Off-lattice and hypercubic-lattice models for diffusion-limited aggregation in dimensionalities 2–8

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Improved algorithms have been developed to simulate both off-lattice and hypercubic-lattice diffusion-limited aggregation (DLA) in dimensionalities d in the range $2 \leq d \leq 8$. For the two-dimensional cases using off-lattice clusters containing up to 10^6 particles, we find that the effective fractal dimensionality is essentially independent of cluster size s for clusters containing more than a few thousand particles and has a value of 1.715 ± 0.004 (significantly higher than the mean-field value of $\frac{5}{3}$). For $d=3$, 4, and 5, it appears to be possible to approach quite close to the asymptotic ($s \rightarrow \infty$) regime and the effective fractal dimensionality for off-lattice clusters is equal to the mean-field value given by $D=(d^2+1)/(d+1)$ (within a few tenths of 1%). For $d \geq 5$ we were not able to approach near enough to the asymptotic limit to make an accurate estimate of the limiting fractal dimensionality. However, the simulation results are consistent with the mean-field theory predictions. For $d \leq 4$ the effects of lattice anisotropy can be seen in the overall shapes of the clusters and the dependence of the cluster radius of gyration on s. For $d \geq 5$ clusters containing 10^5 sites are still in the fluctuation-dominated regime and the dependence of R_g on s is essentially the same for both off-lattice and hypercubic-lattice models. For $d=2$ the effective value of $\bar{\nu}$, which describes how the width of the active zone grows, increases with increasing cluster size and approaches a value of 1/1.715.

INTRODUCTION

Since the introduction of the Witten-Sander[1] model for diffusion-limited aggregation (DLA), considerable interest has developed in a wide variety of nonequilibrium growth and aggregation models. Some of this work has been summarized in recent books,[2–4] conference proceedings,[5–10] and reviews.[11–17] The Witten-Sander DLA model is still of considerable interest because it provides a basis for understanding a broad range of phenomena including random dendritic growth,[18] dielectric breakdown,[19] electrodeposition,[20,21] fluid-fluid displacement in Hele-Shaw cells[22] and porous media,[23,24] thin-film morphology,[25] and the dissolution of porous materials.[26] Interest has also been sustained because the challenge of developing a theoretical understanding of this simple (to define) model has not yet been fully met.

Most of our knowledge concerning the structure of DLA clusters has come from computer simulations. Except for $d=2$, where a significant effort has gone into the development of efficient algorithms,[27–29] most of this knowledge is based on quite-small-scale simulations.[30,31] (However, for $d=3$ an effective fractal dimensionality quite close to 2.50 has been obtained from about a hundred 50 000-site clusters grown on a cubic lattice.[32]) For $d \geq 4$ only a few clusters containing up to 10 000 particles or sites have been generated.[30,31]

The availability of estimates for the fractal dimensionality[33] of DLA for $d=2$–6 (Refs. 1, 30, and 31) stimulated the development of several mean-field[34–36] and position-place renormalization-group[37] models. In particular, the mean-field theories of Muthukumar[34] and of Tokayama and Kawasaki[35] lead to the result

$$D(d)=(d^2+1)/(d+1) , \tag{1}$$

where $D(d)$ is the fractal dimensionality for an embedding space or lattice with a Euclidean dimensionality of d. In the rest of this paper we will describe this estimate for D as D_{MF} and refer to it as the mean-field value. Hentschel[36] obtained a different result from his mean-field theory:

$$D(d)=\begin{cases}(8+5d^2)/(6+5d) & (d \leq 5.1) & \text{(2a)}\\ (8-4d+d^2)/(d-2) & (d \geq 5.1) . & \text{(2b)}\end{cases}$$

Except for $d=2$ where both mean-field theories give results that seem to be well outside the range of large-scale simulation results for off-lattice DLA (or for lattice models in the regime where fluctuations dominate the effects of lattice anisotropy) which indicate that $D(2) \simeq 1.71 \pm 0.005$,[38] the simulation results are not accurate enough to unambiguously distinguish between Eqs. (1) and (2). However, the results obtained from Eq. (1) are in better agreement than those obtained from Eq. (2) for $d=3$.

In recent years a variety of new theoretical approaches to DLA have been developed[39–46] and a new mean-field theory[47] has been developed and extended to a variety of nonequilibrium growth processes more or less closely related to DLA.[48] Because of this increase in theoretical activity, we were motivated to improve the reliability of the simulation results so that a more meaningful comparison with new theoretical predictions could be made. Our objective was to reduce the statistical uncertainty for

$d = 3-6$ by an order of magnitude and, at the same time, reduce the uncertainty associated with finite-size effects by generating about 100 clusters each containing about 10^5 particles or sites using both hypercubic-lattice and off-lattice models. In addition, we wanted to obtain results for $d = 7$ and 8. We were not interested in generating record-size clusters since this would permit us to generate only a few clusters for each model. The numbers and sizes of the clusters were selected as a reasonable compromise between reducing uncertainties due to statistical and finite-size effects in view of the available computer resources (about 2500 h of IBM 3090 computer CPU time).

