Resonance fluorescence of linear and nonlinear oscillators according to classical and quantum theories: An insight into vacuum fluctuations

I. R. Senitzky
Center for Laser Studies and Department of Physics, University of Southern California, Los Angeles, California 90089-1112
(Received 5 May 1989)

The resonance fluorescence spectrum of linear and nonlinear oscillators driven by a monochromatic field of frequency ω_0, near the resonant frequency ω, is calculated both classically and quantum mechanically. The result for the linear oscillator is the same in both calculations: a δ function at ω_0. For the nonlinear oscillator, however, the classical and quantum-mechanical spectra near resonance are qualitatively different: Classically, a δ function at ω_0 is obtained; quantum mechanically, a weak-field calculation leads to a δ function at ω_0 together with finite-width peaks nearby. An intuitive explanation for the difference is suggested by the examination of the quantum-mechanical energy-correlation function for the linear oscillator. This function describes energy oscillations with frequencies within a band centered at $\omega_0-\omega$. The oscillations can be interpreted as the beating of the driven oscillation with those produced by the vacuum fluctuations (the latter having a frequency spread of the order of the relaxation constant). Because of the nonlinearity, the energy oscillation modulates the effect of the driving field to produce sidebands. Higher-order processes produce additional peaks. A heuristic picture of vacuum fluctuations is developed.

I. INTRODUCTION

The resonance fluorescence spectrum (RFS) of a two-level system (TLS) subject to a monochromatic driving field has been investigated both theoretically[1-6] and experimentally,[7,8] and is well known. For a weak driving field, in lowest order of the driving power, the spectrum is a δ function at the driving frequency. This is the same spectrum as that obtained from a harmonic oscillator (HO) at all driving levels, and is consistent with the commonly accepted principle that for sufficiently low excitation oscillators behave linearly.[9] In the next higher order of driving power, the TLS spectrum develops two symmetrical sidebands at $\omega_0\pm\Delta$, where ω_0 is the driving frequency and $\Delta=\omega_0-\omega$, ω being the resonant frequency of the TLS. At the opposite extreme, that of a strong driving field (sufficiently near resonance), the spectrum also exhibits a δ function at ω_0 and two sidebands, but here the sidebands are located at $\omega_0\pm\Omega$, where Ω is the Rabi frequency. The existence of strong-field sidebands can be explained in simple semiclassical terms. When the TLS, in the absence of coupling to the free-space radiation field, is driven at resonance, its energy (expectation value), if initially at the lower level, will oscillate between the two levels in what is known as Rabi oscillation. Consider, now, the imposition of coupling between the TLS and the radiation field. For a sufficiently strong driving field, this coupling may be considered to be a perturbation which will affect the Rabi oscillation only slightly, since the power radiated is small compared to the power interchanged with the driving field. Radiation by the TLS, both spontaneous and induced, is modulated by the Rabi oscillation, and, thus, sidebands at the Rabi frequency are produced. (A more detailed and quantitative discussion of the effect of the radiation field—along the same lines—is given in an earlier paper.[10]) The weak driving-field sidebands, on the other hand, seem to indicate the existence of modulation at the frequency Δ. What is the cause of such modulation, if, indeed, it exists? Since the TLS behavior must become similar to that of a HO as the driving power vanishes, we look for an answer in the differences between the TLS and HO that arise in the low-power regime. In this regime, there appear to be two important differences. One is the nonlinearity of the TLS, and the other is the fact that, for exactness, the TLS must be described quantum mechanically, while the HO can be described, formally, either classically or quantum mechanically. One is thus led to the general problem of investigating the RFS of linear and nonlinear oscillators both classically and quantum mechanically.

While there exists only one type of linear oscillator, there exist many types of nonlinear oscillators, distinguished by the kind of nonlinearity. We choose for detailed analysis a particular type, but we will show later that some significant results are applicable to nonlinear oscillators in general. Since a comparison of classical and quantum-mechanical results is sought, and since a motivation for the present problem comes from the TLS spectrum, we select for consideration a type of nonlinear oscillator that can be described both classically and quantum mechanically, and contains as a limiting (quantum-mechanical) case the TLS. Furthermore, we use a notation that can be read both classically and quantum mechanically, the dynamical variables being either c numbers, classically, or (Heisenberg-picture) operators, quantum mechanically.

II. NONLINEAR OSCILLATOR

The type of nonlinear oscillator to be considered may be called an angular momentum oscillator (AMO). It is described by the three (dimensionless) angular momen-

tum components l_1, l_2, and l_3 that obey the relationship

$$[l_j,l_k]=il_n \ ,$$

where j, k, and n, are the numbers 1, 2, and 3 permuted cyclically, and the bracket stands for the commutator, quantum mechanically, or the Poisson bracket multiplied by i (with respect to dimensionless coordinates), classically. The AMO Hamiltonian is given by

$$H_{\rm AMO}=\hbar\omega l_3 \ ,$$

where $\hbar$ is used for purely dimensional reasons. The equation of motion for any variable V is given by $i\hbar\dot{V}=[V,H]$, which leads to

$$\dot{l}_1=-\omega l_2, \quad \dot{l}_2=\omega l_1, \quad \dot{l}_3=0 \ .$$

From these, one obtains

$$\ddot{l}_i=-\omega^2 l_i, \quad i=1,2 \ .$$

It is seen that both l_1 and l_2 oscillate with frequency ω. The AMO description is made unique by the designation of a total angular momentum

$$l_1^2+l_2^2+l_3^2=l_0(l_0+\lambda) \ , \tag{2.1}$$

where

$$\lambda=\begin{cases}0 & \text{classically}\\ 1 & \text{quantum mechanically} \ .\end{cases}$$

Classically, l_0 is arbitrary, while quantum mechanically, l_0, the total-angular-momentum quantum number, may assume only integral and half-integral values. The TLS corresponds to $l_0=\frac{1}{2}$. The range of l_3 is given by $-l_0\le l_3\le l_0$. From Eq. (2.1), we have

$$l_3=\pm l_0[1+\lambda l_0^{-1}-l_0^{-2}(l_1^2+l_2^2)]^{1/2} \ .$$

It is reasonable to expect that the similarity between the classical AMO and the quantum-mechanical AMO will be greatest for large l_0. We, therefore, consider at first $l_0\gg 1$. We will also consider the driving power sufficiently low so that the energy of the AMO remains well below that for saturation (which is indicated by $\langle l_3\rangle=0$), and is thus negative. Introducing the notation

$$q=l_0^{-1/2}l_1, \quad p=-l_0^{-1/2}l_2, \quad \epsilon=l_0^{-1} \ ,$$

one can write

$$l_3=-l_0[1-\epsilon(q^2+p^2-\lambda)]^{1/2} \ .$$

An expansion of the square root in powers of ϵ leads to

$$E=l_3+l_0=\tfrac{1}{2}(q^2+p^2-\lambda)+\tfrac{1}{8}\epsilon(q^2+p^2-\lambda)^2+\cdots \ ,$$

where E is the excitation energy, that is, the energy above the ground state, in units of $\hbar\omega$. For the commutation relationship of q and p, one obtains

$$[q,p]=i[1-\tfrac{1}{2}\epsilon(q^2+p^2-\lambda)+\cdots] \ .$$

Since, for a HO of (dimensionless) coordinates q and p, we have $E=\frac{1}{2}(q^2+p^2-\lambda)$ and $[q,p]=i$, it is seen that for $\epsilon\to 0$, q and p become the (dimensionless) coordinate and momentum, respectively, of a HO. To complete the notational relationship between the AMO and the HO, we consider the raising and lowering operators for the AMO

$$l_\pm=L_\pm\exp(\pm i\omega t)=2^{-1/2}(l_1\pm il_2) \ ,$$

for which we have

$$[l_+,l_-]=[L_+,L_-]=l_3 \ .$$

$L_\pm$ are the time-reduced variables, and are constant for the free AMO, as is l_3. Setting

$$x=l_0^{-1/2}L_-, \quad x^\dagger=l_0^{-1/2}L_+ \ ,$$

we obtain, as expansions in powers of ϵ,

$$\begin{aligned} l_3&=l_0(-1+\epsilon x^\dagger x+\tfrac{1}{2}\epsilon^2 x^{\dagger 2}x^2+\cdots) \ ,\\ [x,x^\dagger]&=1-\epsilon x^\dagger x-\tfrac{1}{2}\epsilon^2 x^{\dagger 2}x^2+\cdots \ ,\\ E&=x^\dagger x+\tfrac{1}{2}\epsilon x^{\dagger 2}x^2+\cdots \ . \end{aligned} \tag{2.2}$$

It is clear that as ϵ vanishes, $x\exp(-i\omega t)$ and $x^\dagger\exp(i\omega t)$ become the annihilation and creation operators for the HO, and x and $x^\dagger$ are the corresponding time-reduced operators. The parameter ϵ may therefore be regarded as a measure of the nonlinearity of the AMO. As ϵ vanishes, the AMO becomes an HO. In the present analysis, we will consider only the case in which either ϵ or E is sufficiently small so that the expansion of l_3/l_0 can be terminated with the ϵ^2 term.

