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Universal canonical forms for time-continuous dynamical systems

L. Brenig and A. Goriely
Université Libre de Bruxelles, Code Postal 231, Boulevard du Triomphe, 1050 Bruxelles, Belgium
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A natural embedding for most time-continuous systems is presented. A set of nonlinear transformations is shown to split the whole family of dynamical systems into equivalence classes. The degree of nonlinearity is not a relevant characteristic of these classes of ordinary differential equations (ODE's). In each class there exist two particular simple canonical forms into which any such ODE can be cast. We show on an example how to use these canonical forms to find nontrivial integrability conditions.

I. INTRODUCTION

In a recent paper,[1] one of us introduced a class of dynamical systems, the so-called generalized Lotka-Volterra equations:

$$\dot{x}_i=\frac{dx_i}{dt}=\lambda_i x_i+x_i\sum_{j=1}^{n}A_{ij}\prod_{k=1}^{n}x_k^{B_{jk}}\ ,\quad i=1,\ldots,n \tag{1}$$

where x_i are real or complex functions of time t. The coefficients λ_i are real or complex parameters, A and B are $n\times n$ square matrices with real or complex entries.

It was shown in this article that a large category of physically relevant systems belong to that class and that representation (1) provides new methods for finding integrability conditions and for explicitly constructing solutions for these systems.

Nonlinear transformations of the form

$$x_i=\prod_{j=1}^{n}x_j'^{C_{ij}},\quad i=1,\ldots,n \tag{2}$$

where C is a $n\times n$ invertible matrix, were introduced and shown to bring Eqs. (1) into

$$\dot{x}_i'=\lambda_i'x_i'+x_i'\sum_{j=1}^{n}A_{ij}'\prod_{k=1}^{n}x_k'^{B_{jk}'},\quad i=1,\ldots,n \tag{3}$$

with

$$\lambda_i'=(C^{-1}\lambda)_i\ , \tag{4a}$$

$$A_{ij}'=(C^{-1}A)_{ij}\ , \tag{4b}$$

$$B_{ij}'=(BC)_{ij}\ . \tag{4c}$$

Thus, the form invariance of Eqs. (1) under the transformations (2) appears, and moreover, the following quantities are invariant:

$$B'A'=BA\ , \tag{5a}$$

$$B'\lambda'=B\lambda\ . \tag{5b}$$

Hence, the above general family (1) is split into equivalence classes characterized by a given matrix BA and vector $B\lambda$, where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$. It was also shown that when matrix A is invertible, transformation (2) with $C=A$ brings Eq. (1) into a normal form:

$$\dot{x}_i'=(A^{-1}\lambda)_i x_i'+x_i'\prod_{j=1}^{n}x_j'^{(BA)_{ij}},\quad i=1,\ldots,n\ . \tag{6}$$

The essential feature of this form is that only one nonlinear "monomial" term appears in each equation. The powers characterizing these nonlinearities are in general noninteger since they depend on parameters. In Ref. 1 this normal form was transformed into a completely factorized form which lead us to find integrability conditions in parameter space.

Furthermore, if matrix B is invertible, transformation (2) with $C=B^{-1}$, when applied to Eq. (1), leads to the following Lotka-Volterra system:[1]

$$\dot{x}_i'=(B\lambda)_i x_i'+x_i'\sum_{j=1}^{n}(BA)_{ij}x_j',\quad i=1,\ldots,n\ . \tag{7}$$

Thus, two opposite exact normal forms are found for Eqs. (1). The first corresponds to $A'=I$, the second to $B'=I$. The first form is particularly convenient for determining integrability conditions.[1] It also leads to an operating procedure for explicitly solving Eqs. (1) when these conditions are fulfilled. The second normal form contains the lowest degree of nonlinearity. Already here, nonlinearity does not appear to be a relevant parameter, the only ones being the characteristics BA and $B\lambda$ of the equivalence class to which a system belongs.

