Approximate solution of the hydrogenlike atoms in intense laser radiation

Shahid Rashid
RPO 9888, P.O. Box 5063, New Brunswick, New Jersey 08903
(Received 27 April 1989)

The exact solution to the minimally coupled Dirac equation is called the Volkov solution. An operator $\hat{R}$ is developed, which when applied to the free-electron state function gives us the Volkov solution. We apply the operator to the exact solution of the hydrogen atom to find that it is the approximate solution of the hydrogen atom in the presence of intense laser field. The restriction on the laser intensity is that the Kibble parameter ε_K, which is the ratio of the electron's quiver energy E_q to its rest mass energy E_0, is small compared to 1.

I. INTRODUCTION

Volkov[1] is credited with obtaining the exact solution of the Dirac electron in an external electromagnetic field. Later, the exact solution for the minimally coupled Schrödinger equation and the Klein-Gordon equation was used extensively to explore a host of quantum processes[2-7] such as bremsstrahlung. However, the results obtained by using the minimally coupled Klein-Gordon equation were found to be not much different from those obtained using the minimally coupled Schrödinger equation. Simultaneously, a few researchers used the Volkov solution for similar processes to find out if there were any differences.[8] It was discovered that the difference between the results when the minimally coupled Schrödinger equation's solution was used compared to when the Volkov solution was used is large when the intensity of the laser field is sufficient to make the quiver energy E_q of the electron comparable to its rest mass energy E_0.[9]

The problem of photoionization of the hydrogen atom in a laser field has been tackled by many authors[10] and they have investigated the phenomenon in the context of multiphoton absorption. Its importance in the field of laser fusion research lies in the fact that photoionization of atoms is the first important process which will occur when the laser pulse hits the DT-fuel pellet. The use of the Volkov solution in this field was first done by Keldysh,[11] who assumed the electron's final state in the bound-free photoelectric effect to be the solution of the minimally coupled Klein-Gordon equation. Unfortunately, the effect of the laser field on the atom was completely ignored. But, if the laser radiation is intense enough to affect the electron's final-state interaction, then there is a possibility that its effect on the electron's bound state could be important.

In our present analysis, we attempt to obtain a solution of the hydrogen atom in an intense laser field such that we can use the solution to calculate the cross section for the photoionization of the electron from the quasibound to quasifree state. In Sec. II we develop the theory of a new operator $\hat{R}$ which when applied to the free-electron Dirac solution gives us directly the Volkov solution of an electron in a laser field. We then apply this operator $\hat{R}$ to the exact solution ψ_{H} of the minimally coupled Dirac equation for the case of the hydrogen atom and investigate the final resulting wave function ψ_R. It is found that so long as the Kibble parameter ε_K, which is the ratio of the electron's quiver energy E_q to its rest mass energy E_0, is small the wave function ψ_R is the solution of the hydrogen atom in an intense laser field.[12] This is an important solution and its use will be explored in a future paper.

II. THE EXACT SOLUTION OF THE MINIMALLY COUPLED DIRAC EQUATION

We treat the electron relativistically to take into account the relativistic effect of the laser field. We use the Lorentz-Heaviside units[13] with $\hbar=c=1$ and the metric $g^{\mu\nu}=(1,-1,-1,-1)$. The laser beam can be represented by a classical monochromatic field, the amplitude of which is

$$A_\mu=(0,\mathbf{A})=A_0\cos(k\cdot x)\ , \tag{2.1}$$

where

$$k_\mu=(k_0,\mathbf{k})=|\mathbf{k}|(n_0,\mathbf{n})\ , \tag{2.2}$$

and the gauge is $\varepsilon\cdot k=0$. The exact solution for the electron in the intense laser field A_μ is given by the Volkov state[14]

$$\Psi_i=(m/E_i)^{1/2}\left[1+\frac{e}{2n\cdot p_i}(\gamma\cdot n)(\gamma\cdot A)\right]u_i e^{-i(p_i\cdot x-S_i)}\ , \tag{2.3}$$

where u_i is a spinor satisfying the normalization condition

$$u_i^\dagger\cdot u_i=|E_i|/m\ , \tag{2.4}$$

and

$$S_i=(2n\cdot p_i)^{-1}\int_{-\infty}^{+n\cdot x}[2e\,p_i\cdot A-(eA)^2]dy\ . \tag{2.5}$$

The subscript i indicates that the incident electron and the γ are the usual Dirac matrices.[13] The expression (2.3) is an exact solution of the minimally coupled Dirac equation.

