Classical and quantum description of the two-dimensional simple harmonic oscillator in elliptic coordinates

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The Schrödinger equation for the two-dimensional simple harmonic oscillator is solved using elliptic coordinates where it is separable. The separability of the problem in such coordinates is independent of the selection of the focal distance. The solutions are labeled by the total number of quanta N and by a set of characteristic values b corresponding to the eigenvalues of an observable $\hat{B}$, which does not commute with $\hat{L}$, the total angular momentum or $\hat{H}_x$, the energy associated with the x degree of freedom. The well-known quantum energies as well as the characteristic values are obtained by imposing physical polynomial solutions.

I. INTRODUCTION

The harmonic oscillator (HO) is one of the most discussed problems in physics. There is a large number of quantum systems which can be approximated, at least in the limit of small amplitudes, by the HO equations. On the other hand, there are "quasiclassical" states for the quantum HO (coherent states) which illustrate the relation between quantum and classical mechanics when the limit $\hbar \rightarrow 0$ is studied (semiclassical limit).[1,2] Thus, the separability of the HO problem in different coordinate systems as well as the corresponding eigenfunctions and eigenvalues are points of considerable interest.

The procedure to establish the bound states of any physical Hamiltonian generally starts with the choice of an adequate coordinate system in which the time-independent Schrödinger equation can be separated. In fact, the separation constants will be the eigenvalues of different physical observables and will lead to a set of unidimensional differential equations. Finally, physical boundary conditions (or periodicity conditions in the case of angular variables) will restrict the number of allowed eigenvalues which will label the corresponding eigenfunctions.

It is very well known, for example, that Cartesian or polar coordinates are systems where the Schrödinger equation of the two-dimensional simple HO (2D HO) becomes separable. In the former case, one may choose the eigenfunctions

$$\Psi_{N,n_x}(x,y)=\psi_{n_x}(x)\psi_{N-n_x}(y) , \quad (1)$$

where N and n_x are the total number of quanta and the number of quanta associated with the x degree of freedom, respectively.

In polar coordinates since $[\hat{H},\hat{L}]=0$, one may choose

$$\Psi_{N,l}(\rho,\varphi)=R_{N,l}(\rho)\psi_l(\varphi) , \quad (2)$$

where l is the quantum number associated with the angular momentum operator $L=xp_y-yp_x$.

Since both families of eigenfunctions have well-defined energies, one may find the relation between them, for instance, by diagonalization of L on the subspace corresponding to each N in the Cartesian base (1).

The aim of the present paper is to analyze the 2D HO interaction in a different system which also separates the Schrödinger equation: the elliptic coordinate system. The separability of the problem in such coordinates has been discussed from the point of view of group theory[3] and it was shown that the Schrödinger equation reduces to the Ince equation. However, an exhaustive analysis of the physical observables associated with this separation, as well as of the allowed eigenvalues, is still lacking and will be done in this work. This analysis can be very useful when one studies problems with elliptic symmetries such as the elliptical billiard problem.[4-6]

Clearly in the new eigenstates neither $\hat{L}$ nor $\hat{H}_x$, the energy associated with the x degree of freedom, are well defined, but we will show that there exists an observable $\hat{B}$ associated with a new separation constant b. The eigenvalue b of $\hat{B}$ label the resulting eigenfunctions

$$\Psi_{N,b}(\xi,\eta)=R_{N,b}(\xi)\theta_{N,b}(\eta) , \quad (3)$$

which are solutions of the Schrödinger equation in elliptic coordinates.

In Sec. II we study the problem from the point of view of classical mechanics. Section III is devoted to the quantum-mechanical analysis and to the solutions of the Schrödinger equation in elliptic coordinates. In this section we will find a complete set of commuting observables for the problem constituted by $\hat{H}$ and the mentioned operator $\hat{B}$, which in the quantum limit does not commute either with $\hat{L}$ or $\hat{H}_x$. In Section IV we show the behavior of the eigenvalues of the observable $\hat{B}$ and concluding remarks are given in Sec. V.

II. THE CLASSICAL PROBLEM

The elliptical coordinates are defined such that

$$\xi=(r_2+r_1)/2\sigma, \quad \eta=(r_2-r_1)/2\sigma \ , \tag{4}$$

where r_1 and r_2 are distances from the two foci placed at $x=\pm\sigma$. ξ=const defines confocal ellipses while η=const defines confocal hyperbolas. The relation with the Cartesian coordinates is

$$x=\sigma\eta\xi, \quad y=\pm\sigma(1-\eta^2)^{1/2}(\xi^2-1)^{1/2} \ . \tag{5}$$

In this coordinate system the classical Lagrangian $\mathcal{L}=T-V$ is

$$\mathcal{L}(\eta,\xi,\dot\eta,\dot\xi) = \frac{m\sigma^2}{2}\frac{(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}[\dot\xi^2(1-\eta^2)+\dot\eta^2(\xi^2-1)] - V(\xi,\eta) \ , \tag{6}$$

T being the kinetic energy and V the potential energy.

The canonical conjugated momenta result

$$p_\eta=\frac{\partial\mathcal{L}}{\partial\dot\eta}=m\sigma^2\frac{\xi^2-\eta^2}{1-\eta^2}\dot\eta \ , \tag{7}$$

$$p_\xi=\frac{\partial\mathcal{L}}{\partial\dot\xi}=m\sigma^2\frac{\xi^2-\eta^2}{\xi^2-1}\dot\xi \ . \tag{8}$$

Therefore the Hamiltonian is

$$\begin{aligned}H(p_\eta,p_\xi,\eta,\xi)&=p_\eta\dot\eta(p_\eta,\eta,\xi)+p_\xi\dot\xi(p_\xi,\eta,\xi)-\mathcal{L}(\eta,\xi,p_\eta,p_\xi)\\&=\frac{1}{2m\sigma^2(\xi^2-\eta^2)}[(\xi^2-1)p_\xi^2+(1-\eta^2)p_\eta^2]+V(\xi,\eta) \ .\end{aligned} \tag{9}$$

