Nonrelativistic quaternionic quantum mechanics in one dimension

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(Received 12 June 1989)

We present a formalism for treating one-dimensional problems in quaternionic quantum mechanics. As an example, we derive an explicit form for the T matrix for scattering from a square (quaternionic) barrier, and use this result to calculate the transmission and reflection coefficients. We show that the qualitative form of these coefficients is the same as in complex quantum mechanics, even when the barrier has a nonzero value for the quaternionic components of the potential.

I. INTRODUCTION

Quaternionic quantum mechanics (QQM) has been considered as a general structure by a number of authors.[1] Adler has recently described scattering theory in QQM, and has shown that the quaternionic part of the wave function does not contribute to the spatially asymptotic states.[2,3] In particular, he has shown that the optical potential describing scattering restricted to the complex projection of the wave function (where the complex projection is defined by the kinetic Hamiltonian) is a Hermitian operator, and thus that the S matrix on the complex states is unitary. No flux is lost to the quaternionic states. In this paper we analyze the one-dimensional scattering problem in QQM.

We begin with the Schrödinger equation of QQM

$$\frac{\partial\Psi}{\partial t}=-\tilde{H}\Psi \ , \tag{1}$$

where H is an anti-Hermitian quaternionic operator, and Ψ is a quaternionic wave function. It is easy to show that in the time-independent case there always exists a suitable choice of quaternionic phase such that the solution of (1) may be written in the form

$$\Psi=\Phi e^{-iEt} \tag{2}$$

(we set $h/2\pi=1$ throughout) for real E, which satisfies[2]

$$\tilde{H}\Phi=\Phi iE \ . \tag{3}$$

Note that the order of multiplication in these expressions is important due to the noncommutative nature of quaternionic multiplication. For a one-dimensional problem we may write the Hamiltonian as (we have set the mass in the problem equal to $\frac{1}{2}$ in appropriate units[2])

$$\tilde{H}=-i\frac{\partial^2}{\partial x^2}+\tilde{V} \ , \tag{4}$$

so that the kinetic terms and the eigenvalues of $\tilde{H}$ are both in the complex subspace $\mathbb{C}(1,i)$. It is possible to construct conserved currents from Eqs. (1) and (4) in a manner entirely analogous to the usual complex case. One obtains in this fashion the continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial J}{\partial x}=0 \ , \tag{5}$$

with the probability density ρ and current density J given by

$$\rho(x,t)=\Psi^*\Psi, \quad J(x,t)=\Psi^* i\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}i\Psi \ , \tag{6}$$

where the asterisk represents quaternionic conjugation.

We now rewrite the Schrödinger equation using the symplectic representation of quaternions, writing a general quaternion q as[2]

$$q=c_\alpha+jc_\beta \ ,$$

where c_α and c_β are complex [i.e., are elements of $\mathbb{C}(1,i)$], and j is the usual quaternionic unit ($j^2=k^2=-1$, $ij=-ji=k$).

We then write

$$\tilde{V}=i(V_\alpha+jV_\beta) \ , \tag{7}$$

where, because of the anti-Hermiticity of $\tilde{H}$ and thus $\tilde{V}$, V_α is real for local potentials, and V_β may be complex. (This ensures unitarity, since the i component of the potential is set to zero. The j and k components of $\tilde{V}$ do not act as sources and sinks of flux since we have chosen the unit i to describe the time dependence.) We similarly write the wave function

$$\Phi=\Phi_\alpha+j\Phi_\beta \ .$$

The time-independent Schrödinger equation may then be rewritten as a pair of coupled complex equations

$$\begin{aligned}\left[-\frac{d^2}{dx^2}+V_\alpha\right]\Phi_\alpha-V_\beta^*\Phi_\beta&=E\Phi_\alpha \ ,\\ \left[\frac{d^2}{dx^2}-V_\alpha\right]\Phi_\beta-V_\beta\Phi_\alpha&=E\Phi_\beta \ ,\end{aligned} \tag{8}$$

where the reality of V_α has been used. Adler has discussed the scattering theory of (8) in the general case.[2] Here we specialize to the one-dimensional problem, and rewrite (8) as a system of first-order equations:

$$\frac{d}{dx}\Omega=A\Omega \ . \tag{9}$$

Here Ω is the column vector $(\Phi_\alpha,\Phi'_\alpha,\Phi_\beta,\Phi'_\beta)^T$, where the primes indicate differentiation with respect to x, and A is a matrix given by