Computer models

In the Witten-Sander[1] model for diffusion-limited aggregation "particles" are added, one at a time, to a growing cluster or aggregate of particles (or lattice sites) via random-walk trajectories originating from "infinity." In all practical DLA algorithms the particles are actually launched from a randomly selected position on a hyperspherical surface which just encloses the cluster and are terminated if they either contact the cluster (in this event the cluster grows by adding the particle at that position) or move a large distance from the cluster. In the latter event, a new particle is launched from a randomly selected position on the enclosing hyperspherical surface. In our case the launching surface has a radius of $R_{\max} + \delta R$ and is centered on the original seed or growth site. Here $R_{\max}$ is the maximum radius of the cluster and δR has a value of 1.5–3.5 particle diameters or lattice units depending on the model (smaller values were used for δR for larger values of d). The noncontacting-particle trajectories were terminated at a distance of $kR_{\max}$ from the center of the growing clusters. k was given a value of 100 for the two-dimensional simulations, at least 10 for the three-dimensional simulations, and at least 4 for the four-dimensional simulations. Values for k greater than or equal to 2 were used for $d = 5$, 6, and 7. For $d = 8$, k was set to a value of $\frac{5}{3}$ for the lattice models and $\sqrt{3}$ for the off-lattice models. These values for k are quite conservative. Values of k much closer to 1 lead to no detectable distortions in the structures of the clusters. For example, for $d = 2$ values of k as low as 2 seem to give quite good results. In the most simple DLA algorithms[1] the particle trajectories are represented by random walks with a fixed step length of one lattice unit (or a length on the order of the particle radius in off-lattice simulations[30,31]). Using these algorithms, it is not practical to grow clusters containing more than a few thousand sites.[1] Somewhat larger clusters[30,31] can be grown by allowing the particles to take longer steps when they are outside of the region occupied by the cluster. The key to developing more efficient DLA algorithms is to also allow the random walkers to take longer jumps when they are in the empty region between the arms of the cluster. If the particle is at the center of an empty hypersphere, then it will first emerge from the hypersphere at a random position on its surface and the random-walk trajectory inside the hypersphere can be replaced by a single step to a randomly selected position on its surface.[49] Alternatively, if the particle is at the center of a hypercube, it could be randomly moved to a position on the hypercube with a probability given by the Laplacian Green's function.[27] We elected to use the first approach because of the difficulty of calculating the Green's function for d-dimensional hypercubic surfaces with d up to 8.

In order to allow the particle to take large jumps, an efficient way must be found to obtain a good underestimate of the radius of the largest unoccupied hypersphere centered on the current position of the particle. Meakin[28] developed an efficient DLA algorithm by using an underlying lattice to indicate the distance to the nearest particle or occupied site. Each time a new particle is added to the cluster, the information on the underlying lattice is updated in the region surrounding the added particle. Unfortunately, for most computers, this approach is restricted to clusters containing up to a few hundred thousand particles or sites for $d = 2$.[50] For $d = 3$ and for $d = 4$ the maximum cluster size accessible to this approach is less than 10^5. Since this approach is not practical for the growth of very large clusters, we adapted that developed by Ball and Brady.[27] In effect this approach is equivalent to constructing maps of the

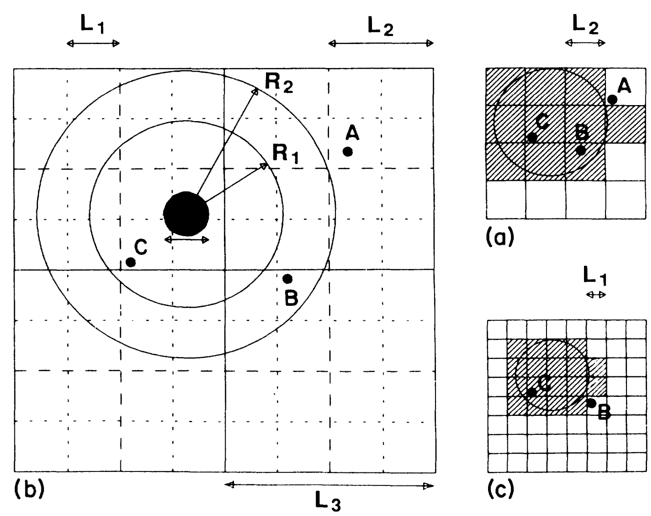


FIG. 1. Here a particle of diameter d_0 is shown in three levels on the hierarchy of maps used to locate the particles in the DLA cluster with respect to the mobile particles. This particle represents a particle that has been just added to the cluster. To indicate the presence of this particle all four elements in the highest-level map with sides of length L_3 will be "marked." (b) shows those elements in the next-lower-level map with sides of length L_2 ($L_3/2$) that will be marked and (c) shows those elements in the lowest-level map that will be marked. If only the particle shown in (a) was present, a random walker at position A would be allowed to move by a distance of L_2 and a random walker at position B would be allowed to move by a distance of L_1. The distance moved by a random walker at position C would be less than L_1 and this distance would be determined from the lower-level more detailed cluster maps.

cluster on different length scales. First the "map" on the coarsest scale is examined and if it indicates that a jump on the scale of this map can be taken, the jump is executed. If, on the other hand, the coarsest map indicates that the particle is near to the cluster, a more detailed map containing information about the location of the cluster in the vicinity of the particle is consulted. This process of consulting more and more detailed maps continues until a jump is executed for the lowest-level (most detailed) map is reached. At this level the map contains information about the exact location of occupied lattice sites or it contains the positions of these sites or particles in a list of coordinates. This provides the information needed to determine if the particle has contacted the cluster or if it can be moved by a small distance (by one lattice unit for the lattice models or by a distance on the order of one particle radius for the off-lattice models).

After a particle has been added to the cluster, the maps in each level of the hierarchy of maps must be updated. This is illustrated in Fig. 1 for the case of a two-dimensional off-lattice DLA model. This figure shows a particle in three levels in the nested hierarchy of maps. At the largest scale all four quadrants with sides of length L_3 will be "marked" (if they are not already marked) to indicate the presence of the new particle. At the next level [Fig. 1(b)] all of the map elements with sides of length L_2 which are intersected by or contained within a circle of radius d_0+L_2 centered on the center of the particle will be marked (here d_0 is the particle diameter). At the lowest level shown in this figure all of the map elements in the map with elements of size L_1 intersected by or contained within a circle of radius d_0+L_1 [Fig. 1(c)] will be marked. The lengths L_1, L_2, and L_3 are related by $L_3=2L_2=4L_1$. In order to conserve memory, only the marked parts of each map are stored.