III. INTERACTION WITH THE FIELD

The AMO will be considered coupled to the electromagnetic field through an electric dipole moment $\mathbf{d}=\boldsymbol{\mu}l_1=\boldsymbol{\mu}_0 q$, where $\boldsymbol{\mu}_0=l_0^{1/2}\boldsymbol{\mu}$. For purposes of studying the RFS, we consider the electromagnetic field to consist of both the free-space radiation field and a prescribed (c-number) driving field near resonance. The free-space field can be described by an infinite set of standing-wave modes inside a large cube of dimension L, made denumerable by the requirement of periodic boundary conditions, with the Hamiltonian of the kth mode given by

$$H_k=\hbar\omega_k a_k^\dagger a_k \ ,$$

where ω_k is the mode frequency, and a_k and $a_k^\dagger$ are the photon annihilation and creation operators, respectively (or classically, complex variables), obeying the relationship $[a_k,a_k^\dagger]=1$. The electric field of the kth mode is specified by

$$\mathbf{E}_k=i(2\pi\hbar\omega_k)^{1/2}\mathbf{u}_k(\mathbf{r})(a_k-a_k^\dagger) \ ,$$

where $\mathbf{u}_k(\mathbf{r})$ describes the spatial dependence of the mode and is normalized over the volume L^3. The free-space field is given by $\mathbf{E}_{\rm rad}=\sum_k\mathbf{E}_k$. With the notation

$$a_k=A_k e^{-i\omega t} \ ,$$

the interaction between the AMO and the free-space field is described by the interaction Hamiltonian

$$H'_{\rm rad}=-\mathbf{E}_{\rm rad}\cdot\mathbf{d}$$
$$=-\frac{i}{2}\sum_k\gamma_k\hbar(A_k e^{-i\omega_k t}-A_k^\dagger e^{i\omega_k t})(x^\dagger e^{i\omega t}+xe^{-i\omega t}) , \tag{3.1}$$

where

$$\gamma_k=\boldsymbol{\mu}_0\cdot\mathbf{u}_k(\mathbf{r}_a)(4\pi\omega_k/\hbar)^{1/2} ,$$

$\mathbf{r}_a$ being the position of the AMO. Strictly speaking, the definition of $H_{\rm rad}$ and of the coupling constant γ_k is good only for coupling to modes of wavelength long compared to the dimensions of the AMO. For coupling to modes of shorter wavelength, the AMO cannot be considered to be a localized dipole, and the coupling becomes weaker. Without going into a discussion of the spatial properties of the AMO, we will assume that for $\omega_k\gg\omega$, γ_k^2 becomes sufficiently small so that integrals over ω_k, which will occur later (when summations over k are converted to integrals over ω_k), converge. The driving field can be specified by

$$\mathbf{E}_d=2\mathbf{E}_0\sin(\omega_0 t+\theta) ,$$

which yields an interaction Hamiltonian

$$H'_d=-i\hbar(\Omega e^{-i\omega_0 t}-\Omega^* e^{i\omega_0 t})(x^\dagger e^{i\omega t}+xe^{-i\omega t}) ,$$

where

$$\Omega=2^{-1/2}(\mathbf{E}_0\cdot\boldsymbol{\mu}_0/\hbar)\exp(-i\theta) .$$

($\mathbf{E}_0\cdot\boldsymbol{\mu}/\hbar$ is known as the Rabi frequency.) The total Hamiltonian is given by

$$H=H_{\rm AMO}+\sum_k H_k+H'_{\rm rad}+H'_d .$$

Since the present analysis will be based on differential and integral equations, initial conditions need to be prescribed. The coupling between the AMO and the free space field will be assumed to have begun sometime in the distant past, say, at $t=-t_0$, when the field was in its ground state. The field ground state will be denoted by the bra and ket vectors $\langle|$ and $|\rangle$, respectively. The coupling to the driving field may be assumed to begin at an arbitrary time. However, since our interest, for purposes of obtaining the RFS, lies in steady states only, and not in transient phenomena, we assume that coupling to the driving field also begins at $-t_0$. In the following, it will be necessary to refer to $A_k(-t_0)$. For notational simplicity, we set $A_k(-t_0)=A_k^{(0)}$, and let t_0 become infinite. It should be noted that in the absence of coupling to the AMO, $A_k(t)=A_k^{(0)}$. The zero-point oscillation of the kth mode is described by $A_k^{(0)}\exp(-i\omega_k t)$, and the superposition of the zero-point oscillations of all the modes constitutes what are known as vacuum fluctuations. Quantum mechanically, $\langle|A_k^{(0)\dagger}=A_k^{(0)}|\rangle=0$, and classically, $A_k^{(0)}=0$.

IV. EQUATIONS OF MOTION

Based on essentially the above Hamiltonian, equations of motion in which the unknowns are AMO variables only have been derived previously,[10] with the utilization of approximations that take note of the special character of the free-space field, and include part, but not all, of the rotating-wave approximation. In the present notation, these equations are

$$\dot{x}=(l_3/l_0)[\Omega e^{-i\Delta t}+F+(\beta-i\beta_2)x]-i\beta_3 x l_3/l_0 , \tag{4.1}$$

and the corresponding H.c. equation. Here, we have

$$\Delta=\omega_0-\omega ,$$
$$\beta=\tfrac{1}{4}\pi\rho(\omega)\gamma^2(\omega) ,$$
$$F=\tfrac{1}{2}\sum_k\gamma_k A_k^{(0)}e^{-i(\omega_k-\omega)t} ,$$
$$\beta_2=\tfrac{1}{4}\int_0^\infty d\omega'\rho(\omega')\gamma^2(\omega')\frac{P}{\omega'-\omega} ,$$
$$\beta_3=\tfrac{1}{4}\int_0^\infty d\omega'\rho(\omega')\gamma^2(\omega')\frac{1}{\omega'+\omega} ,$$

where $\gamma^2(\omega')$ is the value of γ_k^2 averaged over a small frequency range about ω', and $\rho(\omega')$ is the density of modes at ω'. There exists, also an equation for l_3, but it will not be needed in the present analysis, because l_3 will be expressed in terms of x and $x^\dagger$. These equations of motion can be read classically by considering x and l_3 to be c numbers, and setting $F=0$. So far, they are valid for all excitation energies, above saturation as well as below saturation.

It is instructive to examine the physical meaning of F. We note that $\Omega e^{-i\Delta t}$ and F occur as a sum in the equations of motion. Since $\Omega e^{-i\Delta t}$ describes the effect of the driving field on the AMO, one can regard F as the description of the effect of the vacuum fluctuations (VF) on the AMO. These are both fluctuations at a given time with respect to a quantum-mechanical ensemble of identical systems (quantum fluctuations), and fluctuations in time in any one system (since they are a superposition of all the zero-point oscillations). However, one must approach the concept of fluctuations in time with caution, since the VF (or zero-point oscillations) cannot do work. Quantum mechanically, $\langle|F^\dagger=F|\rangle=0$, and classically, $F=0$. Neither $F^\dagger|\rangle$ nor $\langle|F$ vanishes, and there will be need to consider expectation values of products of the form

$$\langle F(t_1)\tilde{F}(t_2),\ldots,\tilde{F}(t_{n-1})F^\dagger(t_n)\rangle ,$$

where $\tilde{F}$ stands for either F or $F^\dagger$. Calculation of these quantities, has been shown to be given, for present purposes, as follows:[10,11]

$$\langle F(t_1)F^\dagger(t_2)\rangle=2\lambda\beta\delta(t_1-t_2) ,$$
$$\langle\tilde{F}(t_1)\cdots\tilde{F}(t_n)\rangle=0,\quad n\ \text{odd} \tag{4.2}$$
$$\langle F(t_1)\cdots F^\dagger(t_n)\rangle=\sum_{a-f}\langle F(t_a)F^\dagger(t_b)\rangle\cdots$$
$$\times\langle F(t_c)F^\dagger(t_d)\rangle\cdots$$
$$\times\langle F(t_e)F^\dagger(t_f)\rangle,\quad n\ \text{even} \tag{4.3}$$

where the summation is taken over all possible combinations of different pairs of the indices $1,\ldots,n$, with the first index of each pair less than the second index, that is, with

the order in each pair the same as in the original product. It can also be shown that $[F(t_1),F^\dagger(t_2)]=2\beta\delta(t_1-t_2)$ holds as an operator relationship.

We return to the equations of motion for the AMO [Eqs. (4.1)] and limit our consideration to energies well below saturation. We can now substitute for l_3 its expansion in terms of x and $x^\dagger$ (up to order ϵ^2) from Eq. (2.2). Writing

$$l_3=l_0(-1+\epsilon E)\ ,$$

where, it is recalled, $E=x^\dagger x+\frac{1}{2}\epsilon x^{\dagger 2}x^2$, we obtain

$$\dot{x}=-\Omega e^{-i\Delta t}-F-[\beta-i(\beta_2+\beta_3)]x +\epsilon\{E[\Omega e^{-i\Delta t}+F+(\beta-i\beta_2)x]-i\beta_3 xE\}\ , \tag{4.4}$$

and the corresponding H.c equation. The constants β_2 and β_3 account for a radiative frequency shift which has both an energy-independent part and energy-dependent part. The energy-independent part can be absorbed into ω by setting $\omega'=\omega-(\beta_2+\beta_3)$ and dropping the prime. The differential equation can then be cast into the following integral equation (recalling that the initial time is taken at $-\infty$)

$$x=-\Omega\frac{e^{-i\Delta t}}{\beta-i\Delta}-\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)}F(t_1) +\epsilon\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)}\{E(t_1)[\Omega e^{-i\Delta t_1}+F(t_1) +(\beta-i\beta_2)x(t_1)] -i\beta_3 x(t_1)E(t_1)\}\ . \tag{4.5}$$

V. RESONANCE FLUORESCENCE SPECTRUM

The RFS can be described by the time average of the expectation value of the power radiated into the free-space field per unit frequency range. An expression for this quantity has been derived previously[12] and is given, for the range $|\omega'-\omega|\ll\omega$, which is the only range we consider, by

$$P(\omega')=\frac{\beta}{\pi}\hbar\omega\int_0^\infty d\tau\langle x^\dagger(t)x(t-\tau)\rangle_{\rm av}e^{-i(\omega'-\omega)\tau}+\text{c.c.}\ , \tag{5.1}$$

where the subscript "av" indicates a time average. This expression is valid both classically and quantum mechanically.