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ V_\alpha - E & 0 & -V_\beta^* & 0 \\ 0 & 0 & 0 & 1 \\ V_\beta & 0 & V_\alpha + E & 0 \end{pmatrix} . \quad (10)$$

We can apply the general theory of linear differential equations to (10).[4] The general solution may be written using spatially ordered exponentials:

$$\Omega(x)=\left[P_x \exp\left[\int_{x_0}^{x} A(s)ds\right]\right]\Omega(x_0)$$
$$=\left[1+\int_{x_0}^{x} A(s)ds + \frac{1}{2!}\int_{x_0}^{x} A(s_1)ds_1 \int_{x_0}^{s_1} A(s_2)ds_2 + \cdots\right]\Omega(x_0) . \quad (11)$$

This solution must be used for potentials that are a function of space, but for the piecewise constant potentials we will henceforth be concerned with there are more straightforward methods available.

II. CONSTANT POTENTIAL SOLUTION

In a region of constant potential we may write the solution (11) as

$$\Omega(x)=\exp[A(x-x_0)]\Omega(x_0)$$
$$=\exp(Ax)B , \quad (12)$$

where B is a constant vector selected to satisfy the initial conditions. One may then diagonalize A

$$A=P\gamma P^{-1} ,$$
$$\exp(Ax)=P\exp(\gamma x)P^{-1} , \quad (13)$$

where $\gamma=\mathrm{diag}(i\mu,-i\mu,\nu,-\nu)$, with μ and ν the eigenvalues of A, given by

$$\mu=[(E^2-|V_\beta|^2)^{1/2}-V_\alpha]^{1/2} ,$$
$$\nu=[(E^2-|V_\beta|^2)^{1/2}+V_\alpha]^{1/2} . \quad (14)$$

This leads to the expression

$$\Omega(x)=P\exp(\gamma x)C\equiv\Theta(x)C \quad (15)$$

for some suitable coefficient vector $C=(C_1,C_2,C_3,C_4)$. The matrix P may be written as

$$P=\begin{pmatrix} 1 & 1 & \frac{-V_\beta^*}{R+E} & \frac{-V_\beta^*}{R+E} \\ i\mu & -i\mu & \frac{-\nu V_\beta^*}{R+E} & \frac{\nu V_\beta^*}{R+E} \\ \frac{-V_\beta}{R+E} & \frac{-V_\beta}{R+E} & 1 & 1 \\ \frac{-i\mu V_\beta}{R+E} & \frac{i\mu V_\beta}{R+E} & \nu & -\nu \end{pmatrix} , \quad (16)$$

where $R=(E^2-|V_\beta|^2)^{1/2}$.

We now make some comments regarding the limit of these equations as $V_\beta \to 0$. One sees from Eqs. (8) that the differential equations for the symplectic components Φ_α and Φ_β decouple when the potential has no j or k component. Thus Φ_α, which is the part of the wave function in $\mathbb{C}(1,i)$, is independent of the purely quaternionic part Φ_β. In a region of free space, where $V_\alpha=0$ as well, Eqs. (8) have the solutions

$$\Phi_\alpha=C_1 e^{ikx}+C_2 e^{-ikx}, \quad \Phi_\beta=C_3 e^{kx}+C_4 e^{-kx} , \quad (17)$$

where $k=\sqrt{|E|}$ is the wave number. From the form of the solutions (17) we see that the quaternionic parts fall off exponentially in the free-space regions, justifying the statement that the asymptotic states are in $\mathbb{C}(1,i)$. When $V_\beta=0$ the matrix P given above takes the block diagonal form (the subscript denotes the free-space case)

$$P_0=\begin{pmatrix} 1 & 1 & 0 & 0 \\ i\mu & -i\mu & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \nu & -\nu \end{pmatrix} . \quad (18)$$

