Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model

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(Received 1 May 1989)

A state of a composite quantum system is called classically correlated if it can be approximated by convex combinations of product states, and Einstein-Podolsky-Rosen correlated otherwise. Any classically correlated state can be modeled by a hidden-variable theory and hence satisfies all generalized Bell's inequalities. It is shown by an explicit example that the converse of this statement is false.

I. INTRODUCTION

Consider a composite quantum system described in a Hilbert space $\mathcal{H}=\mathcal{H}^1\otimes\mathcal{H}^2$. An uncorrelated state of this system is given by a density matrix W [i.e., an operator $W\in\mathcal{B}(\mathcal{H})$ with $W\geq 0$ and tr $W=1$] in $\mathcal{H}$ of the form $W=W^1\otimes W^2$ for two density matrices $W^i\in\mathcal{B}(\mathcal{H}_i)$. This is equivalent to saying that the expectation value $\mathrm{tr}(WA_1\otimes A_2)$ for the joint measurement of observables $A^i\in\mathcal{B}(\mathcal{H}^i)$ $(i=1,2)$ on the respective subsystems always factorizes, i.e.,

$$\begin{aligned}\mathrm{tr}(WA^1\otimes A^2)&=\mathrm{tr}(W\cdot A^1\otimes \mathbb{1})\mathrm{tr}(W\cdot \mathbb{1}\otimes A^2)\\&=\mathrm{tr}(W^1A^1)\mathrm{tr}(W^2A^2)\ .\end{aligned}$$

Such uncorrelated states can be prepared very easily by using two preparing devices for systems 1 and 2, which function independently and yield the states W^1 and W^2, respectively. Then the factorization property means that if the measuring devices described by A^1 and A^2 also operate independently, we are simply conducting two separate experiments at the same time and the classical multiplication rule for probabilities applies.

One way of preparing correlated states is the following: Suppose that each of the two preparing devices has a switch with settings $r=1,\ldots,n$, say, and that with setting r the device i produces systems in the state W_r^i. Suppose we have also a random generator, which produces numbers $r=1,\ldots,n$ with probability p_r. We can combine these three devices into a new preparing apparatus by the following prescription: In each individual experiment one first draws a random number $r\in\{1,\ldots,n\}$. The switches of the two preparing devices are then set according to the result. Clearly then, the expectation of a measurement of observables A^1 and A^2 will be

$$\sum_{r=1}^{n} p_r\mathrm{tr}(W_r^1A^1)\mathrm{tr}(W_r^2A^2)=\mathrm{tr}(W\cdot A^1\otimes A^2) \tag{1}$$

with the density matrix $W=\sum_{r=1}^n p_rW_r^1\otimes W_r^2$, i.e., W is a convex combination of product states. Expectation values for this state W no longer factorize. The physical "source" of these correlations is the random generator, which can be chosen as a purely classical device. Therefore, we shall call a density matrix *classically correlated* if it can be approximated (e.g., in trace norm) by density matrices of the form (1). States that are not classically correlated have been called *EPR correlated*[1] to emphasize the crucial role of such states in the Einstein-Podolsky-Rosen paradox, and for the violations of Bell's inequalities (see below). EPR correlation and classical correlation are defined as a property of the density matrix W. Since there are usually very different ways of preparing the same state W, classical correlation does not mean that the state has actually been prepared in the manner described, but only that its statistical properties can be reproduced by a classical mechanism.

The terminology "classically correlated" is further justified by the observation that in classical probability theory all states have this property. States in probability theory are given by probability measures, and the state of a composite system is given by a probability measure on a product space. Like every probability measure this can be represented as a limit of convex combinations of measures concentrated on a single point. And since the point measures on a product space are product measures, we conclude that any probability measure on a product space can be represented as a limit of convex combinations of product measures, i.e., is classically correlated in the above sense. In the wider context of C^*-algebraic quantum theory,[2] it is known[3] that if one of two subsystems of a composite system is classical (i.e., has a commutative observable algebra), all states of the composite system are classically correlated.