A. Classical derivation

The classical derivation of the RFS will be considered first. A solution of the integral equation, read classically, is given by

$$x_{\rm cl}=x_0e^{-i\Delta t}\ ,$$

where x_0 satisfies the algebraic equation

$$(\epsilon|x_0|^2+\tfrac{1}{2}\epsilon^2|x_0|^4)\{\Omega+x_0[\beta-i(\beta_2+\beta_3)]\} =(\beta-i\Delta)x_0+\Omega\ .$$

The solution of this algebraic equation can be expanded as a power series in ϵ. The zeroth order solution, that for the classical HO, is

$$x_0^{(0)}=-\frac{\Omega}{\beta-i\Delta}\ ,$$

and higher-order solutions will involve nonlinear terms in Ω. It is interesting to note that for $\Delta=\beta_2+\beta_3$ (which brings the driving field into resonance with the uncoupled AMO), an exact solution is given by

$$x_0=-\frac{\Omega}{\beta-i(\beta_2+\beta_3)}\ .$$

It is not of present interest to find higher-order solutions of the algebraic equation for general Δ. What *is* of interest (in view of the quantum-mechanical results to be derived later) is the fact that the RFS of both the linear and nonlinear oscillators, derived classically, is a δ function at the driving frequency, given, according to Eq. (5.1), by

$$P(\omega')=2\beta\hbar\omega|x_0|^2\delta(\omega'-\omega_0)\ .$$

B. Quantum-mechanical analysis: Linear oscillator

In the quantum-mechanical treatment, we consider first the HO, that is, the case $\epsilon=0$. From Eq. (4.5), we have

$$x=-\frac{\Omega e^{-i\Delta t}}{\beta-i\Delta}-\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)}F(t_1)\ . \tag{5.2}$$

Since $\langle|F^\dagger=F|\rangle=0$, this yields

$$\langle x\rangle=x_{\rm cl}\ ,$$

$$\langle x^\dagger(t)x(t-\tau)\rangle=\frac{|\Omega|^2e^{i\Delta\tau}}{\beta^2+\Delta^2}=x_{\rm cl}^\dagger(t)x_{\rm cl}(t-\tau)\ ,$$

and

$$P_{\rm HO}(\omega')=2\beta\hbar\omega\frac{|\Omega|^2}{\beta^2+\Delta^2}\delta(\omega'-\omega)\ ,$$

where the subscript "cl" stands for "classical." The RFS of the HO is, thus, the same quantum mechanically, as it is classically. However, this is not the end of the story. Let us examine the excitation energy of the HO, given by $E=x^\dagger x$ (in units of $\hbar\omega$). We have, of course,

$$\langle E\rangle=E_{\rm cl}=|\Omega|^2(\beta^2+\Delta^2)^{-1}\ , \tag{5.3}$$

as expected from the correlation function for x. However, for the (symmetrized) correlation function of E, we obtain

$$\tfrac{1}{2}\langle E(t)E(t-\tau)\rangle+\text{c.c.}=\frac{|\Omega|^4}{(\beta^2+\Delta^2)^2}+\frac{1}{2}\left[\frac{|\Omega|^2e^{i\Delta\tau}}{\beta^2+\Delta^2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t-\tau}dt_2\,e^{-\beta(2t-t_1-t_2-\tau)}\langle F(t_1)F^\dagger(t_2)\rangle+\text{c.c.}\right]$$
$$=\langle E\rangle^2+\lambda\frac{|\Omega|^2}{\beta^2+\Delta^2}e^{-\beta\tau}\cos\Delta\tau\ , \tag{5.4}$$

where use has been made of Eq. (4.2). Classically, the result is just what would be expected for a precisely defined constant energy. Quantum mechanically, however, the correlation function resembles that for an ensemble of energies, each one of which has not only the classical constant component, but also a superposition of oscillatory components of random phase, with frequencies distributed about Δ with a spread of 2β. It is natural to inquire how quantum theory produces such an oscillation of the energy. The mathematics indicates that the VF are responsible for this oscillation. The VF are, indeed, a random superposition of zero-point oscillations, but these are not ordinary oscillations, since, as mentioned earlier, they cannot do work, and energy cannot be extracted from them.

In order to obtain an insight into the oscillation of the HO energy, we look at the excitation-energy operator,

$$x^\dagger x = \frac{|\Omega|^2}{\beta^2+\Delta^2} + \frac{\Omega^* e^{i\Delta t}}{\beta+i\Delta}\int_{-\infty}^{t} dt_1 e^{-\beta(t-t_1)}F(t_1) + \frac{\Omega e^{-i\Delta t}}{\beta-i\Delta}\int_{-\infty}^{t} dt_1 e^{-\beta(t-t_1)}F^\dagger(t_1) + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 e^{-\beta(2t-t_1-t_2)}F^\dagger(t_1)F(t_2) .$$

Substituting for F, from Eq. (4.1), its definition in terms of the vacuum field, the cross terms can be written as

$$\frac{\frac{1}{2}\Omega^*}{(\beta^2+\Delta^2)^{1/2}}\sum_k \frac{\gamma_k A_k^{(0)} e^{i[\Delta-(\omega_k-\omega)]t-\varphi_k}}{[\beta^2+(\omega_k-\omega)^2]^{1/2}} + \text{H.c.},$$

where φ_k is the phase angle of $(\beta+i\Delta)[\beta-i(\omega_k-\omega)]$. The expectation value of these terms vanishes, of course, since $\langle A_k^{(0)}\rangle=0$, but we see here, in any one member of the ensemble over which the expectation value is an average, exactly what was conjectured on the basis of the correlation function: a superposition of oscillatory components with frequencies distributed about Δ according to a distribution of width 2β.

Let us replace the operator $A_k^{(0)}$ by a complex c number A_k^c, of random phase θ_k, that yields the same energy for each mode as the quantum-mechanical zero-point energy, and has the same expectation value as $A_k^{(0)}$. This constitutes a replacement of the quantum-mechanical zero-point field of each mode by a classical random field, which will be referred to as the "c field", with

$$A_k^c = 2^{-1/2} e^{-i\theta_k} ,$$

where θ_k has a flat probability distribution. Denoting by the subscript c the expression for any operator in which $A_k^{(0)}$ is replaced by A_k^c, we have

$$\langle F_c\rangle = \langle A_k^c\rangle = 0 ,$$

$$\langle A_k^c A_k^{c*}\rangle = \langle A_k^{c*} A_k^c\rangle = \tfrac{1}{2} ,$$

as compared to

$$\langle A_k^{(0)} A_k^{\dagger(0)}\rangle = 1, \quad \langle A_k^{\dagger(0)} A_k^{(0)}\rangle = 0 ,$$

and

$$\langle F_c^*(t_1)F_c(t_2)\rangle = \langle F_c(t_2)F_c^*(t_1)\rangle = \beta\delta(t_1-t_2)$$

as compared to

$$\langle F^\dagger(t_1)F(t_2)\rangle = 0, \quad \langle F(t_2)F^\dagger(t_1)\rangle = 2\beta\delta(t_1-t_2) .$$

The scheme of Eq. (4.3), which is characteristic of a Gaussian random process, remains unaltered (except that the ordering in each pair is of no consequence). A calculation then yields

$$\langle E_c\rangle = \frac{|\Omega|^2}{\beta^2+\Delta^2} + \frac{1}{2} ,$$

$$\langle E_c(t)E_c(t-\tau)\rangle = \langle E_c\rangle^2 + \frac{|\Omega|^2}{\beta^2+\Delta^2} e^{-\beta\tau}\cos(\Delta\tau) + \tfrac{1}{4} e^{-2\beta\tau} . \quad (5.5)$$

The damped oscillatory term in this (classical) correlation function is identical to that in the quantum-mechanical correlation function. As seen more explicitly in the operator expression, the energy oscillation is the result of the beating of the driven oscillation at ω_0 with the oscillation produced by the c field (or, formally, by the VF) that is centered around ω with a width of 2β. The existence of additional terms in Eq. (5.5), in comparison with the quantum-mechanical results of Eqs. (5.3) and (5.4), is accounted for by the fact that the c field can do work while the VF cannot. Thus the c field imparts an average energy of $\frac{1}{2}\hbar\omega$ to the HO, indicated by the additional term in the expression for $\langle E_c\rangle$, with a dispersion indicated by the last term in the expression for $\langle E_c(t)E_c(t-\tau)\rangle$. (This energy is just the quantum-mechanical zero-point energy of the HO. Although this fact may, perhaps, appear coincidental at first glance, it is explained by the observation that the c field causes the HO to come into "thermal" equilibrium with the free-space field.) These results indicate that, as an intuitive tool, one may think of the VF as real (c-field-type) fluctuations, but only with respect to those effects that do not involve, directly or indirectly, work by the VF. Thus the VF can produce a modulation of the HO energy, but no increase in the average energy. More particularly, the VF can produce amplitude oscillations in the HO [represented by the second term in Eq. (5.2)], with the same frequency distribution as that produced by the c field, which can mix with other frequencies to produce beats, but cannot do work on their own, or increase the expectation value of the energy. These amplitude oscillations may be regarded as the zero-point oscillations of the damped HO.