We shall see in Sec. III how the results of complex quantum mechanics are retrieved from this form. One other point requires comment. With $V_\beta=0$ and $V_\alpha \neq 0$ the eigenvalue ν in (14) depends on the *sum* of E and V_α, whereas μ depends on the *difference* $E-V_\alpha$. In complex quantum mechanics (CQM) only the quantity μ appears, and any results derived do not depend on the zero of energy, only the difference of energies. The origin of the dependence on $E+V_\alpha$ has been commented upon by Adler, and may be traced back to the signs in the Schrödinger equation (8).[2,3] In CQM a shift in the zero of the energy can always be compensated for by shifting the phase of the wave function. No such freedom exists in QQM, since the net result of such a phase shift is to introduce a time-dependent phase into the potential. That is, shifting the zero of the energy is not consistent with the time-independent theory. We follow Adler and take the value of the potential to be zero at spatial infinity, and measure the energy E of the particle relative to this.

III. TRANSFER MATRIX

Using the results of Sec. II we are now in a position to calculate the transfer matrix for a square barrier in QQM. We take the most general case of a square barrier with

$$V(x)=\begin{cases} 0, & x<a \\ V_\alpha^0+jV_\beta^0, & a<x<b , \\ 0, & x>b \end{cases} \quad (19)$$

with V_α^0 a real constant and V_β^0 a complex constant. The matching conditions at the boundaries may be written as

$$\Omega(a^-)=\Omega(a^+), \quad \Omega(b^-)=\Omega(b^+) . \quad (20)$$

To calculate the reflection and transmission coefficients for a wave incident from the left of the barrier, we take the wave function in the potential free regions to be of the form

$$\Phi_\alpha(x)=\begin{cases} e^{ikx}+re^{-ikx}, & x<a \\ te^{ikx}, & b<x \end{cases}$$
$$\Phi_\beta(x)=\begin{cases} r'e^{kx}, & x<a \\ t'e^{-kx}, & x>b \;. \end{cases} \tag{21}$$

Note the appearance in (21) of the extra coefficients r' and t', which represent the exponentially decaying parts unique to QQM. In the region $a<x<b$ the solution is given by Eqs. (15) and (16). Denoting by $\Theta_0(x)=P_0\mathrm{diag}(e^{ikx},e^{-ikx},e^{kx},e^{-kx})$ the matrix Θ in the potential free region, the matching conditions (20) may be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \\ t' \end{bmatrix}=\Theta_0^{-1}(b)\Theta(b)\Theta^{-1}(a)\Theta_0(a)\begin{bmatrix} 1 \\ r \\ r' \\ 0 \end{bmatrix} . \tag{22}$$

We can now identify the quantity

$$\Theta_0^{-1}(b)\Theta(b)\Theta^{-1}(a)\Theta_0(a)$$

as the transfer matrix for this problem. Before proceeding we briefly review the transfer matrix for the corresponding CQM problem.[5]

Following a procedure analogous to that outlined above, using the complex Schrödinger equation one may derive the relation

$$C_t=\begin{bmatrix} T_{11} & T_{12} \\ T_{12*} & T_{11*} \end{bmatrix} C_i\equiv TC_i \;,$$
$$T_{11}=e^{-ik(b-a)}\left[\cos(b-a)\mu+\frac{i}{2}\left[\frac{\mu}{k}+\frac{k}{\mu}\right]\sin(b-a)\mu\right] , \tag{23}$$
$$T_{12}=\frac{i}{2}\left[\frac{\mu}{k}-\frac{k}{\mu}\right]e^{-i(b+a)k}\sin(b-a)\mu \;.$$

Here $\mu=\sqrt{E-V_0}$ is the wave number in the region of the potential, where V_0 is the height of the barrier, and the coefficient vectors for the incident and transmitted wave functions, respectively, are given by

$$C_i=\begin{bmatrix} 1 \\ r \end{bmatrix}, \quad C_t=\begin{bmatrix} t \\ 0 \end{bmatrix} . \tag{24}$$