In the case of the two-dimensional lattice models each element in the lowest-level map consists of a 5×5 block of sites represented by a bit map. In the off-lattice models each element of the lowest-level map is associated with a list of particle coordinates. After each move, the position of the particle is examined on successively lower-level maps (starting at the highest level) until it is found to occupy an unmarked part of the map on the nth level and is then moved by the distance L_n or the lowest level of the hierarchy is reached. The efficiency of the algorithms can be improved by going up only one level after each move has been made.

The details concerning the hierarchy of maps depends on the particular model and on the size of the clusters which are required. In the off-lattice models the lowest-level map elements may contain either the coordinates of all of the particles whose centers are in those elements or the coordinates of all of the particles that may be contacted by a particle whose center is in that element. In the former case particle lists associated with adjacent lowest-level map elements must be examined which increases the computer time requirements. However, more stored information is needed for the latter case. In the case of the off-lattice models care was taken to ensure that mobile particles were stopped and added to the cluster at the point along a jump at which they first contacted the cluster. Extensive tests were also carried out to verify that no accidental particle-particle overlaps occurred.

RESULTS

Two-dimensional DLA

Clusters containing more than 10^7 sites[51] have been obtained using the algorithm of Ball and Brady[27] and an investigation of the structure of large two-dimensional square lattice DLA clusters using 4×10^6-site clusters has already been published.[29] We were unable to improve significantly on this algorithm and consequently only results from the off-lattice model are reported here. Prior to the start of this work almost all of the investigations of two-dimensional (2D) off-lattice DLA were carried out using clusters containing 50 000 or fewer particles (more recently[50] a few 250 000–particle clusters have been generated using an algorithm described by Meakin[28]). Using the approach described above, a quite large number of 10^6-particle clusters was generated. After each 5% increment in the cluster mass, the radius of gyration R_g and the width of the active zone ξ were recorded (here ξ is the variance in the deposition radius).

The dependence of the cluster radius of gyration R_g on cluster size s can be described quite well by the power law

$$R_g \sim s^\beta \tag{3}$$

for clusters containing more than a few hundred particles. The effective values of the exponent β and the corresponding dimensionality D_β were obtained by least-squares fitting straight lines to the dependence of $\ln(R_g)$ on $\ln(s)$ over 10 growth increments [i.e., for clusters of sizes s_1, $1.05s_1 \cdots s_2=(1.05)^9 s_1$]. Figure 2 shows the dependence of β on $s[(s_1s_2)^{1/2}]$ obtained from 377 10^6-particle clusters. It is apparent from these results that there is no tendency for the fractal dimensionality D_β to change from the value of about 1.71 obtained from much smaller scale simulations. For clusters in the size range 10^4-10^6 the result $\beta=0.5830\pm0.0014$ ($D_\beta=1.715\pm0.004$) was obtained and for clusters in the size range 10^5-10^6 the result $\beta=0.5832\pm0.0014$ ($D_\beta=1.715\pm0.004$) was obtained. The statistical uncer-

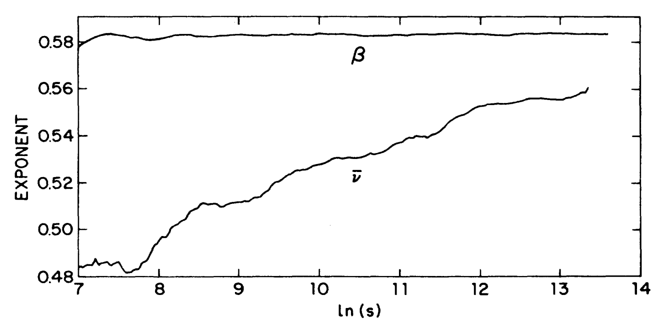


FIG. 2. Dependence of the effective values of the exponents β and ν on the cluster size s obtained from 10^6-particle two-dimensional off-lattice DLA clusters.

tainty ranges given here are 95% confidence limits based on the variance in the values of β obtained from individual clusters. A much smaller (but completely unrealistic) estimate of the statistical uncertainties is obtained from the standard error from least-squares fitting a straight line to the coordinates $(\ln\langle R_g\rangle), \ln(s)$, where $\langle R_g\rangle$ is the mean radius of gyration for clusters of size s. These results are in excellent agreement with those obtained earlier ($\beta \simeq 0.584$) from 1000 50 000–particle off-lattice clusters.[38] Figure 2 also shows the dependence of the exponent $\bar{\nu}$ on $\ln(s)$ obtained in a similar fashion. Here $\bar{\nu}$ is the exponent which describes the dependence width of the active zone of ξ on s:

$$\xi \sim s^{\bar{\nu}} . \tag{4}$$

Here ξ is the variance in the deposition radius for clusters of size s. Plischke and Racz[52] measured the exponent $\bar{\nu}$ using quite-small-scale square-lattice DLA clusters and obtained an effective value of 0.48 ± 0.01. Meakin and Sander[38] then carried out two-dimensional off-lattice simulations and found that $\bar{\nu}$ increased from a value of about 0.48 for small clusters to about 0.54 for clusters containing 25 000–50 000 particles. The results shown in Fig. 2 indicate that $\bar{\nu}$ continues to increase with increasing s and reaches a value of about 0.56 for $s = 10^6$. This is consistent with the idea that $\bar{\nu} \to \beta$ as $s \to \infty$.