C. Quantum-mechanical analysis: Nonlinear oscillator

We proceed to the quantum-mechanical treatment of the nonlinear oscillator, that is, the case $\epsilon\neq 0$. Since the RFS is derived from the correlation function $\langle x^\dagger(t)x(t-\tau)\rangle$, we will utilize expressions for the vectors $\langle|x^\dagger$ and $x|\rangle$, for which the equations are simpler than those for the corresponding operators. An integral equation for $x|\rangle$ is obtained immediately from the operator equation (4.5) by dropping the terms containing F explicitly, since $F|\rangle=0$. Now, the effect of the constants β_2 and β_3 in Eq. (4.5) is the introduction of an energy-dependent shift in the resonant frequency. Since it is not

unreasonable to assume that this shift is small, and will not affect significantly the main features of the RFS, we will ignore the shift, for the sake of simplicity, and approximate by dropping the constants β_2 and β_3. The equation for $x|\rangle$ now reads

$$x|\rangle=-\frac{\Omega e^{-i\Delta t}}{\beta-i\Delta}\Big|\Big\rangle +\epsilon\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)} \times[x^\dagger(t_1)x(t_1)+\tfrac{1}{2}\epsilon x^{\dagger 2}(t_1)x^2(t_1)] \times[\Omega e^{-i\Delta t_1}+\beta x(t_1)]|\rangle \ . \tag{5.6}$$

Formally, this is the same equation as the classical equation for x. One sees immediately that, for $\Delta=0$, the solution is

$$x|\rangle=-(\Omega/\beta)|\rangle \ .$$

This result—and therefore the spectrum—is the same as that for the classical AMO (with $\beta_1=\beta_2=0$), and that for the HO. Inspection of Eq. (5.6) shows that for $\Delta=0$ the nonlinear effect of the driving field is cancelled by the nonlinear effect of the dissipation, so that $x|\rangle$ (and the spectrum) become independent of the nonlinearity. Such independence does not exist for nonvanishing Δ, in which case the solution is much more complicated.

It is apparent from either Eq. (4.4) or (4.5) that for sufficiently small excitation energy the nonlinear term becomes negligible, and the AMO behaves like an HO, in accordance with the previously mentioned principle that nonlinear oscillators behave linearly for sufficiently small excitation. In order to investigate the effect of the nonlinearity on the spectrum (for general Δ), we investigate the transformation of the spectrum as the driving power increases from that in the linear regime. We will use perturbation theory with not only ϵ, but also with the quantity $|\Omega|(\beta^2+\Delta^2)^{-1/2}$, which is the value of $|\langle x\rangle|$ in the linear regime, as perturbation parameters. The perturbation-theory order will be indicated by a parenthetical superscript with two indices, the first referring to the power of $|\Omega|(\beta^2+\Delta^2)^{-1/2}$ and the second to the power of ϵ. Thus $x^{(n,m)}$ will indicate a quantity proportional to $\Omega^r\Omega^{*s}\epsilon^m$, where $r+s=n$. The various orders of the RFS will be given by the expression

$$P^{(n,m)}(\omega')=\frac{\beta}{\pi}\hbar\omega\int_0^\infty d\tau\langle x^\dagger(t)x(t-\tau)\rangle_{\rm av}^{(n,m)} \times e^{-i(\omega'-\omega)\tau}+\text{c.c.}$$

It is obvious that n must be even, since the power emitted into the field cannot depend on the phase of Ω. [It can also be shown that the time average of $\langle x^\dagger(t)x(t-\tau)\rangle^{(n,m)}$ vanished for n odd.]

In the calculation of the function $\langle x^\dagger(t)x(t-\tau)\rangle^{(n,m)}$, there will be need of both the operators x and the vectors $|x\rangle$ of various orders. By inspection, we have, from Eq. (4.5),

$$\begin{aligned} x^{(0,0)}&=-\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)}F(t_1) \ , \\ x^{(1,0)}&=-\frac{\Omega e^{-i\Delta t}}{\beta-i\Delta} \ , \\ x^{(n,0)}&=x^{\dagger(n,0)}=0 \quad \text{for } n>1 \ , \end{aligned} \tag{5.7}$$

the last equation being a statement of the fact that for a linear oscillator, x must be linear in the driving field. We also obtain easily

$$\begin{aligned} &\langle|x^{\dagger(0,m)}=x^{(0,m)}|\rangle=0 \quad \text{for all } m \ , \\ &\langle|x^{\dagger(n,1)}=x^{(n,1)}|\rangle=0 \quad \text{for } n>3 \ , \\ &\langle|x^{\dagger(1,m)}=x^{(1,m)}|\rangle=0 \quad \text{for } m>0 \ . \end{aligned} \tag{5.8}$$

The lowest-order spectrum is $P^{(2,0)}$, which is, of course, the same as the HO spectrum:

$$P^{(2,0)}(\omega')=P_{\rm HO}(\omega')=\frac{2\beta\hbar\omega|\Omega|^2}{\beta^2+\Delta^2}\delta(\omega'-\omega_0) \ . \tag{5.9}$$

The next higher-order spectrum in ϵ is $P^{(2,1)}$, which is determined by the correlation function $\langle x^\dagger(t)x(t-\tau)^{(2,1)}$. Introducing the shorthand notation $(n,m)^\dagger(r,s)$ for $\langle x^{\dagger(n,m)}(t)x^{(r,s)}(t-\tau)\rangle$, we have

$$\langle x^\dagger(t)x(t-\tau)\rangle^{(2,1)}=\sum_{n=0}^{2}\sum_{m=0}^{1}(n,m)^\dagger(2-n,1-m) \ .$$

It follows from Eqs. (5.7) and (5.8) that this function vanishes. In fact, inspection shows that $\langle x^\dagger(t)x(t-\tau)\rangle^{(2,m)}$ vanishes for all $m>0$. This result is, again, an illustration of the fact that in lowest order of the driving power the AMO behaves like an HO. It also follows that $P^{(n,0)}$ vanishes for $n>2$, which is a statement that the HO spectrum is linear in the driving power. Proceeding further, and noting that the only nonvanishing terms in $\langle x^\dagger(t)x(t-\tau)\rangle^{(4,1)}$ are $(3,1)^\dagger(1,0)$ and $(1,0)^\dagger(3,1)$, we obtain

$$x^{(3,1)}|\rangle=-\frac{i\Delta\epsilon\Omega|\Omega|^2e^{-i\Delta t}}{(\beta^2+\Delta^2)(\beta-i\Delta)^2} \ , \tag{5.10}$$

$$\begin{aligned} \langle x^\dagger(t)x(t-\tau)\rangle^{(4,1)}&=-\frac{2\epsilon|\Omega|^4\Delta^2e^{i\Delta\tau}}{(\beta^2+\Delta^2)^3} \\ &=-\frac{2\epsilon|\Omega|^2\Delta^2}{(\beta^2+\Delta^2)^2}\langle x^\dagger(t)x(t-\tau)\rangle^{(2,0)} \ , \end{aligned}$$

which yields

$$P^{(4,1)}(\omega')=-\frac{2\epsilon|\Omega|^2\Delta^2}{(\beta^2+\Delta^2)^2}P^{(2,0)}(\omega') \ .$$

This is a higher-order correction to the lowest-order δ-function spectrum at the driving frequency, and is due to the nonlinearity. This result is the same both classically and quantum mechanically.

We consider, next, $P^{(4,2)}$. The (4,2) correlation function is given by

$$\langle x^\dagger(t)x(t-\tau)\rangle^{(4,2)}=\sum_{n=0}^{4}\sum_{m=0}^{2}(n,m)^\dagger(4-n,2-m) \ .$$

Of the 15 terms on the right side, only three do not vanish, and we obtain

$$\langle x^{\dagger}(t)x(t-\tau)\rangle^{(4,2)}=(3,2)^{\dagger}(1,0)+(2,1)^{\dagger}(2,1)+(1,0)^{\dagger}(3,2)\ .$$

For $\langle |x^{\dagger(2,1)}$, a simple calculation yields

$$\langle |x^{\dagger(2,1)}=\left\langle \left| \frac{\Omega^{*2}\epsilon}{2(\beta+i\Delta)^2}\int_{-\infty}^{t} dt_1 e^{-\beta(t-t_1)}F(t_1)\times(e^{2i\Delta t}-e^{2i\Delta t_1}) \right.\right. , \tag{5.11}$$

and for $\langle |x^{\dagger(3,2)}$, a lengthy calculation yields

$$\langle |x^{\dagger(3,2)}=\frac{\Omega^{*3}\epsilon^2}{(\beta+i\Delta)^3}\left\langle \left| \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 F(t_1)F(t_2) \times \left[e^{-\beta(2t-t_1-t_2)}\left[-\frac{\beta e^{3i\Delta t}}{\beta-i\Delta}+\tfrac{1}{2}e^{i\Delta(t+2t_1)}+\frac{1}{2}e^{i\Delta(t+2t_2)}\right]\right.\right.\right.$$
$$\left.\left.\left. +\tfrac{1}{2}e^{-\beta(t-t_2)}\left[\frac{\beta+i\Delta}{\beta-i\Delta}e^{i\Delta(2t+t_1)}-e^{i\Delta(t_1+2t_2)}\right]\right]+\lambda\frac{i\Delta\epsilon^2\beta|\Omega|^2\Omega^{*}e^{i\Delta t}}{2(\beta^2+\Delta^2)(\beta+i\Delta)^3}\right\langle \right| . \tag{5.12}$$