III. APPROXIMATION SOLUTION FOR THE HYDROGEN ATOM IN THE PRESENCE OF AN INTENSE LASER FIELD

Another way of looking at (2.3) is to propose an operator $\hat{R}$ defined as

$$\hat{R}=\left[1+\frac{e}{2n\cdot\hat{p}}(\gamma\cdot n)(\gamma\cdot A)\right]e^{i\hat{S}}\ , \tag{3.1}$$

where the operators

$$\hat{S}=(2n\cdot\hat{p})^{-1}\int_{-\infty}^{+n\cdot x}[2e\hat{p}\cdot A-(eA)^2]dy\ , \tag{3.2}$$

and

$$\hat{p}^{\mu}=i\partial^{\mu}\ . \tag{3.3}$$

One can easily show that the Volkov solution (2.3) can also be derived from the Dirac solution for a free electron by operating on it with $\hat{R}$, i.e.,

$$\Psi_i=\hat{R}\,\psi_{\text{free}}\ , \tag{3.4}$$

where the free state of the electron is given by

$$\psi_{\text{free}}=(m/E_i)^{1/2}u_i e^{-ip_i x}\ . \tag{3.5}$$

Hence the operator $\hat{R}$ as given by (3.1) is important in the sense that when it operates on the free state it generates a Volkov state for the electron in an intense laser field.

Suppose that the state ψ_{H} is the exact solution for the minimally coupled Dirac equation for a hydrogen atom. Hence, if $A_C=-e/r$ is the ion's Coulomb potential, then ψ_{H} is the solution of the equation

$$[\gamma\cdot(i\partial-eA_C)-m]\psi_{\text{H}}=0\ . \tag{3.6}$$

What we want to do next is to show that an approximate solution ψ_A to the equation

$$[\gamma\cdot(i\partial-eA_C-eA)-m]\psi_E=0\ , \tag{3.7}$$

where A_μ is given by (2.1), can be obtained by the operation of the operator $\hat{R}$, as given by (3.1), on the exact solution ψ_{H} of the hydrogen atom. In other words, the state function

$$\psi_A=\hat{R}\,\psi_{\text{H}} \tag{3.8}$$

is such that

$$[\gamma\cdot(i\partial-eA_C-eA)-m]\psi_A\approx 0\ . \tag{3.9}$$

This is found to be true for the laser intensity range when the Kibble parameter ε_κ, which is the ratio of the electron's quiver energy to its rest mass energy, is smaller than unity. In Eqs. (3.6), (3.7), and (3.9) we are assuming a bound state for the electron.

We begin by substituting (3.1) and (3.8) on the left-hand side of Eq. (3.9) to get

$$[\gamma\cdot(i\partial-eA_C-eA)-m]\times\left[1+\frac{e}{2n\cdot\hat{p}}(\gamma\cdot n)(\gamma\cdot A)\right]e^{i\hat{S}}\psi_{\text{H}}\ . \tag{3.10}$$

We first calculate the $(i\partial)$ term in (3.10) which is given as

$$T_1=(i\gamma\cdot\partial)\left[1+\frac{e}{2n\cdot\hat{p}}(\gamma\cdot n)(\gamma\cdot A)\right]e^{i\hat{S}}\psi_{\text{H}}\ . \tag{3.11}$$

Using the fact that $(\gamma\cdot n)(\gamma\cdot n)=0$ and the commutation rules as given in Ref. 13 in (3.11), we get

$$\begin{aligned}T_1=&\gamma\cdot n/n\cdot p[eAe^{-i\hat{S}}\cdot i\partial-(eA)^2/2e^{-i\hat{S}}]\psi_{\text{H}}\\&+i\gamma^{\mu}[1+e(2n\cdot p)^{-1}(\gamma\cdot n)(\gamma\cdot A)]e^{-i\hat{S}}\partial_\mu\psi_{\text{H}}\ .\end{aligned} \tag{3.12}$$

The rest of the terms in (3.10) are combined into

$$\begin{aligned}T_2=&-\gamma\cdot(eA_C+eA+m)e^{-i\hat{S}}\psi_{\text{H}}-e(2n\cdot p)^{-1}\\&\times[e(\gamma\cdot A)(\gamma\cdot n)(\gamma\cdot A)+e(\gamma\cdot A_C)(\gamma\cdot n)(\gamma\cdot A)\\&+m(\gamma\cdot n)(\gamma\cdot A)]e^{-i\hat{S}}\psi_{\text{H}}\ .\end{aligned} \tag{3.13}$$