In Ref. 1, Eq. (1) was thought to be restrictive. We now show that most of dynamical systems can be cast in this shape and consequently that forms (6) and (7) are universal. We then apply these forms to find integrability conditions and explicit integrals to a model for nonlinear interaction of three waves. This system appears in many domains of physics and, till now, no exact solutions of it were known when dissipation is included. Our approach leads explicitly to such solutions as is shown later.

II. GENERALIZATION

A direct generalization of Eqs. (1) amounts to allow for matrices A and B to be rectangular:

$$\dot{x}_i=\lambda_i x_i + x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} ,i=1,\dots,n, \quad m\geq n \ . \tag{8}$$

It can be seen from (8) that A is a $n\times m$ matrix and B a $m\times n$ matrix.

Most of the relevant dynamical systems can be cast into the form (8) when there is a finite number of polynomial nonlinearities or even when real powers of the dependent variables appear. The transformations (2) are still relevant for the class (8) which is also closed under them:

$$\dot{x}'_i=\lambda'_i x'_i + x'_i \sum_{j=1}^{m} A'_{ij} \prod_{k=1}^{n} x_k'^{B'_{jk}}, \quad i=1,\dots,n, \quad m\geq n \tag{9}$$

with

$$\lambda'=(C^{-1}\lambda) \ , \tag{10a}$$

$$A'=C^{-1}A \ , \tag{10b}$$

$$B'=BC \ , \tag{10c}$$

and C is any invertible $n\times n$ matrix. However, it is clearly impossible to obtain for Eqs. (8) the two canonical forms (6) and (7) unless, as is shown below, the dimension of the system is increased.

III. CANONICAL FORMS

A. First canonical form

This form can be obtained if and only if the $n\times m$ matrix A is of rank n. Let us add to system (8) $m-n$ new equations for variables $x_{n+1},\dots,x_m$ in the following way:

$$\dot{x}_\alpha=\tilde{\lambda}_\alpha x_\alpha + x_\alpha \sum_{\beta=1}^{m} \tilde{A}_{\alpha\beta} \prod_{\gamma=1}^{m} x_\gamma^{\tilde{B}_{\beta\gamma}}, \quad \alpha=1,\dots,m \tag{11}$$

where

$$\tilde{A}=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & & A_{nm} \\ a_{n+1,1} & a_{n+1,2} & & a_{n+1,m} \\ a_{n+2,1} & a_{n+2,2} & & a_{n+2,m} \\ \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{bmatrix}, \tag{12a}$$

$$\tilde{B}=\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} & 0 & \cdots & 0 \\ B_{21} & B_{22} & & B_{2n} & 0 & & 0 \\ \vdots & & & & & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} & 0 & \cdots & 0 \end{bmatrix}, \tag{12b}$$

$$\tilde{\lambda}=\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \\ \rho_{n+1} \\ \vdots \\ \rho_m \end{bmatrix}. \tag{12c}$$

Let us stress that, by construction, the n first equations in system (11) are the same as in system (8) and are not coupled to the $m-n$ remaining ones.

In expression (12c), the parameters ρ_i are arbitrary. This is also the case for the a_{ij} up to the fact that $\tilde{A}$ should be invertible. This last property along with the fact that Eqs. (11) belong to Eqs. (1) in m dimensions allow for transforming the formers and consequently Eqs. (8) to the canonical form (6) in m dimensions. Let us remark that

$$\tilde{B}\tilde{A}=BA \ , \tag{13a}$$

$$\tilde{B}\tilde{\lambda}=B\lambda \ . \tag{13b}$$

B. Lotka-Volterra canonical form

The second canonical form can be obtained if and only if the $m\times n$ matrix B is of rank n. Then another embedding leads to the same system (11) but now with the following expressions for $\tilde{A}$, $\tilde{B}$, and $\tilde{\lambda}$:

$$\tilde{A}=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & & A_{nm} \\ 0 & 0 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{14a}$$