In the derivation of the last term of $\langle |x^{\dagger(3,2)}$, utilization was made of the equality

$$\langle |F(t_1)F^{\dagger}(t_2)=2\lambda\beta\delta(t_1-t_2)\langle | \ ,$$

which follows from Eqs. (4.2) and (4.3). Only this last term contributes to $(3,2)^{\dagger}(1,0)$ and $(1,0)^{\dagger}(3,2)$; the results obtained are

$$\langle x^{\dagger}(t)x(t-\tau)\rangle^{(4,2)}=\frac{\lambda\epsilon^2|\Omega|^4\Delta^2}{(\beta^2+\Delta^2)^3}\left\{\frac{1}{4}e^{-\beta\tau}\left[1+\frac{i\beta}{\Delta}+\left[1-\frac{i\beta}{\Delta}\right]e^{2i\Delta\tau}\right]+\frac{2\beta^2}{(\beta^2+\Delta^2)}e^{i\Delta\tau}\right\} , \tag{5.13}$$

$$P^{(4,2)}(\omega')=\frac{2\beta^2}{\pi}\hbar\omega\frac{\lambda\epsilon^2|\Omega|^4\Delta^2}{(\beta^2+\Delta^2)^2}\left[\frac{1}{[\beta^2+(\omega'-\omega)^2][\beta^2+(\omega'-\omega-2\Delta)^2]}+\frac{2\pi\beta}{(\beta^2+\Delta^2)^2}\delta(\omega'-\omega_0)\right] .$$

It is seen that $P^{(4,2)}(\omega')$ is due entirely to *both* the VF *and* the nonlinearity of the oscillator. The second term in the curly brackets is merely a higher-order correction to the δ function at ω_0, but the first term exhibits new spectral elements, two symmetrical sidebands at $\omega_0\pm\Delta$, of width 2β. How can the presence of these new elements be understood intuitively? Any such understanding must be based, of course, on the combined effect of the VF and the nonlinearity. Here, the energy oscillation produced by the VF in the linear regime becomes significant. Due to the nonlinearity, the forced oscillation produced by the driving field is energy dependent. As was shown previously, the energy oscillates, due to the VF, with a range of frequencies centered at Δ and having a width 2β. This energy oscillation *modulates* the radiation of the oscillator, producing (as modulation does classically) symmetrical sidebands of width 2β at $\omega_0\pm\Delta$. In fact, these sidebands may be regarded as observable evidence of the modulation, and thus, of the energy oscillation.

As was done in the case of the energy oscillation, we investigate mathematically the above intuitive explanation by replacing the VF with the c field. We can expect, of course, extraneous terms associated with work done by the c field, and these should be discarded. In order to have a somewhat formal principle for discarding terms, we derive an expression for the average rate of work dW/dt done by the c field. Considering the c field as prescribed, one obtains [from Eq. (3.1)],

$$\frac{dW}{dt}=\langle \mathbf{E}_c\cdot\dot{\mathbf{d}}\rangle = \frac{i}{2}\sum_k \gamma_k\hbar\left\langle (A_k^c e^{-i\omega_k t}-A_k^{c*}e^{i\omega_k t})\times\frac{d}{dt}(x_c^{*}e^{i\omega t}+x_c e^{-i\omega t})\right\rangle .$$

Neglecting the time variation of x_c compared to that of $\exp(i\omega t)$, and using the rotating-wave approximation, we obtain [noting the defintion of F in Eq. (4.1)],

$$\frac{dW}{dt}=-\hbar\omega\langle F_c(t)x_c^{*}(t)+F_c^{*}(t)x_c(t)\rangle .$$

In lowest order we have

$$\frac{dW^{(0,0)}}{dt}=-\hbar\omega\langle F_c(t)x^{*(0,0)}(t)\rangle+\text{c.c.} =\hbar\omega\int_{-\infty}^{t}dt_1 e^{-\beta(t-t_1)}\langle F_c(t)F_c^{*}(t_1)\rangle+\text{c.c.} =\hbar\omega\beta .$$

If $x_c^{(n,m)}$ has a term $\tilde{x}_c^{(n,m)}$ of the form

$$\tilde{x}_c^{(n,m)}(t)=\int_{-\infty}^{t}dt_1 f(t,t_1)F_c(t_1) ,$$

where $f(t,t_1)$ is a function that does not contain either F_c or F_c^{*}, then the rate of work done by the c field due to this term is

$$\frac{d\tilde{W}^{(n,m)}}{dt}=-\hbar\omega\int_{-\infty}^{t}dt_1[\langle F_c(t)F_c^*(t_1)\rangle f^*(t,t_1)+\langle F_c^*(t)F_c(t_1)\rangle f(t,t_1)]$$
$$=-\hbar\omega\beta\int_{-\infty}^{t}dt_1\delta(t-t_1)[f(t,t_1)+f^*(t,t_1)]=-\tfrac{1}{2}\hbar\omega\beta[f(t,t)+f^*(t,t)]\ .$$

Thus, if $f(t,t)$ has a real part, then terms in the correlation function in which $\tilde{x}_c^{(n,m)}$ or $\tilde{x}_c^{*(n,m)}$ appear are due to higher-order corrections to work done by the c field, and should be discarded.

The correlation function responsible for the sidebands in $P^{(4,2)}$ is $\langle x^{\dagger(2,1)}(t)x^{(2,1)}(t-\tau)\rangle$; the corresponding c function should therefore be examined. From Eq. (4.5) (with $\beta_2=\beta_3=0$) we have

$$x_c^{*(2,1)}=\epsilon\int_{-\infty}^{t}dt_1e^{-\beta(t-t_1)}\{[x_c^*(t_1)x_c(t_1)]^{(1,0)}[\Omega^*e^{i\Delta t_1}+\beta x_c^{*(1,0)}(t_1)]+[x_c^*(t_1)x_c(t_1)]^{(2,0)}[F_c^*(t_1)+\beta x_c^{*(0,0)}(t_1)]\}\ .$$

In accordance with the above "discarding principle," the only term that should *not* be discarded in $x_c^{*(2,1)}$ is

$$\epsilon\int_{-\infty}^{t}dt_1e^{-\beta(t-t_1)}x_c^{*(1,0)}(t_1)x_c^{(0,0)}(t_1)\times[\Omega^*e^{i\Delta t_1}+\beta x_c^{*(1,0)}(t_1)]\ ,$$

which displays explicitly the effect of the oscillation of the energy, and yields

$$x_c^{*(2,1)}(t)=\frac{\epsilon\Omega^{*2}}{2(\beta+i\Delta)^2}\int_{-\infty}^{t}dt_1e^{-\beta(t-t_1)}F_c(t_1)\times(e^{2i\Delta t}-e^{2i\Delta t_1})\ .$$

This is, formally, exactly the same expression as that of $\langle|x^{\dagger(2,1)}(t)$ of Eq. (5.11), with F_c substituted for F. Thus the correlation function $\langle x_c^{*(2,1)}(t)x_c^{(2,1)}(t-\tau)\rangle$ will lead to two symmetrical sidebands similar to those derived quantum mechanically, except that their magnitude will be half as large, since $\langle F_c(t_1)F_c^*(t_2)\rangle=\frac{1}{2}\langle F(t_1)F^\dagger(t_2)\rangle$. (It should be borne in mind that the c field picture is not offered for purposes of exact calculation, but rather as a qualitative intuitive tool.)

The results obtained so far lead to the question whether higher orders of the spectrum will exhibit new spectral features—that is, additional peaks—or merely higher-order corrections to the spectral features already found. In order to obtain an idea of what may be expected in higher order, we look at Eq. (4.5) (with $\beta_2=\beta_3=0$) in terms of the c-field picture. We consider, again, the modulation of the driven oscillation by the excitation energy, and—now, that we are dealing with higher orders—also the modulation of the sideband oscillations. The $x^\dagger x$ term in E will mix the sideband oscillations with those due to the c field to produce bands of energy oscillation, of width 4β, centered at 0 and at 2Δ. The $x^{\dagger 2}x^2$ term will square the lowest-order energy oscillations to produce bands of width 4β also centered at 0 and 2Δ. These higher-order energy oscillations will modulate the driven oscillation of the AMO to produce spectral bands at ω_0 and $\omega_0\pm2\Delta$ of width 4β. Furthermore, the lowest-order energy oscillations will modulate the sideband oscillations and also produce spectral bands at ω_0 and $\omega_0\pm2\Delta$ of width 4β. All of these processes will interfere constructively or destructively, and some of the above results may be extraneous, involving work on the part of the c field. It is reasonable to expect, however, that higher-order spectra will display at least some of these new spectral features. We proceed to the investigation of orders up to (6,4). [The three processes described above are of order (6,4).]

