Characterization of an experimental strange attractor by periodic orbits

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We describe a general procedure to locate periodic saddle orbits in a chaotic attractor reconstructed from experimental data. The method is applied to data from a Belousov-Zhabotinskii chemical reaction. The eigenvalues associated with the saddle orbits are used to estimate the Lyapunov exponents. An analysis of the next amplitude map determines the allowable periodic orbits and yields an estimate of the topological entropy.

Recent theoretical work suggests that periodic saddle orbits determine much of the dynamics on typical attractors.[1-3] In hyperbolic attractors, for example, the natural measure, fractal dimension, and Lyapunov exponents can be expressed as limits involving the periodic saddle orbits.[1,2] Moreover, the saddle orbits can provide a useful characterization of the structure and dynamics of the attractor as a parameter varies.[3] In this paper, we show how to extract the periodic saddle orbits from an attractor reconstructed from experimental data. These orbits are used to estimate the Lyapunov exponents, information dimension, and topological entropy of the attractor. The estimates compare well with those obtained by conventional methods.

The attractor is obtained from a time series from an oscillating Belousov-Zhabotinskii (BZ) chemical reaction.[4,5] The experiment consists of a continuously stirred tank reactor into which various chemical species are fed at a constant rate. The bromide ion concentration in the reactor is recorded at equally spaced intervals, giving a time series of 65 000 values.

A phase-space attractor is reconstructed from the data using the standard method of time delays.[6,7] The experimental data consist of a scalar time series, denoted $\{s_i\}_{i=1}^n$. The attractor is the set of points $x_i = (s_i, s_{i+\tau}, \ldots, s_{i+(d-1)\tau})$ where d is the embedding dimension and τ is the time delay. In this discussion, we choose $d=3$ and $\tau=124$ on the basis of the mutual information criterion discussed in Ref. 8. Figure 1(a) shows a two-dimensional projection of the reconstructed attractor.

The saddle orbits that we consider are periodic, and they appear to have one repelling direction. In this three-dimensional reconstruction, they also have one attracting direction. Trajectories approach the saddle orbit along this direction and remain nearby for a time before they are pushed away. While a point remains near the saddle orbit, it moves with a frequency which is approximately the same as that of the saddle orbit. The saddle

orbits therefore can be located as follows. Let $\epsilon > 0$, and let x_i be a point on the reconstructed attractor. We follow the observed images $x_{i+1}, x_{i+2}, \ldots$ of x_i until we find the smallest index $k > i$ such that $\|x_k - x_i\| < \epsilon$. If such a k exists, we define $m = k - i$ and say that x_i is an (m, ϵ) recurrent point.

In this analysis, we fix $\epsilon = 0.005$ and compute distances in terms of the maximum norm. (The time series is normalized to the unit interval.) The value of m can be calculated using the above procedure for 51 000 out of 65 000 attractor points. We find that over 95% of the recurrence times m are clustered in small intervals around $m = 125, 250, 375, 500, 625, 750, 875,$ and 1000, as illustrated by the histogram in Fig. 2. The scatter in the m
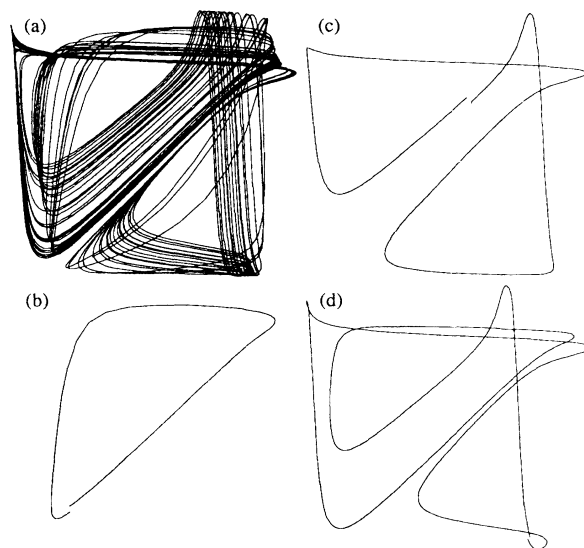

FIG. 1. (a) The BZ attractor. (b)–(d) Trajectories near the period-1, -2, and -3 saddles, respectively.
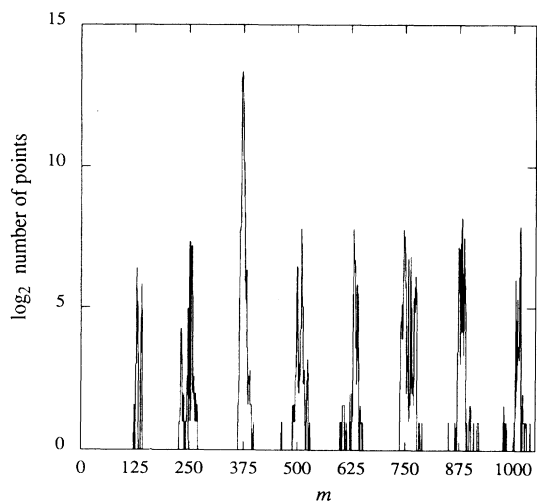
FIG. 2. Histogram showing the number of attractor points associated with each recurrence time m.