For any set of correlations determined in an experiment one can raise the question whether these correlations can be described within a purely classical "hidden-variable" theory. Such a theory is based on some probability space (Ω,Σ,M), called the space of hidden variables, consisting of a σ-algebra Σ of subsets of Ω and a σ-additive normalized measure M on Σ. For any measuring device A with possible outcomes ν, one demands the existence of a measurable *response function* $\omega\mapsto F_A(\nu,\omega)\in\mathbb{R}$ interpreted as the probability that the outcome ν is obtained in an experiment with known value $\omega\in\Omega$ of the hidden variables. Therefore the response functions must satisfy $F_A(\nu,\omega)\geq 0$ and $\sum_\nu F_A(\nu,\omega)=1$ for every $\omega\in\Omega$, where the sum is over all possible outcomes of the measurement of A. A hidden-variable model for some set of correlations is then given by a probability space and a collection of response functions such that the probability in an experiment with a measuring device A on system 1 and a device B on system 2 for obtaining the result ν on A and simultaneously the result μ on B is given by the expression $\int M(d\omega)F_A(\nu,\omega)F_B(\mu,\omega)$.

Any hidden-variable model can be extended to measurements with continuous outcome parameters, and can be modified to a "deterministic model" in which the response functions take only the values 0 and 1. Therefore we shall stay with the above simple definition. The existence of a hidden-variable model is exactly the hypothesis of the usual derivations of Bell's inequalities,[4] the "locality" of the theory being expressed by the fact that the response function for A is independent of B and vice versa. It is known that while these inequalities are necessary conditions for the existence of a hidden-variable model, they are not sufficient.[5] On the other hand, the set of correlations admitting hidden-variable models is a convex set, and as such is completely described by some set of linear inequalities. We shall refer to any one of these inequalities as a (generalized) Bell's inequality. Despite some partial results in this direction,[6] no efficient procedure for obtaining all generalized Bell's inequalities is known.

An interesting question is, then, whether or not the correlations described by a quantum state of a composite system admit a hidden-variable model. The measuring devices of system 1 are then represented by observables, i.e., by Hermitian operators $A\in\mathcal{B}(\mathcal{H}^1)$ with spectral resolution $A=\sum_\nu \alpha_\nu P_\nu$, i.e., eigenvalues α_ν and eigenprojections P_ν. We then say that a state $W\in\mathcal{B}(\mathcal{H}^1\otimes\mathcal{H}^2)$ admits a *hidden-variable model* if there are a probability space (Ω,Σ,M) and response functions, defined for all Hermitian $A=\sum_\nu\alpha_\nu P_\nu\in\mathcal{B}(\mathcal{H}^1)$ and $B=\sum_\mu\beta_\mu Q_\mu\in\mathcal{B}(\mathcal{H}^2)$ with discrete spectrum, such that for all A, B, ν, and μ,

$$\int M(d\omega)F_A(\nu,\omega)F_B(\mu,\omega)=\mathrm{tr}(W\cdot P_\nu\otimes Q_\mu)\ . \qquad (2)$$

We claim that all classically correlated states admit hidden-variable models, and hence satisfy all Bell inequalities. This can be proven quite simply for convex combinations of products as in (1). We then take $\Omega=\{1,\ldots,n\}$, $M(\{r\})=p_r$, $F_A(\nu,r)=\mathrm{tr}(W_r^1\cdot P_\nu)$, and define F_B analogously. Then Eq. (1) implies Eq. (2). We omit the somewhat technical, but straightforward approximation arguments needed to extend this result to all classically correlated states.

We conclude that every state violating some generalized Bell's inequality, i.e., any state not admitting a hidden-variable model, cannot be classically correlated, i.e., is EPR correlated. The well-known experiments demonstrating a violation of Bell's inequalities can thus be taken as direct experimental evidence for the existence of EPR correlated states. The vital importance of such states for quantum theory is further underlined by the fact that they are automatically generated by an interacting time evolution. To be precise, any unitary time evolution, which takes all classically correlated initial states again to classically correlated states, necessarily factorizes into the product of two separate time evolutions. Consequently, the ground state of an interacting system, which is often especially easy to prepare, is usually not classically correlated. The states of a relativistic quantum-field theory are even more universally EPR correlated, since any state of finite energy (in particular, the vacuum) violates Bell's inequalities for suitable spacelike localized observables.[7]

Since any state violating some generalized Bell's inequality is EPR correlated, one might conjecture that the converse holds, i.e., that every state admitting a hidden-variable model is classically correlated. This conjecture is indeed true for pure states of a composite quantum system, given by unit vectors in $\mathcal{H}^1\otimes\mathcal{H}^2$. (This can be demonstrated by constructing violations of Bell's inequalities using techniques from Ref. 8). The purpose of this paper is to show that this conjecture is false for general mixed states. We shall do this by explicitly constructing hidden-variable models for a family of quantum states, which are not classically correlated.