A calculation shows that spectra of order (4,3), (4,4), (6,2), and (6,3) exhibit only higher-order corrections to $P^{(2,0)}$ and $P^{(4,2)}$. In the following discussion, we will ignore such corrections and concentrate only on new features, for which we look in $P^{(6,4)}$. The only nonvanishing terms in the correlation function of order (6,4) are given by

$$\langle x^\dagger(t)x(t-\tau)\rangle^{(6,4)}=(5,4)^\dagger(1,0)+(4,3)^\dagger(2,1)+(4,2)^\dagger(2,2)+(3,3)^\dagger(3,1)+(3,2)^\dagger(3,2)$$
$$+(3,1)^\dagger(3,3)+(2,2)^\dagger(4,2)+(2,1)^\dagger(4,3)+(1,0)^\dagger(5,4)\ .$$

Since the time variation of $x^{(1,0)}(t)|\rangle$ [Eq. (5.7)] is given by $e^{-i\Delta t}$, the terms $(5,4)^\dagger(1,0)$ and $(1,0)^\dagger(5,4)$ will yield only higher-order corrections to the δ function, and are of no present interest. The same argument applies to $x^{(3,1)}(t)|\rangle$ [Eq. (5.10)], and eliminates the terms $(3,3)^\dagger(3,1)$ and $(3,1)^\dagger(3,3)$ from consideration. While terms that produce a correction to the δ function are easy to identify, identification of those that produce a correction to $P^{(4,2)}$ requires some explanation.

The location of peaks in the spectrum is determined by the factor $e^{mi\Delta\tau}$ in the terms of the correlation function $\langle x^\dagger(t)x(t-\tau)\rangle$; a term containing this factor will produce a peak centered at $\omega+m\Delta$. This factor, in turn, is determined by the factors $e^{in\Delta t}$, $e^{in\Delta t_1}$, $e^{in\Delta t_2}$, etc. occurring in the expressions for $\langle|x^\dagger(t)$ and $x(t)|\rangle$, where t_1, t_2, etc., are variables of integration. The terms in $\langle|x^\dagger(t)$ that will be under investigation have—or can be reduced to—the form

$$\langle|\tilde{x}^\dagger=\left\langle\left|\int_{-\infty}^{t}dt_1F(t_1)f(t,t_1)e^{i\Delta(mt+m_1t_1)}\right.\right.\ ,$$

or

$$\langle|\tilde{x}^\dagger=\left\langle\left|\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2F(t_1)F(t_2)f(t,t_1,t_2)\times e^{i\Delta(mt+m_1t_1+m_2t_2)}\right.\right.\ ,$$

where f does not contain oscillatory functions, and the tilde here indicates an individual term in $\langle |x^{\dagger}(t)$. In order that

$$\langle \tilde{x}^{\dagger(r,s)}(t)\tilde{x}^{(p,q)}(t-\tau)\rangle$$

not vanish, $\langle |\tilde{x}^{\dagger(r,s)}$ and $\langle |\tilde{x}^{\dagger(p,q)}$ must have the same number of F factors. We consider first the case of one F factor. This yields

$$\begin{aligned}\langle \tilde{x}^{\dagger(r,s)}(t)\tilde{x}^{(p,q)}(t-\tau)\rangle &= 2\beta\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t-\tau} dt_2 \delta(t_1-t_2) f_1(t,t_1) f_2(t-\tau,t_2) \exp\{i\Delta[mt+m_1t_1-n(t-\tau)-n_2t_2]\} \\ &= 2\beta\int_{-\infty}^{t-\tau} dt_2 f_1(t,t_2) f_2(t-\tau,t_2) \exp\{i\Delta[(m-n)t+(m_1-n_2)t_2+n\tau]\} , \\ &= \varphi(t,\tau)\exp\{i\Delta[(m-n+m_1-n_2)t-(m_1-n_2-n)\tau]\} ,\end{aligned}$$

where φ does not contain oscillatory functions. Since the spectral distribution depends on the time average of the correlation function, we must have

$$m-n+m_1-n_2=0 ,$$

if the term under consideration is to make a contribution. The oscillation with respect to τ then becomes that of $e^{im\tau}$, and the term will contribute to the peak centered at $\omega+m\Delta$. A similar argument applies to the case of two F factors. Thus, merely by noting the exponent of $e^{im\Delta t}$ in the terms of $\langle |x^{\dagger(r,s)}(t)$, we can determine the location of the spectral peaks due to $\langle x^{\dagger(r,s)}(t)x^{(p,q)}(t-\tau)\rangle$. For example, $\langle |x^{\dagger(2,1)}(t)$ [Eq. (5.11)] has one term with $m=2$ and another with $m=0$. The function $\langle x^{\dagger(2,1)}(t)x^{(2,1)}(t-\tau)\rangle$ therefore yields one peak at $\omega+2\Delta$ and another at ω. This result enables one to select, by inspection, only those terms in $\langle |x^{\dagger(r,s)}$ which yield new spectral features.

For $\langle |x^{\dagger(2,2)}$ and $\langle |x^{\dagger(4,2)}$ we obtain, from Eq. (5.6),

$$\langle |x^{\dagger(2,2)} = \frac{1}{4}\frac{\Omega^{*2}\epsilon^2\beta}{(\beta+i\Delta)^3}\left\langle \left| \int_{-\infty}^{t} dt_1 F(t_1) e^{-\beta(t-t_1)} \times (e^{2i\Delta t}-e^{2i\Delta t_1}) ,\right.\right.$$

and

$$\langle |x^{\dagger(4,2)} = -\frac{\Omega^{*2}|\Omega|^2\epsilon^2}{(\beta^2+\Delta^2)(\beta+i\Delta)^2} \times\left\langle \left| \int_{-\infty}^{t} dt_1 F(t_1) e^{-\beta(t-t_1)} \times[\tfrac{1}{2}+i\Delta(t-t_1)](e^{2i\Delta t}-e^{2i\Delta t_1}) .\right.\right.$$

According to the preceding argument, one sees that $(2,2)^{\dagger}(4,2)$ and $(4,2)^{\dagger}(2,2)$ will contribute only higher-order correction for $P^{(4,2)}$. Thus, the only terms in $\langle x^{\dagger}(t)x(t-\tau)\rangle^{(6,4)}$ that may yield new spectral elements are $(4,3)^{\dagger}(2,1)$ and $(3,2)^{\dagger}(3,2)$. We express this statement symbolically as

$$\langle x^{\dagger}(t)x(t-\tau)\rangle^{(6,4)} \Longrightarrow (4,3)^{\dagger}(2,1)+(3,2)^{\dagger}(3,2) ,$$

where the symbol $\Longrightarrow$ indicates terms that may yield new spectral elements.

Equation (5.12) shows that the term $(3,2)^{\dagger}(3,2)$ will contribute toward peaks of finite width at $\omega+3\Delta$, $\omega+\Delta$, ω, and toward the δ function at ω. Although the c-field picture indicated the possibility of a peak at $\omega-\Delta$ ($=\omega_0-2\Delta$), a comparison of the details of the calculation leading to Eq. (5.12) with those of the c-field calculation (neither of which is given here for the sake of brevity) shows that the absence of this peak is due, formally, to the fact that $\langle |x^{\dagger(0,0)}=0$, which is related to the principle that the VF cannot do work. The calculation of $P^{(6,4)}$ can be simplified considerably by considering only the condition $\Delta^2 \gg \beta^2$. Since the width of the lower-order peaks is 2β and that of the higher-order peaks is expected to be 4β, this condition is useful also from an experimental point of view, as it allows a clear separation between peaks. In the following calculation of $P^{(6,4)}$, only the lowest powers of β/Δ necessary for the description of each peak will be retained.

If we ignore the monochromatic term in $\langle |x^{\dagger(3,2)}$ for purposes of the following argument, we can write

$$\langle |x^{\dagger(3,2)} = \frac{\Omega^{*3}\epsilon^2}{(\beta+i\Delta)^3}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 F(t_1)F(t_2) \times f(t,t_1,t_2) ,$$

where f is given by the term in large square brackets in Eq. (5.12). Using Eq. (4.3), we obtain

$$\langle x^{\dagger(3,2)}(t)x^{(3,2)}(t-\tau)\rangle = \frac{4|\Omega|^6\epsilon^4\beta^2\lambda}{(\beta^2+\Delta^2)^3}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t-\tau} dt_3 \int_{-\infty}^{t_3} dt_4 [\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\delta(t_2-t_3)] \times f(t,t_1,t_2)f^*(t-\tau,t_3,t_4) .$$

The second term in the square brackets makes no contribution to the integral, and the first term yields

$$\langle x^{\dagger(3,2)}(t)x^{(3,2)}(t-\tau)\rangle = \frac{4|\Omega|^6\epsilon^4\lambda\beta^2}{(\beta^2+\Delta^2)^3}\int_{-\infty}^{t-\tau} dt_1 \int_{-\infty}^{t_1} dt_2 f(t,t_1,t_2)f^*(t-\tau,t_1,t_2) .$$

Carrying out the integration indicated here, we obtain, to the lowest significant powers in β/Δ,

$$\langle x^{\dagger(3,2)}(t)x^{(3,2)}(t-\tau)\rangle \Longrightarrow \frac{|\Omega|^6\epsilon^4\lambda}{\Delta^6}\left[\frac{1}{4}e^{-2\beta\tau+i\Delta\tau}+e^{-2\beta\tau+3i\Delta\tau}\left[\frac{2}{3}\frac{i\beta^3}{\Delta^3}+\frac{4\beta^4}{\Delta^4}\right]\right] . \quad (5.14)$$