From these expressions one may easily check that $\det(T)=1$ and that $|r|^2+|t|^2=1$. The complex current is given by the quantity

$$J(x)=i\left[\Psi^*\frac{d\Psi}{dx}-\frac{d\Psi^*}{dx}\Psi\right]$$
$$=\left[\Psi^*,\frac{d\Psi^*}{dx}\right]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \Psi \\ \dfrac{d\Psi}{dx} \end{bmatrix} . \tag{25}$$

Now the first of Eqs. (23) and Eq. (25) may be used to derive the necessary condition on T for current conservation on either side of the barrier:

$$T^\dagger\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}T=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \tag{26}$$

We shall see that there exist analogous quaternionic results. The complex T matrix of (23) appears in the limit as $V_\beta\to 0$ as the upper left 2×2 sector of the QQM T matrix to be presented below.

We introduce the QQM T matrix via the relation

$$\begin{bmatrix} t \\ 0 \\ 0 \\ t' \end{bmatrix}=T\begin{bmatrix} 1 \\ r \\ r' \\ 0 \end{bmatrix} . \tag{27}$$

The current is given by

$$J(x)=C^\dagger\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}C\equiv C^\dagger MC \;. \tag{28}$$

Equations (27) and (28) together yield the result $|r|^2+|t|^2=1$, showing that the incoming flux is entirely in the outgoing complex parts of the wave function. Using the current conservation condition (5) in the time-independent case, and the relationship (27), we obtain the current conservation condition

$$T^\dagger MT=M \;. \tag{29}$$

The current conservation condition (29) leads to some relationships between the elements of T. One may derive further relationships by setting $a=-b$ for the positions of the barrier steps, and then using the invariance of the Schrödinger equation under the reflection $x\to -x$. (See Chap. 6 of Ref. 6 for the corresponding treatment in CQM.) In the QQM case we cannot impose further relations from time-reversal invariance, since the Schrödinger equation does not admit a time-reversed solution $\Psi^*(x,-t)$ when V_β is nonzero.

In terms of the components of the 4×4 matrix T the reflection and transmission coefficients are given by

$$r=\frac{T_{21}T_{33}-T_{23}T_{31}}{T_{32}T_{23}-T_{22}T_{33}} ,$$
$$t=T_{11}+\frac{T_{12}(T_{33}T_{21}-T_{23}T_{31})+T_{13}(T_{22}T_{31}-T_{32}T_{21})}{T_{32}T_{23}-T_{22}T_{33}} , \tag{30}$$

with similar expressions for r' and t'. After some straightforward but tedious algebra, using Eqs. (15), (16), (18), and (22), we arrive at an expression for T. We present here only some representative elements, since we do not require the explicit form of T. The remaining elements may be found in the Appendix.

$$T_{11}=\frac{e^{-ik(b-a)}}{4R}\left\{(E+R)\left[2\cos\theta+i\left[\frac{\mu}{k}+\frac{k}{\mu}\right]\sin\theta\right]-\frac{|V_\beta|^2}{(R+E)}\left[2\cosh\phi+i\left[\frac{k}{\nu}-\frac{\nu}{k}\right]\sinh\phi\right]\right\},$$

$$T_{31}=e^{-kb+iak}\frac{V_\beta}{4R}\left[(1-i)\cos\theta+\left[\frac{k}{\mu}+i\frac{\mu}{k}\right]\sin\theta-(1-i)\cosh\phi-\left[\frac{k}{\nu}-i\frac{\nu}{k}\right]\sinh\phi\right], \quad (31)$$

$$T_{33}=\frac{e^{-k(b-a)}}{4R}\left\{(R-E)\left[2\cos\theta+\left[\frac{k}{\mu}-\frac{\mu}{k}\right]\sin\theta\right]+(R+E)\left[2\cosh\phi+\left[\frac{k}{\nu}+\frac{\nu}{k}\right]\sinh\phi\right]\right\},$$

where $\theta=(b-a)\mu$ and $\phi=(b-a)\nu$. The determinant of T can be seen to be unity by construction, and we have also checked that Eq. (29) is satisfied.