$$\tilde{B}=\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} & b_{1,n+1} & \cdots & b_{1,m} \\ B_{21} & B_{22} & & B_{2n} & b_{2,n+1} & & b_{2,m} \\ \vdots & & & & & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} & b_{m,n+1} & \cdots & b_{m,m} \end{bmatrix}, \tag{14b}$$

$$\tilde{\lambda}=\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{14c}$$

In order to ensure the equivalence between this version of system (11) and Eq. (8), the initial conditions on the $m-n$ new variables should be

$$x_\alpha(t=0)=1, \quad \alpha=n+1,\dots,m \ . \tag{15}$$

In expression (14b) the entries (b_{ij}) are arbitrary with the only condition that $\tilde{B}$ is invertible.

We are now able to use this last property to bring Eqs. (11) and consequently the system (8) to Lotka-Volterra form similar to (7). Here also

$$\tilde{B}\tilde{A}=BA \;, \tag{16a}$$

$$\tilde{B}\tilde{\lambda}=B\lambda \;. \tag{16b}$$

Of course, for both cases (A of rank n or B of rank n) the corresponding canonical form is obtained through a transformation (2) in m dimensions with, respectively $C=\tilde{A}$ and $C=\tilde{B}^{-1}$.

In summary we have established the following theorem.

(1) Any n-dimensional system of the form (8) can be brought to a m-dimensional system (11) belonging to the generalized Lotka-Volterra class (1) with $\tilde{B}\tilde{A}=BA$ and $\tilde{B}\tilde{\lambda}=B\lambda$.

(2) The transformations (2) divide the family of systems (8) into equivalence classes characterized by a given matrix BA and a given vector $B\lambda$.

(3) There exists particular transformations (2) leading any member of family (8) into one of the two canonical forms (6) and (7) provided A and B, respectively, are of rank n. The last point leads to a powerful method[1] to determine integrability conditions and explicitly solve system (8).

As an example, let us apply our method to find integrability conditions for a three-dimensional system which modelizes the nonlinear interaction of three waves:

$$\begin{aligned}\dot{x}_1&=\lambda_1x_1+x_1\left[\sum_{j=1}^{3}N_{1j}x_j^2\right]+\gamma_1x_2x_3\;,\\ \dot{x}_2&=\lambda_2x_2+x_2\left[\sum_{j=1}^{3}N_{2j}x_j^2\right]+\gamma_2x_3x_1\;,\\ \dot{x}_3&=\lambda_3x_3+x_3\left[\sum_{j=1}^{3}N_{3j}x_j^2\right]+\gamma_3x_1x_2\;,\end{aligned} \tag{17}$$

where γ_i, N_{ij}, and λ_i are real parameters.

The λ_i's (when negative) describe dissipation. The N_{ij}'s denote modes of competition and the γ_i's resonances between waves. The x_i's are real functions of time t describing the amplitudes of the three interacting modes. System (17) is an approximation of a general description of the coupling of three waves which arises in various fields of physics[2,3] (plasma waves,[4] nonlinear optics,[5] etc.). Solutions of (17) are known[6] if the dissipation terms (i.e., the λ_i's) vanish. We now proceed to find solutions when dissipation is included.

For simplicity, we consider the case where $\gamma_2=\gamma_3=0$, $\gamma_1=\gamma\neq0$. Then, system (17) can easily be cast into form (8) with $n=3$, $m=4$, and

$$A=\begin{bmatrix}N_{11}&N_{12}&N_{13}&\gamma\\ N_{21}&N_{22}&N_{23}&0\\ N_{31}&N_{32}&N_{33}&0\end{bmatrix}, \tag{18}$$

$$B=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\\-1&1&1\end{bmatrix}. \tag{19}$$