The dependence of $M(l)$ on l was also measured where $M(l)$ is the mass contained within a distance l measured from the cluster origin. For a fractal structure we expect to find that

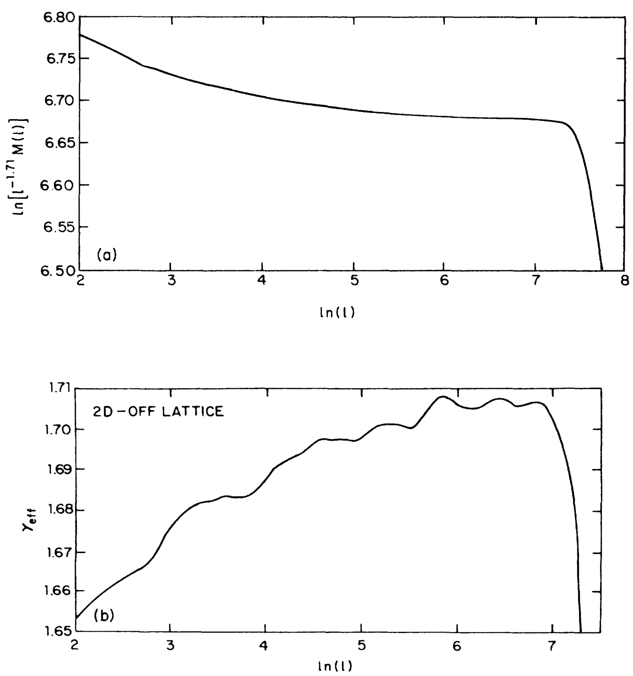


FIG. 3. Estimation of the effective fractal dimensionality D_γ. (a) shows the dependence of $\ln[l^{-1.71}M(l)]$ on $\ln(l)$ and (b) shows the dependence of the effective value of the exponent γ (D_γ) on $\ln(l)$.

$$M(l) \sim l^{\gamma} , \tag{5}$$

where the exponent value provides a measure (D_γ) of the fractal dimensionality. Figure 3(a) shows the dependence of $\ln[M(l)/l^{1.71}]$ on $\ln(l)$ obtained from approximately 400 10^6-particle off-lattice clusters. Similarly, Fig. 3(b) shows the dependence of the effective value of the exponent γ in Eq. (5) on $\ln(l)$. The results shown in Figs. 3(a) and 3(b) are consistent with the idea that in the asymptotic limit $(l \to \infty, s \to \infty)$ $D_\gamma = D_\beta \simeq 1.715$.

With the algorithms used for this work about 2.5 h of CPU time on an IBM 3090 computer is required to grow a 10^6-particle 2D off-lattice cluster. About 1000 h of CPU time was required to obtain the results shown in Figs. 2 and 3.

Three-dimensional results

Figure 4 shows a projection of and a cross section through a 3×10^6-site cubic-lattice DLA cluster. It is apparent from this figure that lattice anisotropy has a similar effect on cubic-lattice DLA clusters as it does on square-lattice DLA clusters. The overall size (about 1000 lattice units) of this cluster is similar to that of a 100 000–site square-lattice DLA cluster which has a more or less diamondlike shape.[27,28] Since a quite large amount of computer time is required to generate a cluster of this size (about 9 h on an IBM 3090 computer) more quantitative results were obtained from smaller clusters. Ninety-eight 1.25×10^6-site cubic-lattice clusters and 169 100 000–particle off-lattice DLA clusters were generated.

Figure 5 shows the cluster-size dependence of the effective fractal dimensionality (D_β) obtained from some of these clusters. For both the lattice model and off-lattice model the effective fractal dimensionality lies close to the mean-field value of 2.50. For the 482 100 000–site cubic-lattice clusters not shown in Fig. 5 the effective dimensionality lies in the range 2.495 ± 0.005 for clusters in the size range $2500 \le s \le 50\,000$ sites. The somewhat smaller value obtained for the larger clusters is consistent with the idea that lattice anisotropy reduces the effective fractal dimensionality. However, it is clear that enormous clusters would be required to reach a value significantly lower than 2.48. For a cluster with six distinct arms and a relatively small (interior) tip angle associated with each arm the theoretical approach of Turkevich and Scher[53] would predict a fractal dimensionality much closer to 2.0 (exactly 2.0 for a zero tip angle). Noise-reduced DLA simulations with relatively large noise-reduction parameters ($m = 100$, where m is the number of contacts required for growth) lead to clusters with an effective fractal dimensionality D_β of 2.20–2.25.[17,54] The results shown in Fig. 5 indicate that no practical scale simulations will lead to effective fractal dimensionalities approaching this range.

Figure 6 shows the cluster-size dependence of the distance δR between the cluster center of mass and the original seed or growth site. It is apparent that δR grows much more slowly with increasing cluster size than does the overall cluster size (represented by the radius of gyration in Fig. 5). The dependence of δR on s was determined for all of the models discussed in this paper. In all cases δR grows much more slowly with increasing cluster size than the cluster radius.

The dependence of the width of the active zone ξ on the cluster mass or number of particles s was measured in all of our simulations. Figure 7(a) shows the dependence of the exponent $\bar{\nu}$ (which describes how ξ grows with increasing cluster size) on the cluster size. These results were obtained from 86 350 000–particle off-lattice clusters. Figure 7(a) also shows the dependence of the effective value of β on $\ln(s)$. For clusters containing more than about 2000 particles $\bar{\nu}$ increases with increasing

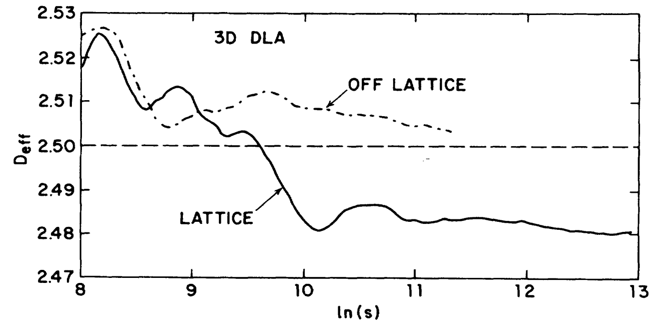


FIG. 5. Dependence of the effective fractal dimensionality D_β obtained from 98 1 250 000–site cubic-lattice DLA clusters (solid curve) and from 169 100 000–particle off-lattice three-dimensional DLA clusters (dashed curve).