We observe that in the CQM limit $V_\beta\to 0$, the above expression for T_{11} reduces to the expression from CQM in (23), as it must, while $T_{31}\to 0$. (In this limit $R\to E$.) In general, in the CQM limit the matrix T becomes block diagonal, with the upper left 2×2 submatrix being the T matrix of Eq. (23). The lower 2×2 matrix describes the quantum mechanics of the decoupled quaternionic part Φ_β of the wave function, and basically is of similar form to the upper part, with hyperbolic functions and real exponential rather than trigonometric functions and complex exponentials.

IV. REFLECTION FROM SQUARE BARRIERS

Using the first of Eqs. (30) and the elements of T, we can calculate the following lengthy expression for the reflection coefficient for a wave incident from the left on a general square barrier, as given in Eq. (19):

$$r=\frac{r_1}{r_2},$$

where

$$r_1=4i|V_\beta|^2\left[1-\cos\theta\left[\cosh\phi+\frac{\beta R}{\Sigma}\sinh\phi\right]\right]+4i\alpha R\,\Sigma\sin\theta\cosh\phi$$
$$+\left[i\alpha\beta\left[\Sigma^2-\frac{D}{\Sigma}|V_\beta|^2\right]-2i|V_\beta|^2\left[\frac{k^2}{\mu\nu}-\frac{\mu\nu}{k^2}\right]\right]\sin\theta\sinh\phi,$$

$$r_2=4|V_\beta|^2+2\cos\theta\left\{2\cosh\phi\left[-2R\Sigma+\frac{D}{\Sigma}|V_\beta|^2\right]-2R\sinh\phi\left[\beta\Sigma+\frac{i}{\Sigma}|V_\beta|^2\left[\frac{k}{\nu}-\frac{\nu}{k}\right]\right]\right\}$$
$$-2|V_\beta|^2\sin\theta\left[\frac{\mu}{k}+\frac{ik}{\mu}\right]\left[(1+i)\cosh\phi+\left[\frac{k}{\nu}+\frac{i\nu}{k}\right]\sinh\phi\right]$$
$$+i\sin\theta\left\{\Sigma^2\left[\frac{\mu}{k}+\frac{k}{\mu}\right](2\cosh\phi+\beta\sinh\phi)-\frac{D}{\Sigma}|V_\beta|^2\alpha\left[2i\cosh\phi+\left[\frac{k}{\nu}-\frac{\nu}{k}\right]\sinh\phi\right]\right\}. \quad (32)$$

Here we have set $\alpha=\mu/k-k/\mu$, $\beta=\nu/k+k/\nu$, $\Sigma=R+E$, and $D=R-E$.

The expressions, while not particularly illuminating, have some interesting features. First we check that the correct limits are obtained. When we let $V_\beta\to 0$, we obtain the result

$$r(V_\beta=0)=\frac{i\left[\frac{\mu}{k}-\frac{k}{\mu}\right]\sin\theta}{2\cos\theta-i\left[\frac{\mu}{k}+\frac{k}{\mu}\right]\sin\theta}, \quad (33)$$

which is the expression for r in CQM. Also, if we set θ and $\phi=0$, corresponding to the case of a finite height barrier of zero width, $r_1=0$ as required.[6] We have also checked that the results (32) reduce to the expression obtained by Adler[3] when we take the δ-function limit $(b-a)V\to\Omega$, where Ω is a constant quaternion, as the width of the barrier $(b-a)\to 0$.

The expression (33) vanishes whenever $\theta=n\pi$, corresponding to transmission resonances in CQM. However, putting $\theta=n\pi$ in (32) produces a nonzero result. To see what is going on we look at (32) for ϕ large (wide barrier or high energy), so that $\sinh\phi\approx\cosh\phi$ and we obtain

$$r(\theta=n\pi)=\frac{i|V_\beta|^2\left[1+\dfrac{\beta R}{R+E}\right]}{R(R+E)(2+\beta)}+O\left[\frac{|V_\beta|^4}{E^4}\right]. \qquad (34)$$