Using the fact that $(\gamma\cdot A)(\gamma\cdot n)(\gamma\cdot A)=-A^2\gamma\cdot n$ and canceling, we get

$$\begin{aligned}T=T_1+T_2=&(eA/n\cdot p)\gamma\cdot ne^{-i\hat{S}}\cdot i\partial\psi_{\text{H}}+i\gamma^{\mu}[1+e(2n\cdot p)^{-1}(\gamma\cdot n)(\gamma\cdot A)]e^{-i\hat{S}}\partial_\mu\psi_{\text{H}}\\&-(e\gamma\cdot A_C+e\gamma\cdot A+m)e^{-i\hat{S}}\psi_{\text{H}}-e(2n\cdot p)^{-1}[e(\gamma\cdot A_C)(\gamma\cdot n)(\gamma\cdot A)+m(\gamma\cdot n)(\gamma\cdot A)]e^{-i\hat{S}}\psi_{\text{H}}\ .\end{aligned} \tag{3.14}$$

Under the approximation that the Kibble parameter ε_K is much smaller than unity, one can easily show that

$$i\gamma^{\mu}e(2n\cdot p)^{-1}(\gamma\cdot n)(\gamma\cdot A)e^{-i\hat{S}}\partial_\mu\psi_{\text{H}}=ie(2n\cdot p)^{-1}e^{-i\hat{S}}\{(\gamma\cdot n)(\gamma\cdot A)(\gamma\cdot\partial)+2[n(\gamma\cdot A)-A(\gamma\cdot n)]\gamma\cdot\partial\}\psi_{\text{H}}\ . \tag{3.15}$$

In (3.15), we have used the commutation rule for the Dirac matrices and the gauge property of the radiation field. Further calculations also show that

$$\begin{aligned}&e(\gamma\cdot A_C)(\gamma\cdot n)(\gamma\cdot A)e^{-i\hat{S}}\\&\quad=e^{-i\hat{S}}e[(\gamma\cdot n)(\gamma\cdot A)(\gamma\cdot A_C)+2|A_C|(\gamma\cdot A)]\ ,\end{aligned} \tag{3.16}$$

and

$$e\gamma\cdot A_C e^{-i\hat{S}}\approx e^{-i\hat{S}}e\gamma\cdot A_C\ . \tag{3.17}$$

Using (3.15), (3.16), and (3.17) in (3.13), we obtain

$$\begin{aligned}T=&e^{-i\hat{S}}[e(2n\cdot p)^{-1}(\gamma\cdot A)(n\cdot\partial)-e\gamma\cdot A]\psi_{\text{H}}\\&-e^2(n\cdot p)^{-1}|A_C|\gamma\cdot A\,\psi_{\text{H}}\ .\end{aligned} \tag{3.18}$$

Under the nonrelativistic approximation, $n \cdot p \approx m$, and therefore (3.18) is very small. Hence (3.9) is true under the approximation that the laser intensity is such that the Kibble parameter ε_K is small compared to unity.

IV. CONCLUSION

As an exact solution for the hydrogen atom and for the case where the electron is in an intense laser field exists, we made an attempt to solve for the combined case where the hydrogen atom is in the presence of an intense laser field. To this end, we developed an operator which when applied to the free-electron state gives us the Volkov state. A logical extension of this is to apply the operator to the exact solution of the hydrogen atom. We tried and discovered that the resulting wave function is indeed an approximate solution for the case when the Kibble parameter ε_K is smaller than unity.

ACKNOWLEDGMENTS

I take this opportunity to thank Rutgers University for allowing me to use the computer facility.

[1]D. M. Volkov, Z. Phys. **94**, 250 (1935).
[2]Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).
[3]H. Brehme, Phys. Rev. C **3**, 837 (1971).
[4]X. Ehlotsky, Opt. Commun. **13**, 1 (1975).
[5]N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1972).
[6]A. Weingartshofer *et al.*, Phys. Rev. Lett. **39**, 269 (1977).
[7]A. Weingartshofer *et al.*, Phys. Rev. A **19**, 2371 (1979).
[8]L. Schlesinger and J. Wright, Phys. Rev. A **20**, 1934 (1979).
[9]S. Rashid, Phys. Rev. A **39**, 4069 (1989).
[10]V. P. Oleinik, Zh. Eksp. Teor. Fiz. **52**, 1049 (1967) [Sov. Phys.—JETP **25**, 697 (1967)].
[11]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys.—JETP **20**, 1307 (1965)].
[12]S. Rashid, Phys. Rev. A **38**, 2525 (1988).
[13]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[14]V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Relativistic Quantum Theory* (Pergamon, New York, 1971), Pt. I.