II. CONSTRUCTION OF THE EXAMPLE

There are two main difficulties in constructing an example of a state with nonclassical correlations, which nevertheless admits a hidden-variable model. The first is to prove that some explicitly given density matrix $W\in\mathcal{B}(\mathcal{H}^1\otimes\mathcal{H}^2)$ is not classically correlated, i.e., that there is no way in which it can be represented as a convex combination of product states. The second difficulty is to verify Eq. (2), which is an infinite system of equations, indexed by the set of all observables. We shall circumvent both difficulties by considering only states of very high symmetry. To be specific, we take $\mathcal{H}^1\cong\mathcal{H}^2\cong\mathbb{C}^d$ as Hilbert spaces of equal finite dimension d. The states considered are those given by density matrices W satisfying $(U\otimes U)W(U^*\otimes U^*)=W$ for all unitary matrices U on $\mathbb{C}^d$. We shall call such states $U\otimes U$ *invariant* for short.

Before embarking on the construction of the hidden-variable model, it is useful to establish some facts about such states. The first is that any operator A commuting with all unitaries of the form $U\otimes U$ is a linear combination of $\mathbb{1}$ and the "flip" V defined by $V\varphi\otimes\psi=\psi\otimes\varphi$. This is seen most easily by considering matrix elements $\langle\varphi_n\otimes\varphi_m, A\varphi_p\otimes\varphi_q\rangle$ for some basis $\varphi_1,\ldots,\varphi_d$. Since for all r there is a unitary operator taking φ_r to $-\varphi_r$ leaving the other basis elements fixed, the matrix element vanishes unless $\{n,m\}=\{p,q\}$. Since any permutation of the basis can also be realized unitarily, A depends only on the three matrix elements with $n=m=p=q$, $n=p\neq q=m$, and $n=q\neq p=m$. These are linked further by considering for U rotations in a two-dimensional plane, making A depend on just two complex parameters, which can be taken as the coefficients of $\mathbb{1}$ and V, or as $\mathrm{tr}A$ and $\mathrm{tr}(AV)$. For density matrices W $\mathrm{tr}W\equiv 1$ is fixed, so that the set of $U\otimes U$ invariant states is parametrized by the single parameter $\Phi:=\mathrm{tr}(WV)$. Since $V^2=\mathbb{1}$ and $V^*=V$ we have $-1\leq\Phi\leq 1$. Note that $\Phi=1$ corresponds to a state with Bose symmetry and $\Phi=-1$ corresponds to a state with Fermi symmetry. In terms of Φ the state is described by

$$W=(d^3-d)^{-1}[(d-\Phi)\cdot \mathbb{1}+(d\Phi-1)\cdot V] , \tag{3a}$$

$$\operatorname{tr}(WA\otimes B)=(d^3-d)^{-1}[(d-\Phi)(\operatorname{tr}A)(\operatorname{tr}B)+(d\Phi-1)\operatorname{tr}(AB)] , \tag{3b}$$

where we have used the formula $\operatorname{tr}(V\cdot A\otimes B)=\operatorname{tr}(A\cdot B)$.

There is a linear projection $\mathbb{P}$ mapping arbitrary density matrices W to $(U\otimes U)$-invariant ones by $\mathbb{P}(W)=\int dU(U\otimes U)W(U^*\otimes U^*)$, where dU denotes the Haar measure of the unitary group of $\mathbb{C}^d$. This integral does not have to be calculated explicitly, since its value depends only on the two parameters $\operatorname{tr}\mathbb{P}W=\operatorname{tr}W$, and

$$\operatorname{tr}[(\mathbb{P}W)V]=\int dU\operatorname{tr}[W(U^*\otimes U^*)V(U\otimes U)]=\operatorname{tr}(WV) .$$

Suppose now that W is classically correlated. Then so is each $(U\otimes U)W(U^*\otimes U^*)$, and hence, by approximating the integral by suitable Riemann sums, $\mathbb{P}W$ is represented as the limit of convex combinations of classically correlated states, and is hence itself classically correlated. Thus in order to determine the set of all $(U\otimes U)$-invariant, classically correlated states we only have to compute the possible values of $\Phi=\operatorname{tr}(WV)$ for classically correlated W. For a product state $W=W^1\otimes W^2$ we find that $\Phi=\operatorname{tr}(WV)=\operatorname{tr}(W^1\cdot W^2)$ is positive. Hence the same is true for convex combinations and for norm limits of such states, i.e., for all classically correlated states. Moreover the extremal value $\Phi=0$ (respectively, $\Phi=1$) is attained by taking W^1 and W^2 to be orthogonal (respectively equal) pure states. In short, a $(U\otimes U)$-invariant density matrix is classically correlated if and only if $\Phi=\operatorname{tr}(WV)$ is positive.