We consider, next, the term $\langle x^{\dagger(4,3)}(t)x^{(2,1)}(t-\tau)\rangle_{\rm av}$, where, from Eq. (5.11), $x^{(2,1)}(t-\tau)|\rangle$ is given by

$$x^{(2,1)}(t-\tau)|\rangle=\frac{\Omega^2\epsilon}{2(\beta-i\Delta)^2}\times\int_{-\infty}^{t-\tau}dt_1 e^{-\beta(t-\tau-t_1)}F^\dagger(t_1)\times(e^{-2i\Delta(t-\tau)}-e^{-2i\Delta t_1})|\rangle ,$$

The calculation of $\langle|x^{\dagger(4,3)}$ is quite complicated and its explicit expression contains many terms. However, only a few of these need be retained for present purposes. In order that a particular term in $\langle|x^{\dagger(4,3)}$ contribute toward

$$\langle x^{\dagger(4,3)}(t)x^{(2,1)}(t-\tau)\rangle_{\rm av} ,$$

it must be of the form

$$\int_{-\infty}^{t}dt_1 f(t,t_1)e^{ni\Delta t+n_1 i\Delta t_1}F(t_1) ,$$

where $f(t,t_1)$ contains no oscillatory functions and $n+n_1=2$. (The condition on the exponents is required to prevent the vanishing of the time average.) Terms with either $n=2$ or $n_1=2$ will yield higher-order corrections to $P^{(4,2)}$ and can be ignored. A number of terms will contain both F's and $F^\dagger$'s as factors in the integral. If the number of F's exceeds the number of $F^\dagger$'s in the entire product by 1, and if the number of $F^\dagger$'s on the left of any factor does not exceed the number of F's, then the product can be reduced to one or more terms which are multiples of single F's. For example, in accordance with Eqs. (4.2) and (4.3)

$$\langle|F(t_1)F^\dagger(t_2)F(t_3)=2\beta\delta(t_1-t_2)\langle|F(t_3) ,$$

$$\langle|F(t_1)F(t_2)F^\dagger(t_3)=2\beta[\delta(t_1-t_3)\langle|F(t_2)+\delta(t_2+t_3)\langle|F(t_1)] .$$

Terms with $n=n_1=1$ can also be ignored, although they contribute to a peak at ω_0. These contributions are of higher order in β/Δ than the lowest-order contributions from $\langle x^{\dagger(3,2)}(t)x^{(3,2)}(t-\tau)\rangle$ to the same peak, as follows from the fact that there exist more oscillatory factors in the integrands of the $n=n_1=1$ terms of $(4,3)^\dagger(2,1)$ than in the integrands of certain terms of $(3,2)^\dagger(3,2)$, where oscillatory factors cancel. We can therefore ignore all terms in $\langle|x^{\dagger(4,3)}$ contributing to peaks at $\omega_0+\Delta$ and ω_0. A lengthy calculation leads to the result, in the lowest powers of β/Δ,

$$\langle x^{\dagger(4,3)}(t)x^{(2,1)}(t-\tau)\rangle \Longrightarrow -\frac{\epsilon^4|\Omega|^6\lambda}{9\Delta^6}\left[\frac{2i\beta^3}{\Delta^3}+\frac{20}{3}\frac{\beta^4}{\Delta^4}\right]e^{-2\beta\tau+3i\Delta\tau} . \quad (5.15)$$

From Eqs. (5.14) and (5.15), we have,

$$\langle x^\dagger(t)x(t-\tau)\rangle^{(6,4)}\Longrightarrow\frac{|\Omega|^6\epsilon^4\lambda}{\Delta^6}\left[\frac{1}{4}e^{-2\beta\tau+i\Delta\tau}+\frac{1}{9}\left[4i\frac{\beta^3}{\Delta^3}+\frac{88}{3}\frac{\beta^4}{\Delta^4}\right]e^{-2\beta\tau+3i\Delta\tau}\right] ,$$

and from Eq. (5.1), we obtain

$$P^{(6,4)}(\omega')\Longrightarrow\frac{2\beta}{\pi}\hbar\omega\frac{|\Omega|^6\epsilon^4\lambda}{\Delta^6}\left[\frac{\frac{1}{2}\beta}{4\beta^2+[\omega'-(\omega+\Delta)]^2}+\frac{4}{9}\frac{15\beta(\beta^4/\Delta^4)+[\omega'-(\omega+3\Delta)](\beta^3/\Delta^3)}{4\beta^2+[\omega'-(\omega+3\Delta)]^2}\right] , \quad (5.16)$$

where $\frac{44}{3}$ has been approximated by 15 in the last expression. In sum, as illustrated schematically in Fig. 1, the total spectrum, up to the (6,4) order, consists of the following: a δ function at ω_0, the lowest order of which is $P^{(2,0)}$ [Eq. (5.9)]; two sidebands at $\omega_0\pm\Delta$, with lowest order given by $P^{(4,2)}$ [Eq. (5.13)], which are symmetrical in this order; and peaks at ω_0 and $\omega_0+2\Delta$, which appear in order (6,4) [Eq. (5.16)]. For $|\beta/\Delta|\ll 1$, the peak at $\omega_0+2\Delta$ is much smaller than that at ω_0.

VI. TWO-LEVEL SYSTEM

So far, we have considered an AMO with $l_0\gg 1$, or $\epsilon\ll 1$. It is instructive to compare the present results to those for a two-level system (TLS), that is, one with $l_0=\frac{1}{2}$, or $\epsilon=2$. The general equations of motion for the AMO, Eqs. (4.1), which hold for arbitrary l_0, can be simplified in the prsent case by the well known angular momentum relationships for $l_0=\frac{1}{2}$ (noting that $x=2^{1/2}L_-$),

$$l_3x=-xl_3=-\tfrac{1}{2}x ,$$

$$l_3x^\dagger=-x^\dagger l_3=\tfrac{1}{2}x^\dagger ,$$

$$l_3=-\tfrac{1}{2}+x^\dagger x ,$$

and become

$$\dot{x}=-\beta x+i(\beta_2-\beta_3)x-(1-2x^\dagger x)(\Omega e^{-i\Delta t}+F) , \quad (6.1)$$

together with the corresponding H.c. equation. (It should be noted that the notation for the TLS is quantum mechanical only; the dynamical variables refer to operators and the formalism is no longer valid classically.) The radiative frequency shift is now given by

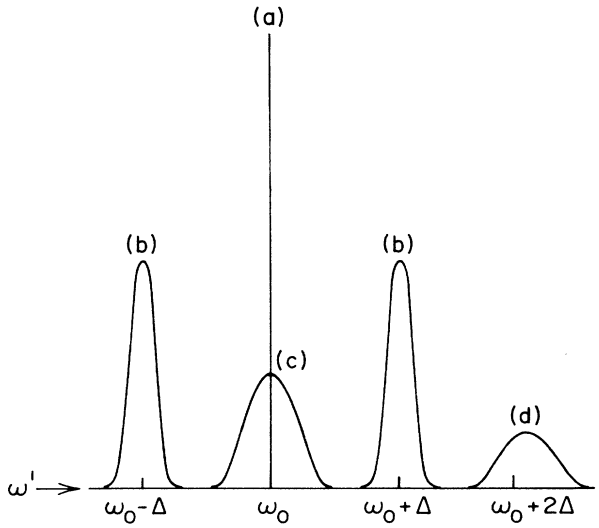

FIG. 1. Schematic illustration of the three lowest orders of the AMO resonance fluorescence spectrum (not to scale). For $l_0 \gg 1$, (a) indicates the (2,0)-order δ-function spectrum, (b) indicates the (4,2)-order spectrum, and (c) and (d) indicate the (6,4)-order spectrum. For the TLS, (a) indicates the second-order δ-function spectrum, (b) indicates the fourth-order spectrum, and (c) indicates the sixth-order spectrum.

$\omega' = \omega - (\beta_2 - \beta_3)$. Replacing ω by ω' (and dropping the prime), we obtain

$$\dot{x} = -\beta x - (1 - 2x^\dagger x)(\Omega e^{-i\Delta t} + F) .$$

If the numbers associated with the excitation energy $x^\dagger x$ are small compared to unity, it is seen that the TLS behaves like an HO. (It is interesting to note, however, that the radiative frequency shift is different from that of the HO, or of the energy-independent part of the AMO of $l_0 \gg 1$, and is independent of the excitation.) The integral form of the equations of motion, recalling that the initial conditions are to be taken at $t = -\infty$, is given by

$$x = -\frac{\Omega e^{-i\Delta t}}{\beta - i\Delta} + \int_{-\infty}^{t} dt_1 e^{-\beta(t-t_1)} \times \{ -F(t_1) + 2x^\dagger(t_1)x(t_1)[\Omega e^{-i\Delta t_1} + F(t_1)] \} ,$$

and the corresponding H.c equation.