It is easy to see that a sufficient condition to separate a given problem in elliptical coordinates is that the potential takes the form

$$V(\xi,\eta)=[f_1(\xi)+f_2(\eta)]/(\xi^2-\eta^2) \ , \tag{10}$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions. As

$$r^2=\sigma^2(\xi^2+\eta^2-1)=\sigma^2(\xi^4-\eta^4-\xi^2+\eta^2)/(\xi^2-\eta^2) \ , \tag{11}$$

the HO interaction satisfies (10), leading to

$$H(\eta,\xi,p_\eta,p_\xi)=\frac{1}{2m\sigma^2(\xi^2-\eta^2)}[(\xi^2-1)p_\xi^2+(1-\eta^2)p_\eta^2]+\frac{m\sigma^2\omega^2}{2}(\xi^2+\eta^2-1) \ . \tag{12}$$

The ξ and η degrees of freedom can be decoupled in the following way:

$$(\xi^2-1)p_\xi^2+m^2\sigma^4\omega^2(\xi^2-1)\xi^2-2m\sigma^2E\xi^2=-A_0 \ , \tag{13a}$$

$$(1-\eta^2)p_\eta^2+m^2\sigma^4\omega^2(1-\eta^2)\eta^2+2m\sigma^2E\eta^2=A_0 \ , \tag{13b}$$

where E ($=H$) and A_0 are the constant energy and the separation constant, respectively.

To explore this further we eliminate E of (13a) and (13b) and we obtain

$$A_0=\frac{\eta^2(\xi^2-1)p_\xi^2+\xi^2(1-\eta^2)p_\eta^2}{(\xi^2-\eta^2)}+m^2\sigma^4\omega^2\xi^2\eta^2 \ .$$

On the other hand, we know that

$$\mathbf{L}_1\cdot\mathbf{L}_2=[(\dot x y-\dot y x)-\sigma^2\dot y^2]m^2=L^2-2m\sigma^2T_y \ ,$$

where $\mathbf{L}_1$ ($\mathbf{L}_2$) is the angular momentum with respect to the focus at $x=\sigma$ ($-\sigma$), $\mathbf{L}$ is the angular momentum with respect to the origin, and we have defined $T_y=\frac{1}{2}m\dot y^2$.

Therefore

$$\begin{aligned}A_0&=\mathbf{L}_1\cdot\mathbf{L}_2+m\sigma^2(p_\eta\dot\eta+p_\xi\dot\xi)+m^2\omega^2\sigma^4\xi^2\eta^2\\&=\mathbf{L}_1\cdot\mathbf{L}_2+2m\sigma^2T+m^2\omega^2\sigma^4\xi^2\eta^2\\&=\mathbf{L}_1\cdot\mathbf{L}_2-m^2\omega^2\sigma^4(\xi^2-1)(1-\eta^2)+2m\sigma^2E\\&=L^2-2m\sigma^2E_y+2m\sigma^2E \ ,\end{aligned}$$

where

$$T=\tfrac{1}{2}m(\dot x^2+\dot y^2), \quad E_y=\tfrac{1}{2}m\dot y^2+\tfrac{1}{2}m\omega^2y^2 \ .$$

Thus we conclude that

$$A_0=B+2m\sigma^2E=L^2+2m\sigma^2E_x \ .$$

As E is constant we can ensure that

$$B=L^2-2m\sigma^2E_y \tag{14}$$

is a constant.

We will use B instead of A_0 in (13a) and (13b) to reduce our two-degree-of-freedom problem to two unidimensional equivalent problems:

$$\frac{p_\xi^2}{2m\sigma^2}+\frac{m\omega^2}{2}\sigma^2\xi^2+\frac{B}{2m\sigma^2(\xi^2-1)}=E \ , \tag{15a}$$

$$\frac{p_\eta^2}{2m\sigma^2}+\frac{m\omega^2}{2}\sigma^2\eta^2-\frac{B}{2m\sigma^2(1-\eta^2)}=E \ . \tag{15b}$$

The solutions $\eta(t)$ and $\xi(t)$ can be easily obtained through the Cartesian solutions $x(t)$ and $y(t)$ and the relations (5). However, there are some properties of (15) that are worthwhile remarking upon.

It is well known that the Poincaré sections (x,p_x) and (y,p_y) of our problem are ellipses for any initial condition. When we work in elliptic coordinates we have two regions in the Poincaré sections and one separatrix varying B with E fixed. This fact originates from the dependence of the equivalent unidimensional potentials on B, displayed in Fig. 1. In Fig. 2 we show the Poincaré sec-

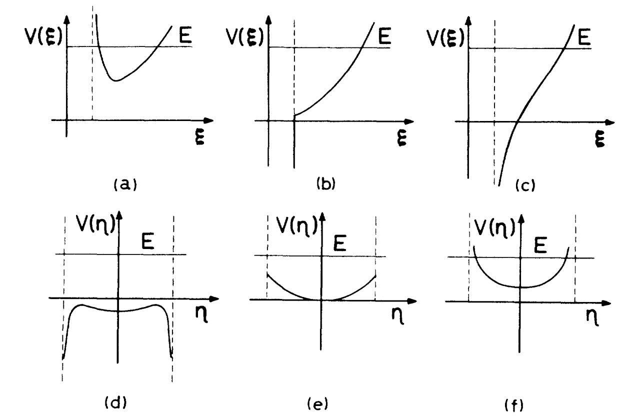

FIG. 1. Equivalent unidimensional potentials. (a) $V(\xi)$, $B>0$; (b) $V(\xi)$, $B=0$; (c) $V(\xi)$, $B<0$; (d) $V(\eta)$, $B>0$; (e) $V(\eta)$, $B=0$; (f) $V(\eta)$, $B<0$.