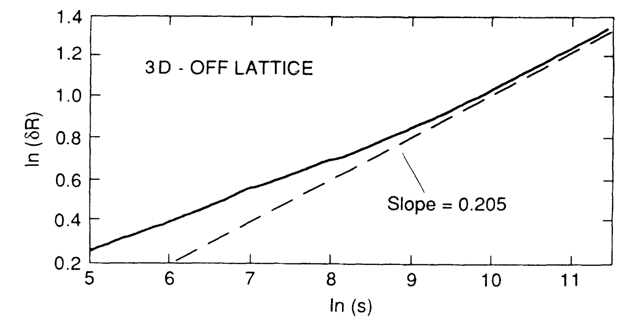
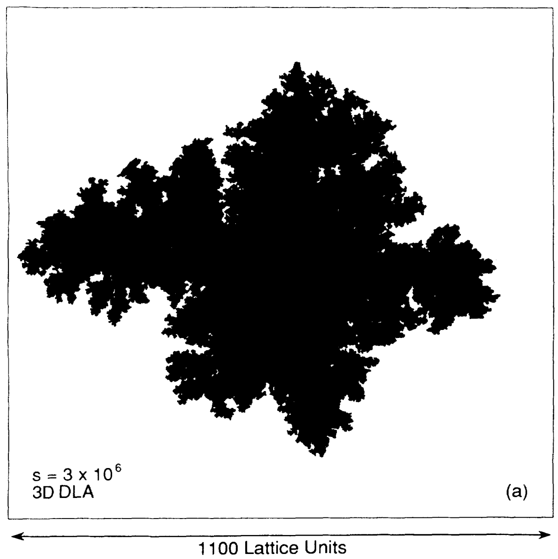


FIG. 6. Growth of the distance δR between the cluster center of mass and the cluster origin obtained from the 169 3D off-lattice clusters.



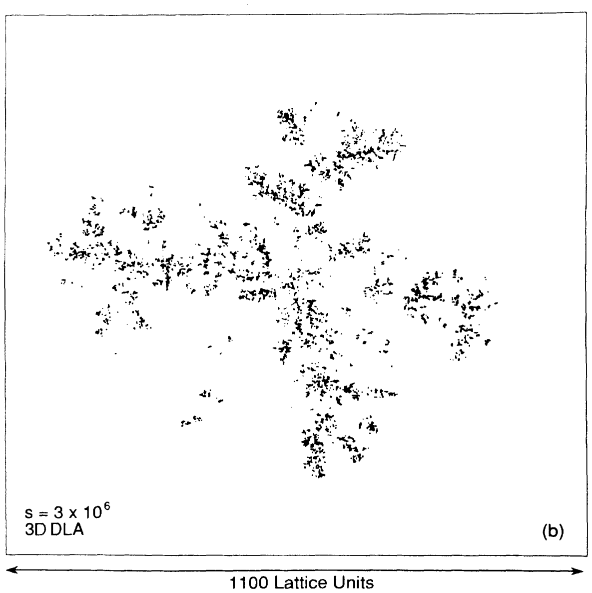


FIG. 4. (a) shows a projection of a 3×10^6-site cubic-lattice DLA cluster and (b) shows a cross section through the cluster origin along the plane of projection.

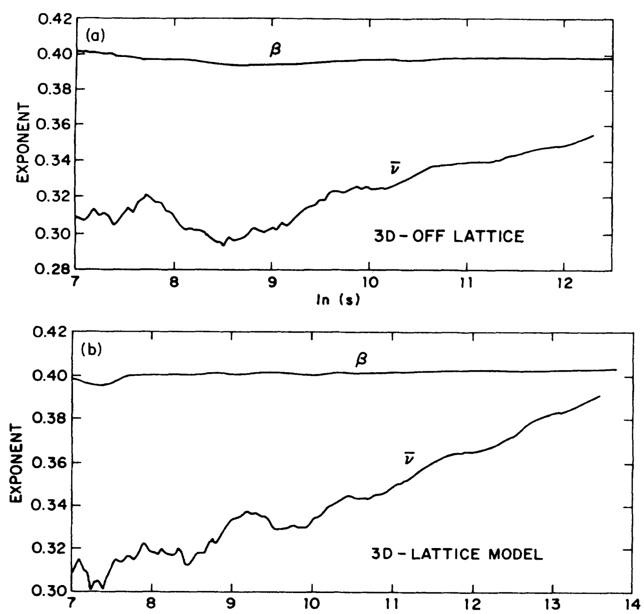


FIG. 7. Dependence of the width of the active zone ξ on cluster size for three-dimensional DLA clusters. (a) shows the dependence of the effective exponents β and ν obtained from the 350 000–particle off-lattice clusters. (b) shows the cluster-size dependence of ν and β obtained from the 1.25×10^6-site lattice-model clusters.

cluster size but has only reached a value of about 0.35–0.36 for the largest cluster sizes. Figure 7(b) shows similar results obtained from the 85 1.25×10^6-site clusters (different from those used to obtain the lattice model results shown in Fig. 5). In this case also there is also an increase in $\bar{\nu}$ with increasing cluster size and the results are consistent with an asymptotic value of about 0.4.

The dependence of $M(l)$ on l was also measured for the 350 000–particle off-lattice clusters and a value of about 2.485±0.005 was obtained by D_γ from the off-lattice clusters.