For constructing the hidden-variable model we shall utilize the $U\otimes U$ symmetry by choosing the space Ω as the unit sphere $\{\omega\in\mathbb{C}^d \mid \|\omega\|=1\}$ and M as the unique probability measure on Ω invariant under all unitary rotations of the sphere. The response functions F_A of an observable $A=\sum_\nu \alpha_\nu P_\nu$ will be taken to depend only on the family $\{P_\nu\}$ of orthogonal projections, but not on the eigenvalues α_ν, or the labeling of the projections P_ν. Moreover, it suffices to consider only the case where each P_ν is one dimensional, since for projections of higher dimension the response function can be chosen as a sum of response functions of one-dimensional projections. The symmetry under unitaries of $\mathbb{C}^d$ will be imposed by the relation

$$F_{U^*AU}(\nu,\omega)=F_A(\nu,U\omega) . \tag{4}$$

The simplest choice for such functions is the one we shall adopt for system 2:

$$F_B(\mu,\omega)=\langle \omega,Q_\mu\omega\rangle \quad \text{where } B=\sum_\mu \beta_\mu Q_\mu . \tag{5}$$

Note that for any positive integrable function $\rho:\Omega\to\mathbb{R}$ there is a unique positive operator $\hat{\rho}=\int M(d\omega)|\omega\rangle\langle\omega|$ such that

$$\int M(d\omega)\rho(\omega)F_B(\mu,\omega)=\operatorname{tr}(\hat{\rho}\cdot Q_\mu) .$$

This will allow us to compute the integrals (2). A characteristic feature of the response functions F_B is that $F_B(\mu,\omega)$ depends only on Q_μ, but not on the remaining projections in the spectral family of B. It is vital for our construction that the choice of response functions F_A for system 1 does not have this property. Suppose to the contrary, that $F_A(\nu,\omega)=f(P_\nu,\omega)$ depends only on P_ν. Then for each fixed ω the map $P\mapsto f(P,\omega)$ is additive on families of mutually orthogonal projections. Hence (assuming $d\geq 3$) by Gleason's theorem[9] there is a density matrix W_ω such that $f(P,\omega)=\operatorname{tr}(W_\omega P)$. Then Eq. (2) holds for the density matrix $W=\int M(d\omega)W_\omega\otimes|\omega\rangle\langle\omega|$, but this state W is explicitly represented as an average of product states and is thus classically correlated. We note in passing that the reason not to settle for the simplest case $d=2$ in the present paper was exactly to exclude the suspicion that the result is a spurious effect due to the failure of Gleason's theorem for $d=2$.

Using only Eqs. (4) and (5), we can reduce the calculation of all the integrals in Eq. (2) to the computation of a single integral. By the choice of F_B there exists, for each A and ν a positive operator $\hat{F}_A(\nu)$ such that

$$\int M(d\omega)F_A(\nu,\omega)F_B(\mu,\omega)=\operatorname{tr}[\hat{F}_A(\nu)\cdot Q_\mu] . \tag{6}$$

Using the U_d invariance of M, we find

$$\begin{aligned}\operatorname{tr}(U\hat{F}_A(\nu)U^*\cdot Q_\mu)&=\int M(d\omega)F_A(\nu,\omega)\langle\omega,U^*Q_\mu U\omega\rangle\\&=\int M(d\omega)F_A(\nu,U^*\omega)\langle\omega,Q_\mu\omega\rangle\\&=\int M(d\omega)F_{UAU^*}(\nu,\omega)F_B(\mu,\omega)\\&=\operatorname{tr}[\hat{F}_{UAU^*}(\nu)\cdot Q_\mu] .\end{aligned}$$