TABLE I. Eigenvalues associated with the periodic saddle orbits for the BZ attractor.

Period	Eigenvalue
1	3.66
2	5.34
3	3.18
4	4.17
5	7.10
6	3.40
7	9.56
8	5.56

values occurs for two reasons. First, the recurrence time of a trajectory depends on how closely it approaches a saddle orbit. Second, we record the time when the trajectory first returns to the ϵ neighborhood of the reference point x_i. This time may differ slightly from the time the trajectory most closely approaches x_i.

The trajectory through each point x_i in Fig. 1(b) comes back within $\epsilon = 0.005$ of x_i approximately 125 time steps later. This orbit has the shortest period, so we call it a period-1 saddle. (In what follows we define a "period" as 125 time steps.) Figures 1(c) and 1(d) show portions of the trajectories which lie near the period-2 and -3 saddles, respectively. In this paper we consider orbits up to period 8. It is important to include periodic saddles of sufficiently high period in order to capture most of the attractor points.

The stability of each saddle orbit is estimated from a linear approximation of the dynamics at points on nearby trajectories. Let x_{ref} be an (m, ϵ) point, and let $\{x_i\}_{i=1}^{n}$ be a collection of points in a 6ϵ neighborhood of x_{ref}.[9] We assume that the dynamics in this neighborhood is nearly linear; that is, we write the map f which takes x_i to x_{i+m} as $f(x) = Ax + b$ for some 3×3 matrix A and a three-vector b. A least-squares procedure similar to that described in Refs. 7 and 10 is used to calculate A and b. Here A is an approximation of the Jacobian matrix $Df(x_{\text{ref}})$. The absolute value of the largest eigenvalue of A provides an estimate of the stability (more precisely, the strength of the repulsion) of the saddle orbit near x_{ref}.

Table I contains a list of the periodic saddle orbits and their associated eigenvalues, which are calculated as follows. Each (m, ϵ) point associated with the orbit of period p is used as a reference point x_{ref}. The Jacobian matrix $Df^p(x_{\text{ref}})$ and the magnitude L of its largest eigenvalue is computed using least squares whenever 50 or more points can be found in a 6ϵ neighborhood of x_{ref}.[11] The median value of L over the total number of reference

points is listed in the table.

One's ability to estimate the eigenvalues associated with a given saddle orbit depends on the number of trajectories that lie nearby. Saddle orbits in densely populated regions of the attractor (where the natural measure is large) are easier to characterize than those in regions which are rarely visited.

The Lyapunov exponents (also called characteristic exponents) measure the rate of separation of nearby initial conditions on the attractor (see Ref. 7 for a precise discussion and additional references). An attractor is chaotic if the largest Lyapunov exponent λ_1 is positive. Roughly speaking, this implies the distance between a typical pair of nearby points on the attractor grows as $2^{\lambda_1 t}$ for small t.

The basic idea behind existing algorithms[12,13] to estimate Lyapunov exponents from experimental data is to follow sets of trajectories for short intervals to measure the observed rates of separation and average them. (Another, more ambitious method is described in Ref. 14.) We now consider a different approach, where we evaluate the Lyapunov exponents in terms of the eigenvalues of the periodic orbits.[2]

Most of the points on the BZ attractor are within $\epsilon = 0.005$ of the periodic saddles listed in Table I. Thus an estimate of the Lyapunov exponents can be obtained from a weighted average of the eigenvalues of the saddle orbits. In this case we weight the eigenvalues according to the number of associated (m, ϵ) points. This yields the estimate $\lambda_1 = 0.56$ bits/period, which agrees well with the estimate $\lambda_1 = 0.50$ bits/period using the algorithm of Wolf et al.[12]

In this experiment it appears that an embedding dimension of 3 is sufficient to reconstruct the attractor (no trajectory crosses another). Under this assumption, there are three Lyapunov exponents for the BZ attractor: a positive exponent (λ_1, estimated above), a zero exponent, and a negative exponent (λ_3).[7] The negative exponent measures the rate at which points near the attractor approach it. Negative Lyapunov exponents are difficult to estimate from experimental data often because one cannot observe how different parts of the attractor (or the associated return map) contract onto each other. If we suppose that the data in this experiment are accurate to 0.1%, then two points whose initial separation contracts by a factor of 1000 after one period will be indistinguish-

able because of the noise. In other words, we cannot measure the negative Lyapunov exponent if $\lambda_3 \lesssim -10$ bits/period.

This consideration leads to an approximate value of the information dimension D_I.[15] Assuming that the Kaplan-Yorke[16] conjecture holds for this attractor, then

$$D_I = 2 + (\lambda_1 + \lambda_2)/|\lambda_3| \; .$$

Using the estimates for the Lyapunov exponents obtained above, we have $D_I \sim 2 + 0.6/10 = 2.06$. Although this estimate of λ_3 is somewhat speculative, it is consistent with calculations of the information dimension using the method of nearest neighbors,[17] which gives $D_I = 2.12 \pm 0.04$. (The estimates depend on which nearest neighbors are used; this is reflected in the variance.)