Results from four-dimensional simulations

Clusters containing up to 10^6 sites were grown using the four-dimensional lattice model. This is two orders of magnitude larger than the largest previously reported 4D clusters.[30,31] Figure 8 shows two projections and a cross

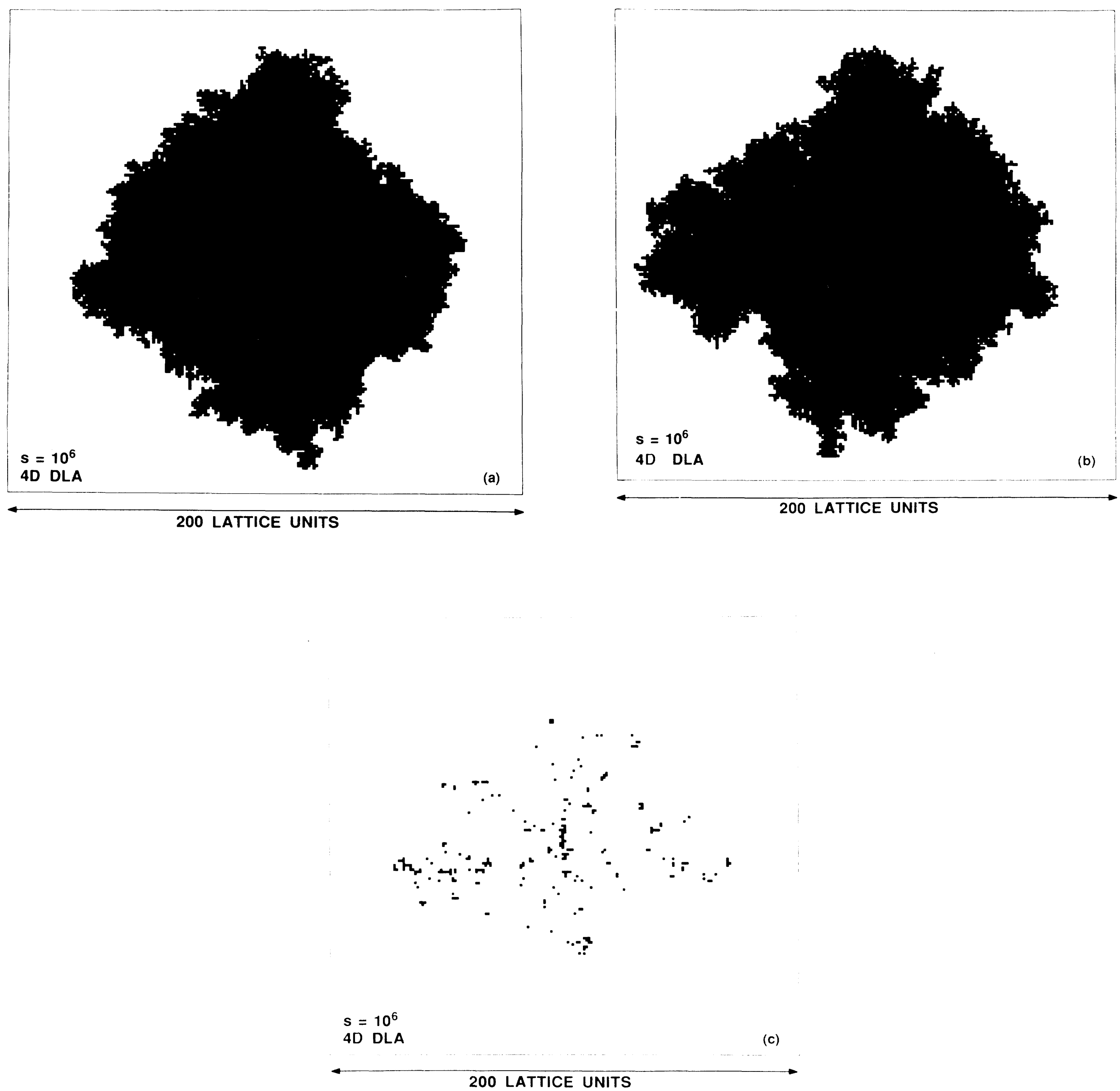


FIG. 8. Two mutually perpendicular projections and a cross section through a 10^6-site DLA cluster grown on a four-dimensional hypercubic lattice.

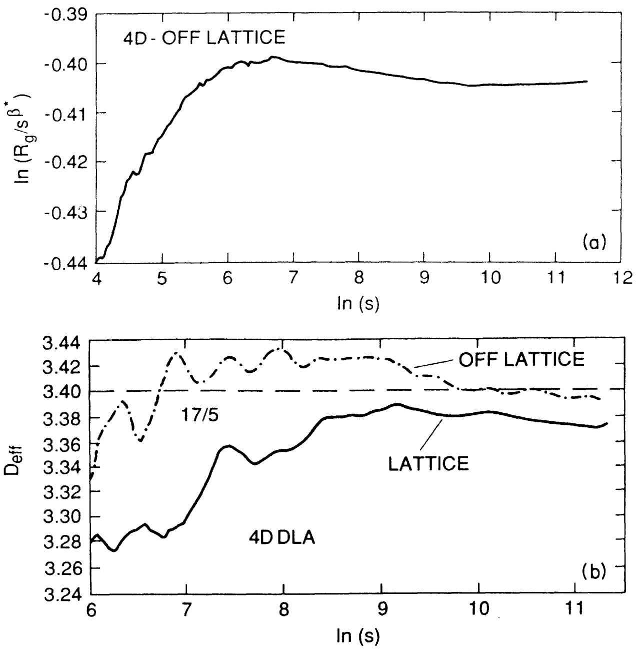


FIG. 9. Growth of the radius of gyration for the four-dimensional DLA clusters. (a) shows the dependence of R_g/s^{β^*} ($\beta^*=1/3.40$) on the cluster size obtained from the off-lattice clusters and (b) shows the cluster size dependence of the effective fractal dimensionality D_β for both lattice-model and off-lattice clusters.

