P(n,\beta)$. In the large-n limit this is a very small change having, however, quite drastic consequences.

The largest eigenvalue $\tilde{\lambda}(n,\beta)$ of $C(n,\beta)$ has been found to approach, for large n, the eigenvalue $\lambda_1(\beta)$ of the GFP operator in the whole β region:

$$\tilde{\lambda}(n,\beta)\xrightarrow[n\to\infty]{}\lambda_1(\beta) . \tag{13}$$

When calculating $\tilde{\lambda}(n,\beta)$, *no* critical slowing down shows

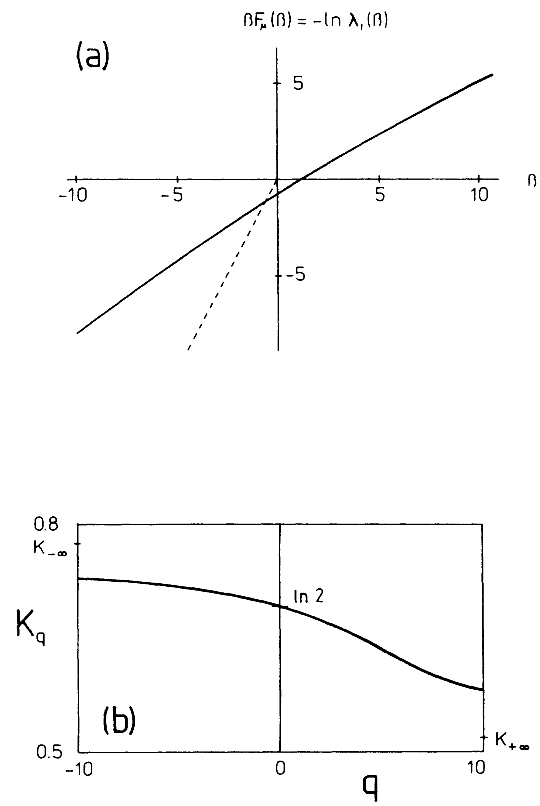

FIG. 5. The logarithm of the eigenvalue $\lambda_1(\beta)$ $\equiv\exp[-\beta F_\mu(\beta)]$ (a) and the K_q spectrum (b) obtained for the map $x'=1-2x^4$ by calculating the largest eigenvalue of the truncated matrix and using relations (3) and (13). The dashed line in (a) represents the branch $-\ln\lambda_0(\beta)=\beta\ln c$. The deviation of the curve in (b) from the exact K_q values is within the line thickness.

up at β_c since there is no degeneracy in the largest eigenvalues of $C(n,\beta)$. The corresponding eigenvector agrees, for $n \gg 1$, with the one belonging to the largest (second largest) eigenvalue of $P(n,\beta)$ outside (inside) the condensed phase, i.e., it approaches the eigenfunction of (1) belonging to $\lambda_1(\beta)$.

Comparing Eqs. (3) and (13) we conclude that the truncated matrix $C(n,\beta)$ provides a powerful tool for calculating the entropy spectrum of fully developed chaotic maps. Figure 5 shows the results obtained in this way for the map $x'=1-2x^4$. The deviation of the curve on Fig. 5(b) from the exact K_q values is of order 1% in relative error in the range shown, and cannot be resolved within the line thickness used to plot. It is worth noting that, when evaluating $\tilde{\lambda}(n,\beta)$ for sufficiently large β values, the second largest (complex) eigenvalues are also to be taken into account since their modulus might then be close to that of $\tilde{\lambda}$.

V. COMPARISON WITH OTHER METHODS: THE INTERPRETATION OF $Q^{(\beta)}(x)$

Based on the concept of the scaling function,[29] a transfer matrix $T(n,\beta)$ has been defined in Refs. 4 and 7. Using the facts that the scaling function is the daughter-to-mother ratio of the length scales of the cylinders (the daughters of a cylinder are the two subintervals into which it is divided when refining the partition) and that the nonzero elements of $T(n,\beta)$ are just the values of the scaling function raised to a power β, a detailed comparison of $T(n,\beta)$ and $P(n,\beta)$ shows that these matrices are equivalent. (The formal difference between them is that the elements of T are indexed by the symbol sequences of the cylinders rather than by their actual position along the x axis.)

The largest eigenvalue of $T(n,\beta)$ was shown[4,7] to approach, for large n, $\exp[-\bar{G}(\beta)]$ where $\bar{G}(\beta)$ is defined by the growth rate of the "partition" sum over all the length scales $l_j \equiv l(I_j^{(n)})$ raised to a power β, i.e., by $\sum_{i=1}^{2^n} l_i^\beta \sim e^{-\bar{G}(\beta)n}$. The quantity $-\bar{G}(\beta)$ is often called pressure.[2,3,5] Note the difference between this coverage and that obtained by the Δ_i's [see the lines below Eq. (4)]. Nevertheless, since $T(n,\beta)$ and $P(n,\beta)$ are equivalent and (10) holds, we find that $G(\beta)=\bar{G}(\beta)$, i.e., the two coverages are also equivalent at least from the point of view of the growth rates of their partition sums. As a consequence of (4), $-\beta F(\beta)$ turns out to be the pressure.