We thus see that the transmission resonances of CQM are "filled in" when $V_\beta \neq 0$. What actually happens is that the behavior of the transmission function as a function of energy is qualitatively the same as the CQM case. However, the positions of the zeros in $|r|^2$ are shifted in energy from the CQM case. In other words, there are still transmission resonances (at least for relatively small values of $|V_\beta|$), but they are no longer exactly at $\theta=n\pi$. In Figs. 1(a) and 1(b) we present some typical numerical results. The curves marked C are $|r|^2$ as a function of energy in CQM for a square barrier. For the curves labeled Q we have taken $V_\alpha^0=10$ units, and $|V_\beta^0|=1$ unit and 10 units, respectively. To set the height of the barriers for the complex cases we have taken a height such that the threshold energy, where the energy of the incoming particle becomes equal to the barrier height, or alternately, the wave number in the region $a<x<b$ becomes real, is the same for each curve. From Eq. (14) we see that in the quaternionic case this requires

$$E^2>(V_\alpha^0)^2+|V_\beta^0|^2 . \qquad (35)$$

Thus for Fig. 1(a), when $V_\alpha^0=10$ units and $|V_\beta^0|=1$ unit, the threshold energy is simply $\sqrt{101}$ units. Similarly, for Fig. 1(b) the threshold energy is $\sqrt{200}$ units. These are then taken to be the respective heights of the barriers for the comparative complex cases.

It is interesting to see in Fig. 1 that a value of $|V_\beta^0|$ equal to 10% of the real part of the potential produces an effect in the reflection coefficient that is quite small. We see a larger effect in Fig. 1(b) for $|V_\beta^0|=V_\alpha^0$, but qualitatively there are no obvious differences between QQM and CQM.

One may then ask if this result carries over to the case of a sequence of square barriers of different heights (of both the real and quaternionic parts). In fact, this was the initial motivation for this work. We were unable to obtain tractable expressions for the double-barrier case. The T matrix for such a compound barrier, as calculated using REDUCE,[7] resulted in an output over 3000 lines long. We thus turned to numerical evaluation of the reflection and transmission coefficients using products of the T matrix we calculated for the single barrier. We found that the reflection coefficient was, in general, phase shifted when the order of the two barriers was reversed, but that the magnitude was unchanged. Unfortunately, this is also true, in general, of CQM, so we cannot make any qualitative predictions for possible experiments to determine the existence or otherwise of quaternionic potentials in nature on the basis of these simple considerations. We also investigated the effect of introducing a third square barrier, again finding no change in the magnitude of $|r|^2$ when the order of the barriers was reversed.