Since this holds for all one-dimensional projections Q_μ, we find $U\hat{F}_A(\nu)U^*=\hat{F}_{UAU^*}(\nu)$. In particular, $\hat{F}_A(\nu)$ commutes with all unitaries commuting with A, that is to say it has a representation $\hat{F}_A(\nu)=\sum_\mu\lambda(\nu,\mu)P_\mu$. Since F_A is not to depend on the labeling of the projections P_μ, we conclude that $\lambda(\nu,\mu)$ depends only on whether or not $\nu=\mu$. Hence $\hat{F}_A(\nu)=\lambda_1P_\nu+\lambda_2\mathbb{1}$ for some $\lambda_1,\lambda_2\in\mathbb{R}$, which are independent of A. Since $\sum_\nu\hat{F}_A(\nu)=\mathbb{1}$ we must have $\lambda_1+d\lambda_2=1$. Hence for computing all $\hat{F}_A(\nu)$ it suffices to compute the integral (6) with $Q_\mu=P_\nu$, for which the right-hand side of (6) is $\lambda_1+\lambda_2$. Comparing Eqs. (6) and (3) we find that Eq. (2) holds with Φ determined by

$$\Phi=-1+(d^2+d)\int M(d\omega)F_A(\nu,\omega)\langle\omega,P_\nu\omega\rangle . \tag{7}$$

It remains to construct a family of response functions $F_A(\nu,\omega)$ satisfying (4), for which Φ becomes negative. Thus we have to make the integral (7) as small as possible under the constraints $F_A(\nu,\omega)\geq 0$ and $\sum_\nu F_A(\nu,\omega)=1$ for all ω and all A. Since for every ω and every A these constraints single out a convex set in $\mathbb{R}^d$, we expect that the smallest values of Φ is attained for response functions taking only the values 0 and 1. This suggests the following choice:

$$F_A(\nu,\omega)=\begin{cases}1 & \text{if } \langle\omega,P_\nu\omega\rangle<\langle\omega,P_\mu\omega\rangle \text{ for all } \nu\neq\mu\\ 0 & \text{if } \langle\omega,P_\nu\omega\rangle>\langle\omega,P_\mu\omega\rangle \text{ for some } \mu\neq\nu ,\end{cases} \tag{8}$$

Note that we have left $F_A(\nu,\omega)$ unspecified at all points where $\langle\omega,P_\nu\omega\rangle$ is the minimum of the numbers $\langle\omega,P_\mu\omega\rangle$, but not the unique minimum. However, since this set is of measure zero, it will not contribute to the integral (7) anyway, and we may choose on this subset any measurable function satisfying the constraint. We have written Eq. (8) in such a form that the covariance property postulated in Eq. (4) is manifest. Moreover, F_A is independent of the labeling of the P_ν in the sense that it only depends on $\langle\omega,P_\nu\omega\rangle$ and the set of numbers $\langle\omega,P_\mu\omega\rangle$, but not on their ordering.

Since with Eq.(8) the integrand in Eq. (7) is a function of the d numbers $\langle\omega,P_\mu\omega\rangle$, we need to compute the probability distribution of these special random variables with respect to the measure M. Choosing a basis ψ_μ, in which the P_μ are diagonal, we have $\langle\omega,P_\mu\omega\rangle=x_\mu^2+y_\mu^2$, where $\omega=\sum_\mu(x_\mu+iy_\mu)\psi_\mu$. A function f depending only on $u_\mu(\omega)=x_\mu^2+y_\mu^2$ has expectation value

$$\begin{aligned}\int M(d\omega)f(u_1(\omega),\dots,u_d(\omega))\\ &=N\int dx_1dy_1\cdots dy_d\delta\left[1-\sum_\mu u_\mu(\omega)\right]\\ &\quad\times f(u_1(\omega),\dots,u_d(\omega))\\ &=\pi^dN\int du_1\cdots du_d\delta\left[1-\sum_\mu u_\mu(\omega)\right]f(u_1,\dots,u_d),\end{aligned}$$

where N denotes a normalizing factor, and we have substituted $dx_\mu dy_\mu=\frac{1}{2}du_\mu d\varphi_\mu$ and have integrated out the angle variables φ_μ. We thus obtain a probability measure on the simplex (generalized tetrahedron) $\{(u_1,\cdots u_d)|\sum_\mu u_\mu=1\}$, which is just the restriction of the Lebesgue measure on the hyperplane $\sum_\mu u_\mu=1$ to the simplex. Remarkably, this measure depends on the choice of the complex number field for the Hilbert spaces $\mathcal{H}^1$, $\mathcal{H}^2$. For the real and quaternion fields, which axiomatic quantum mechanics has been forced to consider[10] as well, we would obtain an additional factor $(\prod_\mu u_\mu)^\alpha$, with $\alpha=-\frac{1}{2}$ for the reals, emphasizing the corners of the simplex, and $\alpha=1$ for the quaternions, emphasizing the center.