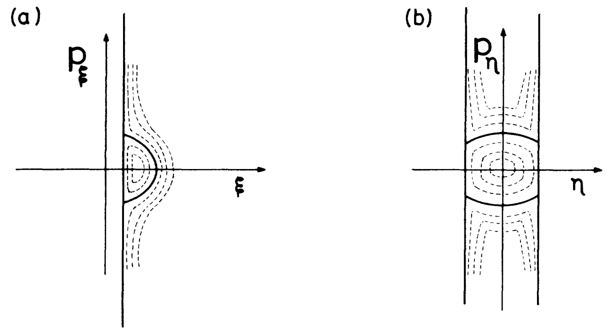

FIG. 2. Poincaré sections (a) p_ξ vs ξ; (b) p_η vs η.

tions (ξ,p_ξ) and (η,p_η). For $B>0$ there are two turning points for ξ, while η takes all possible values $(-1\leq\eta\leq 1)$. For $B<0$, there are two turning points for η, and ξ varies from 1 to $\xi_{\max}$. For $B=0$, the hyperbolic potential vanishes and $-1\leq\eta\leq 1$ and $1\leq\xi\leq\xi_{\max}$ which correspond to the separatrices.

Finally we show (Fig. 3) some trajectories in the real plane for each case. We remark that the separatrices correspond to $B=L^2-2m\sigma^2E_y=0$. When the particle crosses the x axis ($y=0$) this becomes

$$L^2=m^2\sigma^2\dot{y}^2 . \tag{16}$$

Therefore the particle passes over the foci [Fig. 3(b)].

There is one particular case of separatrix which in addition to (16), the condition

$$E=\tfrac{1}{2}m\omega^2\sigma^2$$

is fulfilled. This corresponds to linear oscillations on the x axis just between the foci [Fig. 3(d)].

We want to remark that the HO is the only central potential (apart from the case $V=$const) separable in elliptical coordinates. Moreover, in opposition to other physical situations which are also separable in elliptical coordinates (two-center problems, elliptic billiards) in the present case the parameter σ is not determined by the geometrical properties of the system and the separability exists whatever the focal distance. This fact can be easily understood since the HO potential is a second-degree homogeneous function. Thus changes in the focal distance by a factor δ (i.e., $\sigma'=\delta\sigma$), change the coordinate system $(\xi,\eta)\rightarrow(\xi',\eta')$ so that according to (6) the new Lagrangian becomes

$$\mathcal{L}'(\xi',\eta')=\delta^2\mathcal{L}(\xi',\eta') ,$$

conserving the separability.

That is, σ is not a global scale factor [which would imply $\mathcal{L}'(\xi',\eta')=\delta^2\mathcal{L}(\xi,\eta)$] but an arbitrary parameter which defines a determined coordinate system. As a particular case, in the limit $\sigma\rightarrow 0$, $\xi\rightarrow\infty$ we obtain the well-known polar coordinates, $\sigma\xi\rightarrow r$, $\eta\rightarrow\cos\varphi$, $B\rightarrow L^2$.

III. THE QUANTUM-MECHANICAL PROBLEM

In elliptic coordinates the Laplacian can be easily written[7] and the time-independent Schrödinger equation results:

$$\left\{-\frac{\hbar^2}{2m\sigma^2}\frac{1}{(\xi^2-\eta^2)}\left[(\xi^2-1)^{1/2}\frac{\partial}{\partial\xi}\left[(\xi^2-1)^{1/2}\frac{\partial}{\partial\xi}\right]+(1-\eta^2)^{1/2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)^{1/2}\frac{\partial}{\partial\eta}\right]\right]+V(\xi,\eta)\right\}\Psi_E(\xi,\eta)=E\Psi_E(\xi,\eta) . \tag{17}$$

We can assume eigenfunctions

$$\Psi(\xi,\eta)=R(\xi)\theta(\eta) , \tag{18}$$

resulting in

$$\left[(\xi^2-1)^{1/2}\frac{d}{d\xi}\left[(\xi^2-1)^{1/2}\frac{d}{d\xi}\right]-\frac{m^2\sigma^4\omega^2}{\hbar^2}\xi^2(\xi^2-1)+\frac{2m\sigma^2E}{\hbar^2}(\xi^2-1)\right]R(\xi)=bR(\xi) , \tag{19a}$$

$$\left[(1-\eta^2)^{1/2}\frac{d}{d\eta}\left[(1-\eta^2)^{1/2}\frac{d}{d\eta}\right]-\frac{m^2\sigma^4\omega^2}{\hbar^2}\eta^2(1-\eta^2)+\frac{2m\sigma^2E}{\hbar}(1-\eta^2)\right]\theta(\eta)=-b\theta(\eta) , \tag{19b}$$

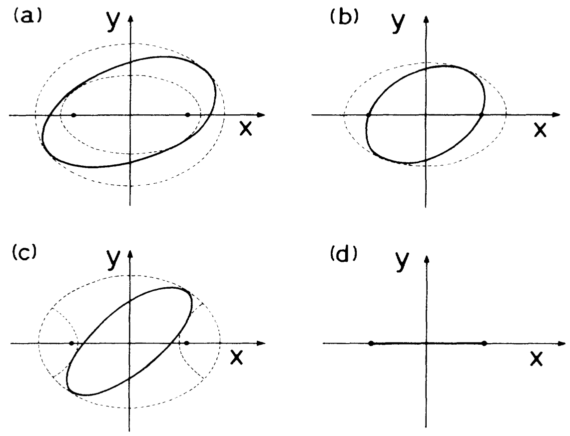

FIG. 3. Trajectories of the two-dimensional harmonic oscillator. (a) $B>0$; (b) $B=0$; (c) $B<0$; (d) $B=0$ and $E=\frac{1}{2}m\omega^2\sigma^2$.

where b is the separation constant. The allowed values of b are the eigenvalues of a quantum operator $\hat{B}$ which is just the physical observable corresponding to the classical magnitude defined in (14). So that

$$\hat{B}=\hat{L}^{2}-2m\sigma^2\hat{H}_y \; . \tag{20}$$

Since $[\hat{H},\hat{B}]=0$, the eigenfunctions (18) will have E (i.e., the total number of quanta) and B well defined.