Thus, system (17) can be brought to the Lotka-Volterra canonical form (7) by a transformation (2) in four dimensions with $C=\tilde{B}^{-1}$, where $\tilde{B}$ is given by

$$\tilde{B}=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\-1&1&1&1\end{bmatrix}. \tag{20}$$

The new system is given by the canonical form (7) in four dimensions with

$$BA=\begin{bmatrix}2N_{11}&2N_{12}&2N_{13}&2\gamma\\ 2N_{21}&2N_{22}&2N_{23}&0\\ 2N_{31}&2N_{32}&2N_{33}&0\\ -N_{11}+N_{21}+N_{31}&-N_{12}+N_{22}+N_{32}&-N_{13}+N_{23}+N_{33}&-\gamma\end{bmatrix}, \tag{21}$$

$$B\lambda=\begin{bmatrix}2\lambda_1\\2\lambda_2\\2\lambda_3\\-\lambda_1+\lambda_2+\lambda_3\end{bmatrix}. \tag{22}$$

Obviously, this system can be integrated if

$$N_{1i}=N_{2i}+N_{3i},\quad i=1,2,3 \tag{23}$$

and

$$N_{12}=N_{13}=0\;. \tag{24}$$

Let us note that these conditions do not imply any decoupling of the original system (17) and any conditions on the dissipation coefficients.

Under conditions (23), the fourth equation of the new system reads

$$\dot{x}'_4=\nu x'_4-\gamma x'^2_4\;, \tag{25}$$

with $\nu=-\lambda_1+\lambda_2+\lambda_3$.

This equation can be integrated and expressed in term of the original variables:

$$x_4'(t)=x_1^{-1}x_2x_3=\left[\frac{\gamma}{\nu}+K_1e^{-\nu t}\right]^{-1} . \qquad (26)$$

In (26) we have used the fact that by the conditions (14) and (15) on the embedding we have

$$x_4=1 \qquad (27)$$

for all t.

The first equation of the considered canonical form is a Bernouilli equation for x_1'

$$\dot{x}_1'=2[\gamma x_4'(t)+\lambda_1]x_1'+2N_{11}x_1'^2 , \qquad (28)$$

for which the solution is

$$x_1'(t)=[K_2(t)]^{-1}\exp\left[2\int^t d\tau[\gamma x_4'(\tau)+\lambda_1]\right] , \qquad (29)$$

$$K_2(t)=2N_{11}\times\int_{t_0}^t d\tau\exp\left[-\int^\tau d\theta\, 2[\gamma x_4'(\theta)+\lambda_1]\right] . \qquad (30)$$

This integral is connected to the original variables by

$$x_1'(t)=[x_1(t)]^2 . \qquad (31)$$

Then, inserting relations (26), (29), and (30) in the second equation of (17), we are led to

$$\dot{x}_2=[\lambda_2+N_{21}x_1^2(t)]x_2+N_{22}x_2^3+N_{23}[x_1(t)]^2x_2^{-1} . \qquad (32)$$

Multiplying both sides by x_2 and introducing

$$y=x_2^2 , \qquad (33)$$

we find the following Riccati equation for y:

$$\dot{y}=2[\lambda_2+N_{21}x_1^2(t)]y+2N_{22}y^2+2N_{23}[x_1(t)]^2 , \qquad (34)$$

which is equivalent to a second-order linear equation by a well-known transformation. Thus, we have found integrability conditions for system (17) which are not at all trivial from the physical point of view and which allow for dissipation (there is indeed no constraint on the λ_i's). Other cases of integrability for system (17) can be found by using transformation (2). These results will be published elsewhere.[7]

To conclude, our approach shows that the degree of the nonlinearity is not a fundamental characteristic of a dynamical system. Its equivalence class characterized by the two objects BA and $B\lambda$ is the only relevant property. This property can be used to find integrability conditions on the canonical forms and explicit integrals. Furthermore, recent investigations indicate a close connection between our transformation theory and Poincaré's normal forms. For these reasons, we are convinced that this theory is worthy of exploration.

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