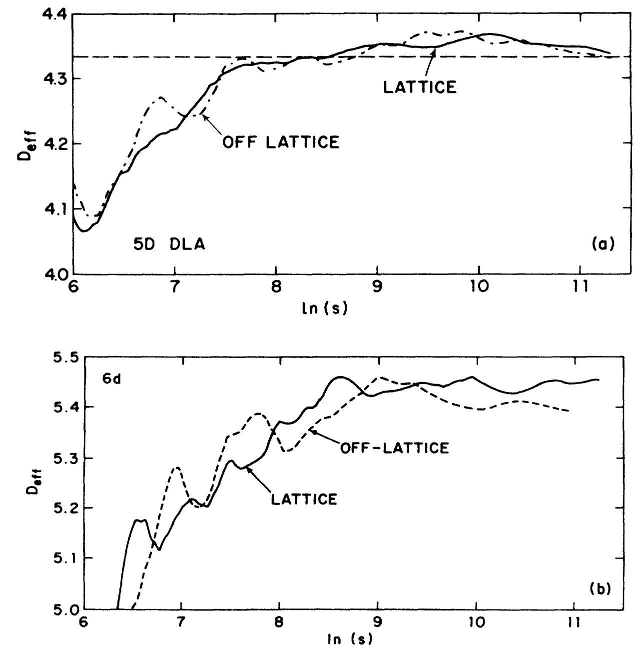


FIG. 10. Cluster-size s dependence of the effective fractal dimensionality D_β obtained from the five-dimensional (a) and six-dimensional (b) DLA models. In (a) the horizontal dashed line corresponds to $D_{\text{eff}}=\frac{13}{3}$.

section through one of these clusters. For clusters of this size the effects of the hypercubic-lattice anisotropy are quite apparent.

Figure 9(a) shows the dependence of $\ln(R_g/s^{\beta^*})$ on $\ln(s)$ obtained from 69 100 000–particle off-lattice DLA clusters. Here β^* is the mean-field-theory value for β (1/3.4). The results shown in Fig. 9(a) show that the effective value for the fractal dimensionality D_β is close to 3.4 (D_{MF}) for clusters containing more than about 1000 particles. Figure 9(b) shows the cluster-size dependence of D_β obtained from the off-lattice clusters and from 340 100 000–site clusters grown on a four-dimensional hypercubic lattice. The lattice-model simulations lead to clusters with an effective fractal dimensionality that rises to a value close to 3.38 with increasing cluster size. For the off-lattice clusters a slightly higher value (quite close to 3.40) is obtained. Results were also obtained from 46 300 000-site lattice-model clusters. The effective fractal dimensionality obtained from these clusters was also quite close to 3.40 (D_{MF}). Overall, these results are consistent with the idea that $D\simeq 3.40$ for off-lattice model clusters. The effective fractal dimensionality of the lattice-model clusters may be slightly reduced by the effects of lattice anisotropy.

Five- and six-dimensional simulation results

Using the improved algorithms described above we were able to grow clusters to a size of 100 000 particles or sites for $d=5$. Eighty-two off-lattice clusters and 241 lattice-model clusters were grown to this size. For $d=6$ clusters were grown to a size of 100 000 sites using the lattice model and 47 clusters were grown to a size of 70 000 particles using the off-lattice model. The dependence of the effective fractal dimensionality D_β on the cluster size is shown for all of these models in Fig. 10. For $d=5$ both models give results which are quite close to the mean-field value of $\frac{26}{6}$ (4.333. . .). For $d=6$ the effective values obtained for D_β are distinctly larger than the mean-field value of $\frac{37}{7}$ or 5.285. . . . For $d=6$ a cluster containing 100 000 sites is still quite small (the radius of gyration is about 9.2 lattice units) so that the effective fractal dimensionalities shown in Fig. 10(b) may not be close to their asymptotic ($s\rightarrow\infty$) values.

Seven- and eight-dimensional models

No previous results exist for $d>6$. Twenty-eight 70 000–site clusters were generated using the 7D lattice model and 30 25 000–particle clusters were grown using the off-lattice model. For $d=8$ the clusters were even smaller. Using a lattice model 56 10 000–site clusters were grown and 38 4500-particle off-lattice model clusters were grown. Figure 11 shows the results obtained for the size dependence of the effective fractal dimensionality from the 7D lattice model and 8D off-lattice model. In both cases the effective fractal dimensionality starts out at a value much smaller than the mean-field value $D_{\text{MF}}=6.25$ for $d=7$ and $D_{\text{MF}}=7.222. . .$ for $d=8$) but increases with increasing cluster size and has reached a value substantially in excess of the mean-field value for the largest cluster sizes.

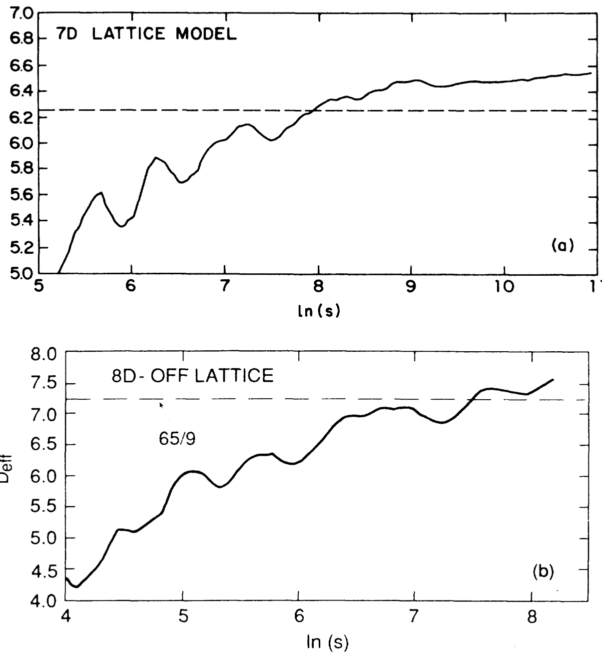


FIG. 11. Cluster-size dependence of the effective fractal dimensionality D_β obtained from the seven-dimensional lattice model (a) and eight-dimensional (b) off-lattice model for DLA.