According to the discussion of Sec. II, it is not surprising that the eigenvalues of the observable $\hat{B}$ [see Eq. (20)] depend not only on the intrinsic properties of the system (i.e., the mass m and the frequency ω) but also on σ. In the cylindrical limit the eigenvalues of $\hat{B}$ are $\hbar^2 m^2$, m being an integer number.

We can obtain the eigenfunctions and the corresponding eigenvalues of $\hat{B}$ by means of diagonalization of $\hat{B}$ in the Cartesian basis. On the other hand, we can solve (19) by using standard techniques of numerical analysis.

A. Diagonalization of $\hat{B}$ in the Cartesian basis

Let us first discuss the general results of the diagonalization procedure in order to establish the solutions of (19) in a more simple way.

We recall that

$$\begin{aligned} a_x^{\dagger}&=2^{-1/2}(\beta\hat{x}+i\hat{p}_x/\beta\hbar)\;,\\ a_y^{\dagger}&=2^{-1/2}(\beta\hat{y}+i\hat{p}_y/\beta\hbar)\;, \end{aligned} \tag{21}$$

with $\beta=(m\omega/\hbar)^{1/2}$ so that

$$\hat{L}=i\hbar(a_x a_y^{\dagger}-a_x^{\dagger}a_y)\;, \tag{22a}$$

$$\hat{H}_y=\hbar\omega(a_y^{\dagger}a_y+\tfrac{1}{2})\;. \tag{22b}$$

Then

$$\begin{aligned}\hat{B}=-\hbar^2[&(a_y^{\dagger})^2a_x^2+(a_x^{\dagger})^2a_y^2+(2\alpha-1)a_y^{\dagger}a_y-a_x^{\dagger}a_x\\ &-2a_x^{\dagger}a_xa_y^{\dagger}a_y+\alpha]\;,\end{aligned} \tag{23}$$

where $\alpha=\beta^2\sigma^2$. The matrix elements are

$$\langle n_x-2,n_y+2|\hat{B}|n_x,n_y\rangle =-\hbar^2[(n_y+1)(n_y+2)(n_x)(n_x-1)]^{1/2}\;,$$

$$\langle n_x+2,n_y-2|\hat{B}|n_x,n_y\rangle =-\hbar^2[(n_x+1)(n_x+2)(n_y)(n_y-1)]^{1/2}\;,$$

$$\langle n_x,n_y|\hat{B}|n_x,n_y\rangle=-\hbar^2[(2\alpha-1)n_y-n_x-2n_xn_y+\alpha]\;.$$

The only nonzero off-diagonal elements verify

$$\Delta n_x=\pm 2,\quad \Delta n_y=\mp 2\;. \tag{24}$$

As a result of the diagonalization we obtain the coefficients $C_{N,b}^{n_x}$ that determine the eigenfunction (18) as a linear combination of Cartesian eigenfunctions:

$$\langle\xi\eta|Nb\rangle=\sum_{n_x}C_{N,b}^{n_x}\langle\xi\eta|n_x,n_y=N-n_x\rangle\;, \tag{25}$$

and using (5),

$$\langle\xi\eta|n_x,n_y=N-n_x\rangle=\frac{\beta\exp\left[-\frac{\alpha}{2}(\xi^2+\eta^2-1)\right]}{[\pi 2^N n_x!(N-n_x)!]^{1/2}}\mathrm{H}_{n_x}(\sqrt{\alpha}\eta\xi)\mathrm{H}_{N-n_x}(\alpha^{1/2}[(\xi^2-1)(1-\eta^2)]^{1/2})\;, \tag{26}$$

where $\mathrm{H}_n(z)$ is the Hermite polynomial of degree n.

As expected from the parity of the potential, the eigenfunctions (25) will have well-defined x parity and y parity. This fact will be used to establish some properties of the wave functions (18).

We may distinguish the even-N and odd-N cases. For even N the $\hat{B}$ matrix can be reduced to two blocks. One corresponds to n_x and n_y even (namely, $++$ states) and the other to n_x and n_y odd (namely, $--$ states). Let us consider one $++$ state. All Cartesian eigenfunctions appearing in the expansion (25) have the same asymptotic form:

$$\exp[-(\alpha/2)(\xi^2+\eta^2-1)]\;.$$

On the other hand, each term of (25) is a product of two even polynomials in $\eta\xi$ and in $(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}$, respectively. We conclude that the eigenfunctions (18) corresponding to $++$ states are of the form

$$\begin{aligned}\langle\eta\xi|Nb\rangle^{++}=&\mathcal{N}_{N,b}e^{-(\alpha/2)(\xi^2-1)}A_{N,b}^{++}(\xi)\\ &\times e^{-\alpha\eta^2/2}B_{N,b}^{++}(\eta)\;,\end{aligned} \tag{27}$$

where $A_{N,b}^{++}(\xi)$ and $B_{N,b}^{++}(\eta)$ are even polynomials of degree N and $\mathcal{N}_{N,b}$ is a normalization constant.

Similar considerations for the $--$ states lead to eigenfunctions

$$\langle \eta\xi|Nb\rangle^{--}=\mathcal{N}_{N,b}e^{-(\alpha/2)(\xi^2-1)}(\xi^2-1)^{1/2}A_{N,b}^{--}(\xi)\times e^{-\alpha\eta^2/2}(1-\eta^2)^{1/2}B_{N,b}^{--}(\eta)\ , \quad (28)$$

where $A_{N,b}^{--}(\xi)$ and $B_{N,b}^{--}(\eta)$ are odd polynomials of degree $N-1$.

For N odd the $\hat{B}$ matrix can be also reduced to two blocks. One corresponding to n_x even and n_y odd (namely, $+-$ states) and the other corresponding to n_x odd and n_y even (namely, $-+$ states).