Additional information about the periodic saddle orbits can be obtained from an analysis of the next amplitude map $f(x_n) = x_{n+1}$, shown in Fig. 3. Here we have plotted the $(n+1)$st relative minimum in the time series as a function of the nth relative minimum. For convenience, we normalize the time series to the unit interval so that f is defined on [0,1]. This return map has a single critical point: an absolute maximum at x_c. Let $A = [0, x_c)$ and $B = (x_c, 1]$. A careful examination of the experimental data reveals that points in A map only to points in B [i.e., if $x < x_c$ then $f(x) > x_c$], but points in B map to points in $A \cup B$. We represent this rule with the transition matrix[18]

$$M = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

containing a zero entry for the disallowed transition AA.

This observation yields an estimate of the topological entropy. Let N_p be the number of periodic orbits of period p in the attractor. The topological entropy h_t is given as[19]

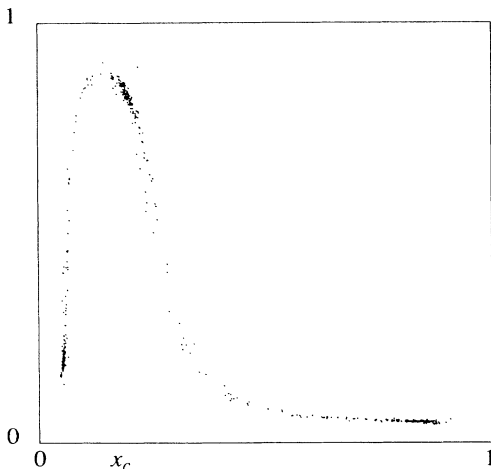$$h_t = \lim_{p \to \infty} \frac{1}{p} \log_2 N_p \; .$$

FIG. 3. Return map for the BZ attractor.

For example, the map $f(x) = 4x(1-x)$ has 2^p periodic orbits of period p.[18,20] (The initial conditions can be represented as binary strings and the dynamics as a left shift by one bit. The number of points of period p is the number of different binary strings of length p.) The topological entropy of this map is 1.

In this BZ attractor, not all binary sequences (strings of A's and B's) are possible; in particular, the sequence AA is disallowed. (This exclusion principle is called pruning[21].) The easiest way to determine the possible sequences is to use the transition matrix M given above. It can be shown[18] that the number of orbits of period p for a map of the interval is given by $\text{tr} M^p$.[22] Assuming that the dynamics is described by M, then

$$h_t = \lim_{p \to \infty} \frac{1}{p} \log_2 \text{tr} M^p \; .$$

In this case, $h_t \to \log_2 \lambda_{\max}$ as $p \to \infty$, where $\lambda_{\max} = (1 + \sqrt{5})/2$ is the largest eigenvalue of M. Hence $h_t \approx 0.696$ bits/orbit.

This estimate of the topological entropy of the BZ attractor is an upper bound because we have used only the lowest-order transition matrix. A better upper bound can be obtained by examining longer sequences of strings. For example, if we examine the data further to determine which pairs of three-digit strings occur, we find the transition matrix[23]

$$M' = \begin{array}{c} \\ \begin{array}{c} ABA \\ ABB \\ BAB \\ BBA \\ BBB \end{array} \end{array} \begin{array}{c} \begin{array}{ccccc} ABA & ABB & BAB & BBA & BBB \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array} \; .$$

Proceedings as before, we find $\lambda_{\max} \approx 1.48$, so that $h_t \approx 0.563$ bits/orbit.

The topological entropy h_t is an upper bound for the metric entropy, which in this case should be equal to the positive Lyapunov exponent λ_1 since there is only one expanding direction on the attractor.[7] Our estimate of $\lambda_1 = 0.56$ bits/orbit is in excellent agreement with the value of h_t obtained from the transition matrix. (Another approach to the estimation of topological entropy is discussed in Refs. 24 and 25.)

The determination of M' depends somewhat on the choice of the critical point x_c in Fig. 3, because there are a few orbits which pass very close to x_c. (In addition, there are two orbits which lie away from the main curve of the return map. We attribute them to experimental artifacts.) Although the dynamics described by M' suggests that at least two different period-6 orbits are possible, only one is observed in the experiment. This implies that either additional pruning occurs (which is not apparent from the binary triples), or the other period-6 orbit is visited too infrequently to be detected in these data.

The determination of the periodic saddle orbits from experimental data is possible in principle as long as the

underlying dynamics is relatively low dimensional. In addition, a low noise level and a long time record are important so that the recurrent points can be located easily. Finally, the stability of the saddle orbits determines their visibility; saddle orbits whose positive eigenvalues are small can be found more easily than those whose eigenvalues are large.

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