DISCUSSION

By developing improved DLA algorithms for $d > 2$ and using quite large amounts of computer time, we have been able to substantially reduce the uncertainties concerning the effective dimensionality D_β describing the relationship between the overall cluster size and its mass. For $d = 2$ our results for off-lattice DLA are in good agreement with earlier work. The fractal dimensionality of 1.715 ± 0.004 obtained from the off-lattice simulations is 25 standard deviations larger than the mean-field value [$D_{MF} = \frac{5}{3}$ (Refs. 34, 35, and 47)] and about 17 standard deviations below the value of 1.75 obtained by Hentschel using a different mean-field theory.[36] Assuming a diamondlike shape, Turkevich and Scher[39] obtained a fractal dimensionality of $\frac{5}{3}$ from the strength of the singularity in a Laplacian field gradient normal to the surface of a diamond with the boundary condition $\phi = 0$ on the diamond surface and $\phi = 1$ at infinity. (Here ϕ is the scalar field which obeys the Laplace equation $\nabla^2 = 0$.) However, the diamondlike shape does not seem to be appropriate for either square-lattice or off-lattice DLA. Using related ideas Ball[43] has presented arguments leading to the result

$$D = 1 + \sqrt{2}/2 = 1.707\ldots \tag{6}$$

for two-dimensional off-lattice DLA. This value for D is 4 standard deviations from the simulation results. Because of the possibility of finite-size effects, even for very large (10^6 particles) clusters, this theoretical prediction is not inconsistent with the simulation results. Similarly, the value of 1.7099. . . obtained theoretically by Nagatini[42,55,56] is also in good agreement with the simulation results.

Although the origin (growth site) of two-dimensional DLA clusters appears to be at the center of anomalously high density,[57] the results shown in Fig. 3 indicate that the average density $\rho(r)$ at a distance r from the origin decreases asymptotically according to the power law

$$\rho(r) \sim r^{-\alpha'}, \tag{7}$$

where the exponent α has a value of $d - D$. Although we did not measure $\rho(r)$ for other positions in the cluster, the two point density-density correlation function $C(r)$ also has the form[28]

$$C(r) \sim r^{-\alpha} \tag{8}$$

and it appears that the exponents α and α' are equal in the limit $r \to \infty$. For $d = 3$, 4, and 5 the off-lattice simulations give values of D_β very close to the mean-field value of $(d^2+1)/(d+1)$ and for $d = 3$ and 4 the hypercubic-lattice simulation gives effective values for D that are about 2.48 and 3.38, respectively (just below the mean-field values of 2.50 and 3.40). This is probably an effect of lattice anisotropy (the effects of lattice anisotropy on the overall cluster shapes can be seen clearly in Figs. 5 and 8). The mean-field theory of Hentschel[36] [Eq. (2)] leads to the prediction $D \simeq 2.524$ for $d = 3$ and $D = 3.385$ for $d = 4$. For $d = 3$ our simulation results are in better agreement with the predictions of the mean-field theories, but the value for D obtained from Eq. (1) is closer to the simulation results than that obtained from Eq. (2). For $d = 4$ the simulation results and both mean-field theories [Eqs. (1) and (2)] are in quite good agreement.

For $d < 6$ the effective fractal dimensionality at first increases quite rapidly with increasing cluster size, reaches a broad peak and then decreases very slowly towards its asymptotic value. For $d = 6$ the effective fractal dimensionality increases with increasing cluster size and reaches a plateau at a value slightly larger than the mean-field value. It seems most probable that in this case the effective value for D_β would decrease as it does in the other cases. The position (cluster size) of the peak in D_β increases with increasing d and from the results obtained for $d < 6$ we would expect, for clusters containing about 10^5 particles, that D_β would have a value close to its maximum. Similarly, for $d = 7$ and $d = 8$ the effective value of D_β increases continuously with increasing s reaching a size substantially larger than the mean-field value for the largest attainable cluster sizes. If these results were taken at face value, they would indicate that $d - D$ decreases with increasing d. This seems unlikely in view of the fact that $d - D$ increases with increasing d for small d and that the asymptotic ($d \to \infty$) value for D is $d - 1$ (Refs. 58 and 59) for off-lattice DLA. Overall, these simulations indicate that the fractal dimensionality is distinctly different from the mean-field values of $\frac{5}{3}$ [Eq. (1)] and $\frac{7}{4}$ [Eq. (2a)] for two-dimensional off-lattice DLA but for $d \geq 3$ the simulation results are consistent with the mean-field predictions of both Eqs. (1) and (2). However, Eq. (1) seems to give values for the fractal dimensionality that are in slightly better agreement with the simulation results. In addition, Eq. (1) is consistent with the idea that $D \rightarrow d-1$ as $d \rightarrow \infty$ (Refs. 58 and 59), whereas Eq. (2) indicates that $D \rightarrow d-2$ as $d \rightarrow \infty$.

The dependence of the effective fractal dimensionality on cluster size is almost the same for small off-lattice- and lattice-model clusters with sizes up to s^* where s^* has a value of about 1000 for $d=2$, 10 000 for $d=3$, and $>10^5$ for $d \geq 4$. Beyond this size the effects of lattice anisotropy become more important than the statistical fluctuations for the numbers of clusters which we were able to grow. It would be possible to distinguish between the off-lattice and lattice models by growing much larger numbers of smaller clusters for $d \leq 3$.

Although the results given here are consistent with the predictions of Eq. (1) for $d>2$, they should not be regarded as an affirmation of the mean-field theories used to obtain this equation. At this stage Eq. (1) should be regarded as a successful empirical formula which gives values of D that are in good agreement with the simulation results for $d \geq 3$ and the theoretical asymptotic limit ($D \rightarrow d-1$ as $d \rightarrow \infty$).[58,59] A comprehensive theory for DLA should be capable of explaining the failure of Eq. (1) for $d=2$ as well as its success for $d>2$.

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