In the same way as in the N-even case, we conclude that

$$\langle \eta\xi|Nb\rangle^{+-}=\mathcal{N}_{N,b}e^{-(\alpha/2)(\xi^2-1)}(\xi^2-1)^{1/2}A_{N,b}^{+-}(\xi)\times e^{-\alpha\eta^2/2}(1-\eta^2)^{1/2}B_{N,b}^{+-}(\eta)\ , \quad (29)$$

where $A_{N,b}^{+-}(\xi)$ and $B_{N,b}^{+-}(\eta)$ are even polynomials of degree $N-1$ and

$$\langle \eta\xi|Nb\rangle^{-+}=\mathcal{N}_{N,b}e^{-(\alpha/2)(\xi^2-1)}A_{N,b}^{-+}(\xi)e^{-\alpha\eta^2/2}\times B_{N,b}^{-+}(\eta)\ , \quad (30)$$

where $A_{N,b}^{-+}(\xi)$ and $B_{N,b}^{-+}(\eta)$ are odd polynomials of degree N. In fact, as will be shown, $A_{N,b}(\xi)$ and $B_{N,b}(\eta)$, are the Ince polynomials and in the following will be called $I_{N,b}(z)$.

B. The differential equation in elliptic coordinates

Now, we will find the eigenvalues and eigenfunctions of the problem by solving the differential equations which are solutions of (19a) and (19b). In fact, both are the same equation

$$\left[(z^2-1)\frac{d^2}{dz^2}+z\frac{d}{dz}-\alpha^2(z^2-1)z^2+\sigma^2k^2(z^2-1)-b\right]F(z)=0\ , \quad (31)$$

where $k^2=2mE/\hbar^2$ and $\alpha=m\omega\sigma^2/\hbar^2$.

The solutions of (31) can be written as

$$F(z)=\exp(-\alpha z^2/2)I^1(z)\ , \quad (32a)$$

$$F(z)=\exp(-\alpha z^2/2)(z^2-1)^{1/2}I^2(z)\ , \quad (32b)$$

where

$$I^1(z)=\sum_{l=0} a_l z^l \quad \text{(first kind)} \quad (33a)$$

$$I^2(z)=\sum_{l=0} c_l z^l \quad \text{(second kind)}\ . \quad (33b)$$

As expected, these solutions are consistent with the results of the previous discussions where only the symmetry properties of B were invoked.

The recurrence relations for the coefficients are

$$a_l(l^2+\alpha(2l+1)-\sigma^2k^2-b)+a_{l-2}[\sigma^2k^2+2\alpha(1-l)]=a_{l+2}(l+2)(l+1)\ , \quad (34a)$$

$$c_l[2l^2+\alpha(2l+1)-\sigma^2k^2-b+l+1]+c_{l-2}[2\sigma^2k^2+b+3\alpha-1-l(l+4\alpha-2)]+c_{l-4}[2\alpha(l-2)-\sigma^2k^2]=c_{l+2}(l+2)(l+1)\ . \quad (34b)$$

If k^2 and b are given, these recurrence relations determine the solutions for $I(z)$. However, we are interested in physical solutions which are bound states. This fact implies that I^1 and I^2 must be polynomials to ensure convergence as $z\to\infty$. In other words, we need k^2 and b values such that a_l (or c_l) must vanish for all l greater than a given n.

Let us first consider (34a). There are even and odd solutions (with even and odd n, respectively). Assuming n even ($n=2p$, $p=0,1,2\ldots$) we will have one even polynomial of degree n and $a_l=0$ for all $l>n$. Then, using (34a) for $l=n+2$, and as we will establish $a_{n+2}=0$,

$$a_n[\sigma^2k^2-2\alpha(n+1)]=a_{n+4}(n+4)(n+3)\ , \quad (35)$$

with $a_n\neq 0$ (for hypothesis); therefore in order to ensure that a_{n+4} vanishes:

$$\sigma^2k^2=2\alpha(2p+1)\ ,\qquad E=\hbar\omega(2p+1)\ , \quad (36)$$

for $p=0,1,2,\ldots$, which is the well-known quantum energy for the two-dimensional simple HO. Condition (36) provides $a_{n+4}=0$ and it is independent of b.

We must ensure $a_{n+2}=0$ now. For this we insert (36) in (34a) and we obtain

$$a_{2p}(4p^2-\alpha-b)+a_{2p-2}4\alpha=0\ . \quad (37)$$

By recurrency, starting from $a_0=1$ a polynomial expression can be found whose roots give the allowed values of b. We will call them characteristic values b_l^{1+}. It is easy to see that there are $p+1$ real characteristic values for each $n=2p$. It is important to remark that the resulting polynomial is just the equivalent to the eigenvalue equation of the operator $\hat{B}$ in the $++$ subspace, as described in Sec. III A. In this way the quantification condition for the observable $\hat{B}$ is obtained.

A similar analysis of (34a) for the odd solutions ($n=2p+1$, $p=0,1,2,\ldots$) leads to

$$\sigma^2k^2=2\alpha[(2p+1)+1]\ ,\qquad E=\hbar\omega[(2p+1)+1]\ . \tag{38}$$

In this case, starting from $a_1=1$ and using (34a), another $(p+1)$-degree polynomial can be obtained which corresponds to the eigenvalue equation of the Hermitian matrix $\hat{B}$ in the $-+$ subspace. As in the even-n case there exist $p+1$ characteristic values b_i^{1-} for each $n=2p+1$.