With this probability distribution of $\langle\omega,P_\mu\omega\rangle$, the computation of the integral in (7) can now be made without pencil and paper. After the above-mentioned substitution, we have to compute an integral over a regular simplex S with d vertices, embedded into $\mathbb{R}^{d-1}$, and which is best imagined to "stand" on the face with $u_\nu=0$, so that u_ν represents the "vertical" coordinate. By definition of the measure, the total volume of S is 1. The subset in which $u_\nu\le u_\mu$ for all μ is again a simplex S' with the same base, whose apex is the barycenter of S, which is at height $u_\nu=d^{-1}$. The volume of S' is d^{-1} since S is the disjoint union of d pieces congruent to S'. We have to compute the integral of u_ν over S'. This is equal to the height of the barycenter of S', which is d^{-1} times the height of S', which is equal to d^{-2}, multiplied by the volume of S', which is d^{-1}. Hence the integral is d^{-3}, and we obtain our final result

$$\Phi=-1+d^{-2}(d+1)\ . \tag{9}$$

The positive zero of this expression, considered as a polynomial of d^{-1}, is at the inverse golden ratio, hence $\Phi<0$ and, in fact, $\Phi\le-\frac{1}{4}$ for all $d\ge2$. Hence we have constructed a hidden-variable model for a U_d-invariant state [see Eq. (3) with Φ given by Eq. (9)], which is not classically correlated since $\Phi<0$. A surprising aspect of Eq. (9) is that for large d, Φ approaches -1, which is as far removed from the classically correlated states $\Phi\ge0$ as is compatible with positivity of W. Note also that there is no *a priori* reason why a hidden-variable model should give correlations compatible with a quantum-mechanical positivity condition. This is shown by a strange phenomenon happening for large values of Φ: If one chooses $F_A(\nu,\omega)=\langle\omega,P_\nu\omega\rangle$ instead of (8), one obtains a hidden-variable model for the U_d-invariant state $\mathbb{P}W$, where W is a pure state with wave vector $\varphi\otimes\varphi$. As shown above this is the classically correlated state with $\Phi=+1$. Suppose, however, that we define F_A as in Eq. (8), but with reversed inequality signs. Then it is easy to see that the integral in Eq. (7) is larger than for $F_A(\nu,\omega)=\langle\omega,P_\nu\omega\rangle$. Hence we obtain a hidden-variable model for a "state" with $\Phi>1$ (e.g., $\Phi=\frac{5}{4}$ for $d=2$). Thus the correlations are given by (positive) expressions of the form $\mathrm{tr}(W\cdot P_\nu\otimes Q_\mu)$ as in Eq. (2), but W is not a positive operator. Thus there must be some positive operator $C\in\mathcal{B}(\mathcal{H}^1\otimes\mathcal{H}^2)$ with $\mathrm{tr}(WC)<0$, and the only "paradox" here is that by the positivity of all $\mathrm{tr}(W\cdot P_\nu\otimes Q_\mu)$ this C cannot be a positive linear combination of products of positive operators, i.e., C cannot be classically correlated.

Finally, we would like to mention a possible extension of this result. Up to now we have identified quantum-mechanical observables with decompositions $\mathbb{1}=\sum_\nu P_\nu$ of the identity into mutually orthogonal projections P_ν. However, for many purposes this concept turns out to be too narrow,[11] and it is useful to allow, instead of projections, arbitrary positive operators G_ν with $\mathbb{1}=\sum_\nu G_\nu$, where $\mathrm{tr}(WG_\nu)$ is interpreted as the probability of the outcome ν of some measuring device described by the decomposition $\{G_\nu\}$. In order to also represent observables of this type in the hidden-variable model one would have to define the corresponding response functions F_G, and compute the integrals of Eq. (6) also for these. Due to the lack of symmetry, it is a much more difficult task than for the projection valued case to do this explicitly. Even the case $d=2$ seems to involve some nontrivial geometrical estimates. We conjecture, however, that the existence of hidden-variable models for some nonclassically correlated states can be demonstrated also in this wider setting.

ACKNOWLEDGMENTS

The author would like to thank Guido Raggio for stimulating discussions. The support by the Alexander von Humboldt Foundation is gratefully acknowledged.

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