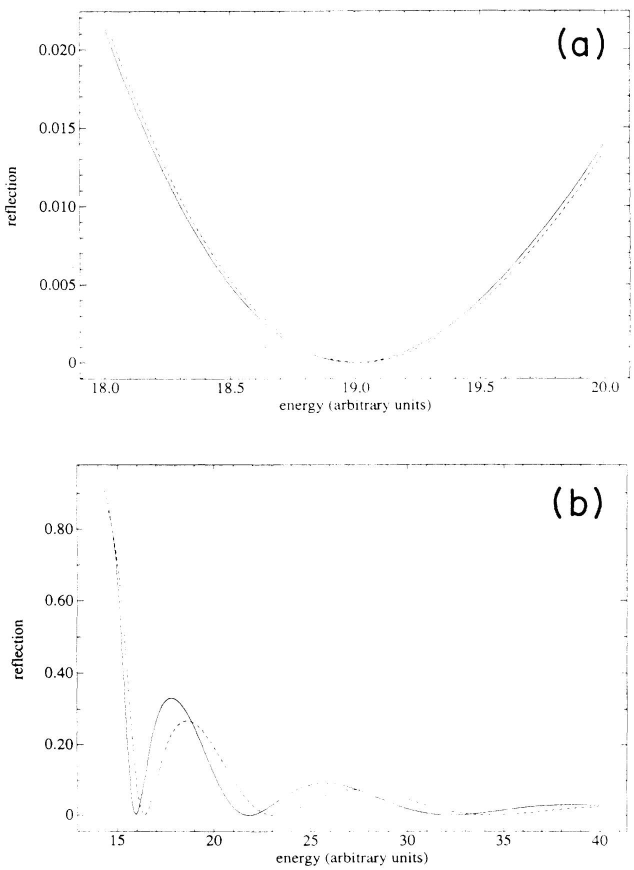

FIG. 1. (a) Reflection $|r|^2$ as a function of energy for CQM (dashed curve) with a square barrier of height $\sqrt{101}$ units, and the same quantity in QQM (solid curve) with $V_\beta^0=1$ unit and $V_\alpha^0=10$ units, to give the same threshold energy. Here we have shown the curves only near one of the minima in order to show the differences. The overall shape of the curves is similar to those in (b). We see that the position of the minimum (transmission resonance) is shifted slightly. (b) Same as (a) with a square barrier of height $\sqrt{200}$ units in the complex case and $V_\beta^0=10$ units and $V_\alpha^0=10$ units in the QQM case. Now the minima have been shifted considerably, but the qualitative form of the curves is still similar.

V. CONCLUSION

We have presented a simple formalism for treating wave-mechanical problems in quaternionic quantum mechanics. We have constructed the transfer matrix for a square barrier, and shown explicitly how complex quantum mechanics emerges in the case of a real potential. We have shown how one may calculate reflection and transmission coefficients for square barriers using the transfer matrix, and shown some typical results.

ACKNOWLEDGMENTS

One of us (A.J.D.) would like to thank G. C. Joshi and C. G. Nash for some useful discussions during the course of this work. (B.H.J.M.) would like to thank S. L. Adler and C. G. Nash for helpful discussions.

APPENDIX

Here we present the remaining 13 elements of the matrix T, as introduced in the text. The elements T_{11}, T_{31}, and T_{33} appear as Eq. (31) in the main text:

$$T_{12}=\frac{ie^{-i(b+a)k}}{4R}\left[(R+E)\left[\frac{\mu}{k}-\frac{k}{\mu}\right]\sin\theta+\frac{|V_\beta|^2}{(R+E)}\left[\frac{k}{\nu}+\frac{\nu}{k}\right]\sinh\phi\right],$$

$$T_{13}=\frac{V_\beta^* e^{k(a-ib)}}{4R}\left[-(1+i)\cos\theta+\left[\frac{\mu}{k}-\frac{ik}{\mu}\right]\sin\theta+(1+i)\cosh\phi+\left[\frac{\nu}{k}+\frac{ik}{\nu}\right]\sinh\phi\right],$$

$$T_{14}=\frac{V_\beta^* e^{-k(a+ib)}}{4R}\left[(1-i)\cos\theta+\left[\frac{\mu}{k}+\frac{ik}{\mu}\right]\sin\theta-(1-i)\cosh\phi+\left[\frac{\nu}{k}-\frac{ik}{\nu}\right]\sinh\phi\right],$$

$$T_{21}=T_{12}^*,\quad T_{22}=T_{11}^*,\quad T_{23}=-\frac{V_\beta^*}{V_\beta}T_{13}^*,\quad T_{24}=-\frac{V_\beta^*}{V_\beta}T_{14}^*,\quad T_{32}=-\frac{V_\beta}{V_\beta^*}T_{31}^*,$$

$$T_{34}=\frac{e^{-k(b+a)}}{4R}\left[(E-R)\left[\frac{k}{\mu}+\frac{\mu}{k}\right]\sin\theta-(R+E)\left[\frac{k}{\nu}-\frac{\nu}{k}\right]\sinh\phi\right],$$

$$T_{41}=\frac{V_\beta e^{k(b+ia)}}{4R}\left[-(1+i)\cos\theta+\left[\frac{k}{\mu}-\frac{i\mu}{k}\right]\sin\theta+(1+i)\cosh\phi-\left[\frac{k}{\nu}+\frac{i\nu}{k}\right]\sinh\phi\right],$$

$$T_{42}=-\frac{V_\beta}{V_\beta^*}T_{41}^*,\quad T_{43}=-e^{2k(a+b)}T_{34}^*,$$

$$T_{44}=\frac{e^{k(b-a)}}{4R}\left\{(R-E)\left[2\cos\theta-\left[\frac{k}{\mu}-\frac{\mu}{k}\right]\sin\theta\right]+(R+E)\left[2\cosh\phi-\left[\frac{k}{\nu}+\frac{\nu}{k}\right]\sinh\phi\right]\right\}.$$

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