We turn about (34b) now. As before, there are even solutions and odd solutions. Assuming n even ($n=2p$, $p=0,1,2,\ldots$) so that $c_l=0$ for all $l>n$, using (34b) for $l=n+4$ and taking into account that we will require $c_{n+2}=c_{n+4}=0$, we obtain

$$c_n[2\alpha(n+2)-\sigma^2k^2]=c_{n+6}(n+6)(n+5)\ , \tag{39}$$

with $c_n\neq0$ (for hypothesis); therefore, for $p=0,1,2,\ldots$,

$$\sigma^2k^2=2\alpha(2p+2)=2\alpha[(2p+1)+1]\ ,\qquad E=\hbar\omega[(2p+1)+1]\ . \tag{40}$$

This is the same condition (38). However, (38) leads to odd polynomials of degree $2p+1$, while (40) leads to even polynomials of degree $2p$ times the factor $(z^2-1)^{1/2}$ [see (32b)].

We want $c_{n+2}=0$ now. That is, using (34b) and (40)

$$c_{2p}(8p^2-3\alpha+2p+1-b)+c_{2p-2}(11\alpha+b-1-4p^2+4p)-c_{2p-4}8\alpha=0\ . \tag{41}$$

By recurrency, starting from $c_0=1$ we obtain the equation that determines the $p+1$ real characteristic values b_i^{2+} which is equivalent to the eigenvalue equation of $\hat{B}$ in the $+-$ subspace.

Let us remark that the condition $c_{n+4}=0$, that is,

$$c_{2p}[3\alpha+b-4(p+1)p-1]-c_{2p-2}4\alpha=0\ , \tag{42}$$

is automatically fulfilled by the characteristic values b_i^{2+} determined through (41).

Analyzing (34b) for odd-n values (let us write $n=2p-1$, $p=1,2,\ldots$) and requiring $c_{n+2}=c_{n+4}=c_{n+6}=0$ it can be concluded that the allowed energies must satisfy, for $p=1,2,\ldots$,

$$\sigma^2k^2=2\alpha(2p+1)\ ,\qquad E=\hbar\omega(2p+1)\ , \tag{43}$$

which is the same condition (36). However, (36) leads to even polynomials of degree $2p$, while (43) leads to odd polynomials of degree $2p-1$ times the factor $(z^2-1)^{1/2}$ [see 32(b)]. As in the previous cases the condition $c_{n+2}=0$ leads to a p-degree polynomial whose roots will determine the characteristic values b_i^{2-}.

Table I summarizes the results of this section.

IV. BEHAVIOR OF THE CHARACTERISTIC VALUES

Table II shows the analytic expressions for the characteristic values corresponding to the states with $N=0$, 1, 2, and 3.

The ground state is obviously a $++$ state which also has $\hat{L}$ and $\hat{H}_x$ well defined. In this case $\hat{B}$ can be represented by a 1×1 matrix and the only characteristic value b_0^{1+} is trivially obtained.

For the first excited state, the $\hat{B}$ matrix is also diagonal in the Cartesian base. The $+-$ state corresponds to the $n_x=0$, $n_y=1$ state with a characteristic value $b_1^{2+}=1-3\alpha$ and the wave function is the product of two even polynomials $I^2_{1,1-3\alpha}(z)$. On the other hand, the $-+$ state corresponds to the $n_x=1$, $n_y=0$ Cartesian state and has a characteristic value $b_1^{1-}=1-\alpha$, while the associated polynomials are odd polynomials of the first kind.

TABLE I. Summary of the properties of the wave functions obtained for the two-dimensional HO and their associated eigenfunctions. See the Appendix for the evaluation of the normalization constants.

	Parity x	Parity y	Wave Functions	Degree of P	Characteristic values
N even	$+$	$+$	$I^1_{N,b}(\xi)I^1_{N,b}(\eta)\exp\left[(\xi^2+\eta^2-1)\left[-\frac{\alpha}{2}\right]\right]$	N	b_i^{1+}, $i=0,\ldots,N/2$
	$-$	$-$	$I^2_{N,b}(\xi)(\xi^2-1)^{1/2}I^2_{N,b}(\eta)(1-\eta^2)^{1/2}\exp\left[(\xi^2+\eta^2-1)\left[-\frac{\alpha}{2}\right]\right]$	$N-1$	b_i^{2-}, $i=1,\ldots,N/2$
N odd	$-$	$+$	$I^1_{N,b}(\xi)I^1_{N,b}(\eta)\exp\left[(\xi^2+\eta^2-1)\left[-\frac{\alpha}{2}\right]\right]$	N	b_i^{1-}, $i=1,\ldots,(N+1)/2$
	$+$	$-$	$I^2_{N,b}(\xi)(\xi^2-1)^{1/2}I^2_{N,b}(\eta)(1-\eta^2)^{1/2}\exp\left[(\xi^2+\eta^2-1)\left[-\frac{\alpha}{2}\right]\right]$	$N-1$	b_i^{2+}, $i=1,\ldots,(N+1)/2$

TABLE II. Characteristic values corresponding to the states with $N=0$, 1, 2, and 3.

N	Number of states	Characteristic values	Cylindric values
0	1	$b_0^{1+}=-\alpha$	0
1	2	$b_1^{1-}=1-\alpha$	1
		$b_1^{2+}=1-3\alpha$	1
2	3	$b_0^{1+}=-3\alpha+2-2(\alpha^2+1)^{1/2}$	0
		$b_1^{2-}=4-3\alpha$	4
		$b_1^{1+}=-3\alpha+2+2(\alpha^2+1)^{1/2}$	4
3	4	$b_1^{1-}=-3\alpha+5-2(\alpha^2-2\alpha+4)^{1/2}$	1
		$b_1^{2+}=-5\alpha+5-2(\alpha^2+2\alpha+4)^{1/2}$	1
		$b_2^{1-}=-3\alpha+5+2(\alpha^2-2\alpha+4)^{1/2}$	9
		$b_2^{2+}=-5\alpha+5+2(\alpha^2+2\alpha+4)^{1/2}$	9

The $N=2$ state has also a trivial solution for the $--$ subspace. The corresponding Cartesian state $n_x=1$, $n_y=1$ has the characteristic value $b_1^{2-}=4-3\alpha$ and the associated polynomials are odd polynomials of the second kind. On the other hand, the $++$ subspace has dimension 2 and the corresponding characteristic values must be obtained by diagonalization of $\hat{B}$ which can be easily carried out leading to the b_0^{1+} and b_1^{1+}. These are the lowest states without well-defined n_x.