In the case of the TLS, it is possible to obtain a closed-form solution for the correlation functions—and thus, the spectrum—from the equations of motion.[2-5] However, for the present purpose of comparison with the perturbation-theory solution for $l_0 \gg 1$, it is more instructive to look at the perturbation-theory solution for the TLS spectrum. Since ϵ is not small compared to unity in the case of the TLS, we use only a single small parameter $|\Omega|(\beta^2+\Delta^2)^{-1/2}$, with the order being indicated by a single parenthetical superscript. The equation for the vector $x|\rangle$ reads

$$x|\rangle = -\frac{\Omega e^{-i\Delta t}}{\beta - i\Delta} + 2\Omega \int_{-\infty}^{t} dt_1 e^{-\beta(t-t_1) - i\Delta t_1} x^\dagger(t_1)x(t_1)|\rangle ,$$

which yields, immediately, $x^{(0)}|\rangle = 0$, and

$$x^{(1)}|\rangle = -\frac{\Omega e^{-i\Delta t}}{\beta - i\Delta} .$$

In the lowest order, therefore, the spectrum is given by

$$P^{(2)}(\omega') = 2\beta\hbar\omega|\Omega|^2(\beta^2+\Delta^2)^{-1}\delta(\omega'-\omega_0) ,$$

which is the same as the HO spectrum obtained for arbitrary field strength. In next higher order, a calculation leads to

$$x^{(2)}|\rangle = \frac{i\Omega^2}{\Delta(\beta - i\Delta)} \int_{-\infty}^{t} dt_1 F^\dagger(t_1) e^{-\beta(t-t_1)} \times (e^{-2i\Delta t} - e^{-2i\Delta t_1})|\rangle ,$$

and yields

$$\langle x^{\dagger(2)}(t)x^{(2)}(t-\tau)\rangle = \frac{|\Omega|^4}{(\beta^2+\Delta^2)^2} e^{-\beta\tau} \times \left[\left[1 - \frac{i\beta}{\Delta} \right] e^{2i\Delta\tau} + 1 + \frac{i\beta}{\Delta} \right] .$$

The correlation functions $\langle x^{\dagger(3)}(t)x^{(1)}(t-\tau)\rangle$ and $\langle x^{\dagger(1)}(t)x^{(3)}(t-\tau)\rangle$ contribute toward the spectrum only higher-order corrections to the δ function, which we ignore, as was done eventually for the case $l_0 \gg 1$. We can therefore write

$$P^{(4)}(\omega') = \frac{\beta}{\pi}\hbar\omega \int_0^\infty d\tau \langle x^{\dagger(2)}(t)x^{(2)}(t-\tau)\rangle_{\text{av}} \times e^{-i(\omega'-\omega)\tau} + \text{c.c.}$$
$$= \frac{8\beta^2\hbar\omega|\Omega|^4}{\pi(\beta^2+\Delta^2)} \times \frac{1}{[\beta^2+(\omega'-\omega-2\Delta)^2][\beta^2+(\omega'-\omega)^2]} .$$

This result describes two symmetrical peaks at $\omega_0 \pm \Delta$. It is interesting to compare it with $P^{(4,2)}$ [Eq. (5.13)] for the case $l_0 \gg 1$. Inspection shows that if we set $\epsilon = 2$ in $P^{(4,2)}$, we have, with respect to the two symmetrical sidebands,

$$P^{(4)} = \frac{\beta^2+\Delta^2}{\Delta^2} P^{(4,2)}_{\epsilon=2} .$$

It is seen that, while $P^{(4,2)}$ vanishes for $\Delta \to 0$, $P^{(4)}$ does not vanish under the same conditions. The reason for this difference is the fact that, for the TLS, the relative decay rate of x is independent of the energy, in contrast to its energy dependence for $l_0 \gg 1$. [Equation (4.4) contains a nonlinear β term, which accounts for the vanishing of all higher-order spectra for $\Delta = 0$, while Eq. (6.1) contains no such term.] Thus, unlike the case $l_0 \gg 1$, the spectrum of the TLS broadens—from that of a δ function—with the increase of power on exact resonance. On the other hand, for $\Delta^2 \gg \beta^2$, the spectra for the two

cases become identical. This indicates that for $\Delta^2 \gg |\Omega|^2+\beta^2$, only the first two levels of any AMO participate significantly.

Lastly, we consider $P^{(6)}$. According to previous reasoning,

$$\langle x^{\dagger}(t)x(t-\tau)\rangle^{(6)} \Longrightarrow \langle x^{\dagger(4)}(t)x^{(2)}(t-\tau)\rangle + \langle x^{\dagger(3)}(t)x^{(3)}(t-\tau)\rangle \ .$$

A somewhat lengthy calculation yields only one new spectral element, a peak of finite width at ω_0. For $\beta^2 \ll \Delta^2$, which we considered also in deriving $P^{(6,4)}$, only $\langle x^{\dagger(3)}(t)x^{(3)}(t-\tau)\rangle$ is significant, and the result is

$$\langle x^{\dagger(3)}(t)x^{(3)}(t-\tau)\rangle \Longrightarrow \frac{4|\Omega|^6}{\Delta^6} e^{-2\beta\tau+i\Delta\tau} \ ,$$

$$P^{(6)} \Longrightarrow \frac{16\beta^2\hbar\omega|\Omega|^6}{\pi\Delta^6} \frac{1}{4\beta^2+(\omega'-\omega-\Delta)^2} \ .$$

It is seen that this is the same result as that for the peak at ω_0 in $P^{(6,4)}$ with $\epsilon=2$, which again indicates the participation of only the lowest two levels of an AMO for $\Delta^2 \gg |\Omega|^2+\beta^2$.

VII. GENERAL DISCUSSION

In the preceding analysis, the linear oscillator and only one class of nonlinear oscillators have been considered. However, a number of conclusions drawn from this analysis can be generalized to all types of nonlinear oscillators. We consider the classical results first. It was seen that the RFS is a δ function at the driving frequency for both the HO and the AMO (when the latter is driven below saturation). Now, it is entirely reasonable to expect that any nonlinear oscillator which can achieve an amplitude sufficiently large so that it radiates as much power (averaged over a cycle) as it absorbs from the driving field will oscillate with periodic motion; in other words, that it will achieve a steady state. The fundamental component of this forced oscillation accounts for the δ function in the RFS. The type of spectrum (in the neighborhood of resonance) is therefore the same, classically, for any oscillator, linear or nonlinear (provided the condition concerning saturation is satisfied by the latter).

We consider, next, the quantum-mechanical results. Here, the RFS of the linear oscillation is also a δ function at the driving frequency. There exists an essential difference between the quantum-mechanical and classical results, however, with respect to the excitation energy. While it is a well-defined constant classically, the energy is described, quantum mechanically, by a probability distribution with a rms deviation equal to its square root. Furthermore, the energy correlation function indicates the existence of an oscillation of part of the energy at what may be regarded as the beat frequency produced by the mixing of the driving field and the VF. Since the effect of the VF on the oscillator is centered around ω within a spread of 2β, the beat frequency is centered at $\Delta\ (=\omega_0-\omega)$ and has a spread of 2β; the phase is random, of course. Now, all nonlinear oscillators behave like linear oscillators in the lowest order of the driving-field power. In the next higher order the behavior of a nonlinear oscillator depends on its energy. Since part of the energy oscillates with frequencies centered at Δ and having a spread of 2β, it will *modulate* the radiated field and produce two symmetrical sidebands at $\omega_0\pm\Delta$ of width 2β in the RFS. These sidebands should therefore be a phenomenon associated with all nonlinear oscillators.

Carrying the above argument to further higher order, one may expect the lowest-order energy oscillation to modulate the higher-order radiation, and the higher-order energy oscillation to modulate the lowest-order radiation. Possible locations for peaks resulting from these processes are ω_0 and $\omega_0\pm2\Delta$, and the width may be expected to be 4β. There will be interference between the several processes, and, of course, those that are related to—or involve—work done by the VF cannot occur. In the case of the AMO with $l_0 \gg 1$, next-higher-order peaks appear at ω_0 and $\omega_0+2\Delta$. In the case of the TLS, a next-higher-order peak appears at ω_0 only. If one may generalize from these two examples, a higher-order peak will appear at least at ω_0. Further details depend on the type of nonlinearity of the oscillator.

[1]M. C. Newstein, Phys. Rev. **167**, 89 (1968).

[2]B. R. Mollow, Phys. Rev. **188**, 1969 (1969); Phys. Rev. A **5**, 1511 (1972); **12**, 1919 (1975).

[3]G. Oliver, E. Ressayre, and A. Tallet, Lett. Nuovo Cimento **2**, 777 (1971).

[4]C. Cohen-Tannoudji, in *Frontiers in Laser Spectrocopy,* 1975 Les Houches Lectures, edited by R. Balian, S. Haroche, and S. Liberman (North-Holland, Amsterdam, 1977).

[5]H. J. Kimble and L. Mandel, Phys. Rev. A **13**, 2193 (1976).

[6]H. J. Carmichael and D. F. Walls, J. Phys. B **9**, 1199 (1976).

[7]F. Schuda, C. R. Stroud, and M. Hercher, J. Phys. B **7**, L198 (1974).

[8]F. Y. Wu, R. E. Grove, and S. Ezekiel, Phys. Rev. Lett. **35**, 142 (1975).

[9]This principle is usually based on an examination of a Taylor series expansion of the potential energy of the oscillator about its point of equilibrium. Since the linear term must vanish, the lowest nonvanishing term is usually quadratic in the coordinate (it cannot be of odd power.) If the lowest nonvanishing term is of a higher even power, then the principle does not apply. It will be seen in Sec. II that the principle applies to the TLS, as a special case of the oscillator under consideration there.

[10]I. R. Senitzky, Phys. Rev. A **6**, 1175 (1972).

[11]I. R. Senitzky, Phys. Rev. Lett. **31**, 955 (1973), Eq. (9).

[12]I. R. Senitzky, Phys. Rev. Lett. **40**, 1334 (1978).