It has been stated in Ref. 7 that the two largest eigenvalues of T coincide in the condensed phase. In view of the equivalence of T and P, this statement seems to be dubious since we saw a definite difference between the two first eigenvalues in the condensed phase.

The transfer matrix formalism helps in finding a physical interpretation for $Q^{(\beta)}(x)$, the eigenfunction of (1) associated with the free energy. The sum $\sum_{j=1}^{2^n} l_j^\beta$ can be regarded[4,7] as the partition sum of an Ising chain of length n by considering a spin configuration j to be given, via a simple rule, by the symbol sequence $\epsilon_1, \ldots, \epsilon_n$ of the cylinder $I_j^{(n)}$. The energy of configuration j is $E_j=-\ln l_j$; therefore, l_j^β stands for the Boltzmann factor p_j of the configuration at temperature $1/\beta$.

Consequently, the largest eigenvalue of T (or P) yields, in the thermodynamic limit $n \to \infty$, the free energy of the spin system while the components s_j of the corresponding right eigenvector are related to the configuration probabilities p_j:

$$s_j = Q^{(\beta)}(x_j)p_j, \quad x \in I_j^{(n)}, \tag{14}$$

as it follows from Eq. (9). [Here x_j can be any point inside $I_j^{(n)}$ except for the outermost cylinders where $Q^{(\beta)}(x)$ might diverge.] Thus the following interpretation of $Q^{(\beta)}(x)$ is found: this function measures how strongly the eigenvector components differ from the actual spin-configuration probabilities. The difference is due to the fact that the matrix $e^{\beta F(\beta)}T(n,\beta)$ is, in general, not a stochastic matrix because the relation $\sum_{j=1}^{2^n} T_{ij}(n,\beta) = e^{-\beta F(\beta)}$ need not hold for $\beta \neq 1$, and therefore the components of the eigenvector cannot be regarded as probabilities. It has been shown that for $\beta/\beta_c > 1$ $Q^{(\beta)}(x)$ is always smooth at $x=-1$ while vanishing (singularly) at the other end of the attractor.[12,13,28] This asymmetry can be considered as a new characteristic of the condensed phase.

VI. DISCUSSION

We conclude by a few comments on possible further applications of the method. First, note that although we were interested up to now in general, nonhyperbolic maps, the procedure can be applied also to hyperbolic cases. The main simplifying feature is then the coincidence of $\lambda_*(\beta)$ and $\lambda_1(\beta)$. Consequently, no phase transition exists. The largest eigenvalues of $P(n,\beta)$ and $C(n,\beta)$ both approach then $\exp[-\beta F(\beta)]$. The discretization of the GFP operator can also be worked out for hyperbolic maps exhibiting transient chaotic behavior. Such maps possess a chaotic repeller,[30] rather than attractor, characterized by its own natural measure. An attractive feature of this class is the fact that the generalized dimensions D_q with respect to the natural measure follow from the free energy via the implicit equation[9,19]

$$\beta F(\beta)|_{\beta=(1-q)D_q+q} = qF(\beta)|_{\beta=1} . \tag{15}$$

Other relations (Refs. 5,13,15, and 31) yield the generalized dimensions and entropies with respect to the so-called Gibbs measures[2,5] on the repeller, based again on the knowledge of the free energy alone. In such cases, therefore, the largest eigenvalue of the matrix $P(n,\beta)$ can directly be used to specify different multifractal spectra of dimensions and entropies.

It is to be mentioned that equations similar to the GFP one appear also in other contexts. Important examples are the momenta of the finite-time Lyapunov exponents,[32,10] scaling indices of harmonic measures on certain Julia sets,[14] and multifractal features of non-natural measures obtained by iterating one-dimensional maps backward.[16]

Finally, we mention that GFP operators might be associated also with thermodynamic properties of fractals appearing outside the field of dynamical systems. It has been proven[17,12] that fractals the length scales of which can be obtained by combining a few functions, so-called presentation functions,[17] are characterized by an eigenvalue problem similar to Eq. (1). In particular, if only two presentation functions are needed (the fractal is organized on a binary tree), the eigenvalue problem is equivalent to Eq. (1). The presentation functions correspond then to the two branches of f^{-1}. Equation (4) was shown to hold[17,12] where $G(\beta)$ is the growth rate of the length scale partition function. The largest eigenvalue $\lambda_*(\beta)$ or the free energy are, therefore, characteristics of the length scale distribution of such fractals, and typically undergo also phase transitions.

The discretized version of the GFP operator worked out in this paper might thus be relevant also in the study of general fractals.

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