For the case $N=3$, both $-+$ and $+-$ subspaces have dimension 2 and we obtain b_1^{1-},b_2^{1-} and b_1^{2+},b_2^{2+} by diagonalization of two 2×2 matrices.

Let us remark the correct behavior of the characteristic values when the limit $\alpha\to0$ is taken (cylindric limit). As $\hat{B}\to\hat{L}^2$ and the cylindric eigenstates have L well defined ($L=m\hbar$ where $m=\pm N,\pm(N-2),\ldots,0$ if N even or $m=\pm N,\pm(N-2),\ldots,\pm1$ if N odd) the characteristic values will be as follows.

(i) N even:

$$b_i^{1+}=0,4,16,\ldots,N^2\ ,$$

$$b_i^{2-}=4,16,\ldots,N^2\ .$$

(ii) N odd:

$$b_i^{1-}=1,9,25,\ldots,N^2\ ,$$

$$b_i^{2+}=1,9,25,\ldots,N^2\ .$$

We can see that the characteristic values (except the 0 value b_0^{1+}) are doubly degenerate [see also Fig. 4(a)].

Figures 4(b)–4(d) display the distribution of the characteristic values for a wide range of N for $\alpha=1$, 2, and 4. One remarkable point is the increasing number of negative characteristic values when α increases. We recall that α is a measure of the deformation of the used elliptical system (in fact, it is a ratio between the focal distance and the size parameter β^{-1} of the HO).

It is also of interest to note the behavior of the characteristic values when they cross the $b=0$ axis that classically correspond to the separatrix $B=0$ which was mentioned in Sec. II. For large positive b_i values ($i\to N/2$), they appear to be degenerate, that is $b_i^{1+}\approx b_i^{2-}$ for N even and $b_i^{1-}\approx b_i^{2+}$ for N odd. These values are split near the $b=0$ axis and this effect is illustrated in Fig. 4(d) where the lines of constant i were drawn. For even N, $b_i^{1+}>b_i^{2-}$ and for odd N, $b_i^{1-}>b_i^{2+}$. Finally for negative values of b the characteristic values corresponding to first-kind eigenfunctions b_i^{1+} and b_{i+1}^{1-} display both the same linear dependence with N, whereas another linear function is followed by the b_i^{2+} and b_i^{2-} second-kind characteristic values.

V. CONCLUDING REMARKS

In the present work we have analyzed the problem of the two-dimensional simple harmonic oscillator when it is separated in elliptic coordinates. We have seen that this is the only physical system, with a central potential (apart from the case $V=$const) which results in being separable in such coordinates and this property remains whatever the chosen focal distance. We solved the classical problem finding the appropriate constant of motion that allows us to display the Poincaré sections. From the quantum-mechanical point of view, the Schrödinger equation leads to the differential equation

$$(z^2-1)F''(z)+zF'(z)+[-\alpha^2(z^2-1)z^2+\sigma^2k^2(z^2-1)-b]F(z)=0\ ,$$

which admits physical solutions associated with polynomials if the condition

$$\sigma^2k^2=2\alpha(N+1)$$

is fulfilled, N being an integer number (providing the quantification energy condition) and the separability constant b takes only certain values, the characteristic values, that provides the quantum condition for a new observable $\hat{B}$ which commutes with the Hamiltonian. These characteristic values, whose general behavior was also displayed in this work, can be determined by diago-

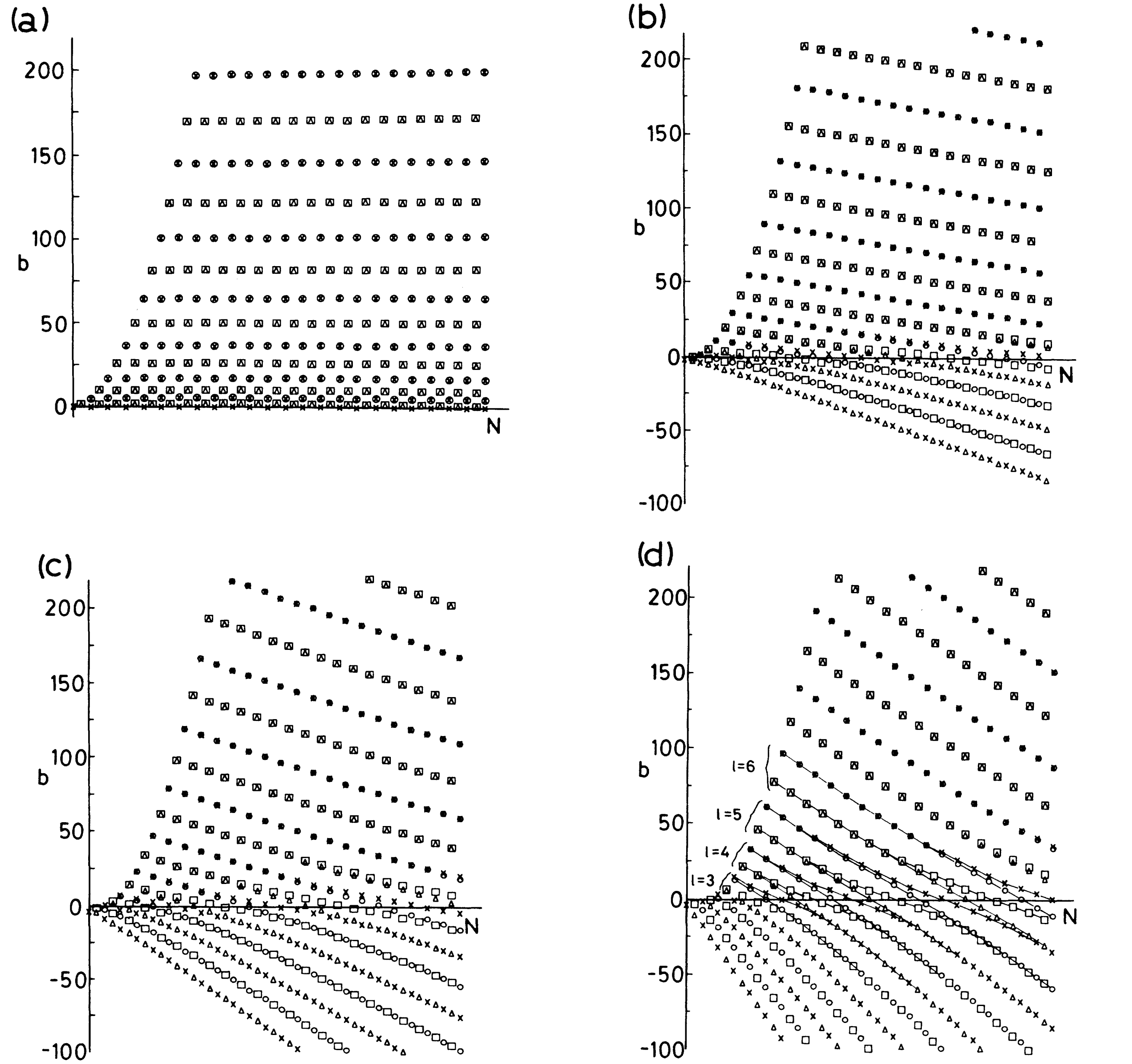

FIG. 4. Characteristic values b vs N. The cross corresponds to b^{1+}, the circle to b^{2-}, the square corresponds to b^{2+}, and the triangle to b^{1-}. (a) $\alpha=0$, (b) $\alpha=1$, (c) $\alpha=2$, (d) $\alpha=4$.

nalization of $\hat{B}$ in the Cartesian base or by using recurrence relations and a truncation condition which ensures convergence when $z \to \infty$.

Let us finally remark that analogous procedures can also be applied to other two-dimensional problems with elliptic symmetry or even to three-dimensional physical systems such as the three-dimensional simple HO problem which leads to separable equations if one uses prolate (or oblate) ellipsoidal coordinates.

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APPENDIX: NORMALIZATION OF THE WAVE FUNCTIONS

If we want to use the discussed wave functions as an orthonormal base of the Hilbert space, we must normalize them, that is,

$$\Psi(\xi,\eta)=\mathcal{N}_{N,b}R_{N,b}(\xi)\theta_{N,b}(\eta) \ , \tag{A1}$$

where $\mathcal{N}_{N,b}$ are normalization constants which can be determined by integration over all the space, that is,

$$1=\int_{\xi=1}^{\infty}\int_{\eta=-1}^{1}|\Psi(\xi,\eta)|^2\sigma^2(\xi^2-\eta^2)d\xi\, d\eta \ . \tag{A2}$$

However, as we will show, the normalization constants can also be determined as follows.

Let us consider states with even N whose wave functions belong to the first class:

$$\langle \xi\eta|Nb\rangle = \mathcal{N}_{N,b}\exp[-(\alpha/2)(\xi^2+\eta^2-1)]I^1_{N,b}(\xi) \times I^1_{N,b}(\eta) . \quad (A3)$$

As the $I^1_{N,b}$ polynomials have been constructed by setting $a_0=1$, the independent term of the total wave function (A3), aside from the exponential factor, is just the normalization constant $\mathcal{N}_{N,b}$. On the other hand, the coefficients $C^{n_x}_{N,b}$ of the expansion (25) verify

$$\sum_{n_x} |C^{n_x}_{N,b}|^2 = 1 . \quad (A4)$$

Therefore we can determine $\mathcal{N}$ by searching for the independent term of (25) which for N even and solutions of the first kind results in

$$\mathcal{N}_{N,b} = \frac{\beta}{(\pi 2^N)^{1/2}} \sum_{n_x} C^{n_x}_{N,b} \frac{1}{\sqrt{n_x!(N-n_x)!}} h_0^{(n_x)} \times D_{N-n_x}(\sqrt{\alpha}) , \quad (A5)$$

where D_n are even n-degree polynomials defined as

$$D_n(z) = \sum_{l=0}^{n} h_l^{(n)}(-1)^{l/2} z^l , \quad (A6)$$

and $h_l^{(n)}$ is the l coefficient of the n-degree Hermite polynomial.

In the way we can find the normalization constants for the even-N second-kind solutions. In this case we must search for the term of degree 1 and therefore,

$$\mathcal{N}_{N,b} = \frac{\beta\alpha}{(\pi 2^N)^{1/2}} \sum_{n_x} C^{n_x}_{N,b} \frac{1}{\sqrt{n_x!(N-n_x)!}} h_1^{(n_x)} \times E_{N-n_x-1}(\sqrt{\alpha}) , \quad (A7)$$

where the even polynomials $E_m(z)$ are defined as

$$E_m(z) = \sum_{l}^{m} h_l^{(m+1)}(-1)^{l/2} z^l . \quad (A8)$$

For the odd-N first-kind solutions we obtain

$$\mathcal{N}_{N,b} = \frac{\beta\sqrt{\alpha}}{(\pi 2^N)^{1/2}} \sum_{n_x} C^{n_x}_{N,b} \frac{1}{\sqrt{n_x!(N-n_x)!}} h_1^{(n_x)} \times D_{N-n_x}(\sqrt{\alpha}) , \quad (A9)$$

and for the odd-N second-kind we find

$$\mathcal{N}_{N,b} = \frac{\beta\sqrt{\alpha}}{(\pi 2^N)^{1/2}} \sum_{n_x} C^{n_x}_{N,b} \frac{1}{\sqrt{n_x!(N-n_x)!}} h_0^{(n_x)} \times E_{N-n_x-1}(\sqrt{\alpha}) . \quad (A10)$$

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