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Propagation of images in an inhomogeneous fluid

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A new class of exact scalar Green's functions is presented. These functions solve fluid dynamics problems analogous to lens design problems in optics. In particular, given a point signal at a specified location, the Green's functions discussed here describe waves that propagate asymptotically to specified image points. The geometry of the target determines the structure of the background medium, which may be interpreted to be an acoustic or optical fluid, or a curved space-time. There is no restriction to spherical or axial symmetry. Included in the class are the known propagators for massless fields in Rindler space and in a space of constant negative curvature. In the latter case, the Green's function derived here is the analog of the Green's function that describes free-particle nonrelativistic quantum mechanics in a space of constant negative curvature. In general, the waveforms of this class are multiple bubbles that separate from a spherical initial pulse and converge monotonically on point singularities or zero-equipotential surfaces of a pair of underlying electrostatic potentials.

I. INTRODUCTION

The fundamental problem of fluid acoustics is to find the pressure field due to a localized sound source in an inhomogeneous medium. There are exact solutions for a variety of patterns of inhomogeneity that vary in one dimension.[1] However, absent radial or certain cylindrical symmetries, problems with multidimensional variations are generally treated with standard approximations.[2] Here, as an addition to the library of exact solutions, I present an infinite class of Green's functions in which all spatial dimensions can enter. As I shall explain, this class enables exact solution of the wave propagation analogs of certain boundary-value problems in electrostatics to which the traditional method of images applies. They also provide an implicit way to sum, albeit in a limited context, Hadamard series for an infinite class of massless fields propagating in curved spaces. These functions also provide possible end points for "transmutations"[3] of systems with inhomogeneous media into systems with known solutions.

The solutions of interest arise from the following problem: Design a fluid medium such that a point acoustic signal at a specified location asymptotically "illuminates" images at a specified set of points, and asymptotically leaves all other points "dark." I address this problem using the scalar wave equation

$$n^2(\mathbf{r})\frac{\partial^2\psi}{\partial t^2}-\nabla^2\psi=-4\pi\delta^3(\mathbf{r}-\mathbf{r}_1)\delta(t) \ . \tag{1.1}$$

In (1.1), ψ may be interpreted as the acoustic pressure due to a unit velocity pulse at $\mathbf{r}_1$ in an inhomogeneous medium characterized by the refractive index $n(\mathbf{r})$. Units are chosen such that the reference value $n(\mathbf{r}_1)=1$. As we shall see, the signaling problem is a convenient construct that allows one to find solutions of an infinite class of wave equations of the form (1.1) using methods of electrostatics.

II. GENERAL FORM OF GREEN'S FUNCTIONS

A simple one-dimensional analog of the three-dimensional medium in the problem above is a string stressed and weighted in such a way that the propagation velocity $c(x)=1/n(x)$ vanishes at a pair of prespecified points. Depending on the relative position of the source, either or both of these points may then be interpreted as the "target" of the initial pulse. The general solution for the $(1+1)$-dimensional homogeneous wave equation in the singular case with a double zero, and therefore only one target point in the velocity profile, $c(x)=1/n(x)=(ax+b)^2$, has been derived for other purposes by Synge.[4] Synge's wave in one spatial dimension has properties that suggest the existence of similar solutions under more general conditions and in higher-dimensional spaces. The structure of his solution is preserved in the presence of derivative and mass terms in the wave equation.[5] The double-zero profile for $c(x)$ allows an exact WKBJ solution in Schrödinger theory[6] and admits an infinite invariance group of point transformations.[7] For completeness, the $(1+1)$-dimensional Green's function for the more general two-target profile $c(x)=(ax+b)(cx+d)$ is derived in the Appendix. As shown there, the derivation follows easily from a combined rescaling of the wave function, conformal transformation of the metric, and imposition of translational invariance. The two-target wave function has an underlying Hamiltonian structure which determines the form of the refractive index and again hints at generalization.

To create a three-dimensional analog of the weighted elastically supported string, I proceed as follows. Given a unit impulse at the end point, a semi-infinite, uniformly weighted and tensioned string, embedded in a uniform elastic medium, takes the shape $\Theta(t-x)J_0(\mu(t^2-x^2)^{1/2})$, where $\Theta(x)$ is the unit step function, J_0 the Bessel function of order zero, and μ^2 the ratio of the elasticity of the background to the string tension. To describe a three-dimensional excitation, it is natural to make three changes in the $(1+1)$-dimensional Green's function. First, to impose an initial condition appropriate to three spatial dimensions, I introduce an initial unit velocity pulse via a spatial derivative.[8] Second, I introduce a mapping that smoothly associates each spatial string point with a three-dimensional wave surface $f(\mathbf{r})$. In light of Fermat's principle, the magnitude of f plays the role of a geodesic coordinate measured along the rays of geometrical optics. This interpretation will be used below to make a connection with analogous problems in quantum mechanics. Third, to maintain energy flux conservation I introduce an overall spatially dependent multiplicative factor $A(\mathbf{r})$.

On implementing these changes, the general trial form for a class of solutions of (1.1) is the d'Alembert wave function

$$\psi=\frac{A}{2}\frac{\partial}{\partial f}[\Theta(t-f)J_0(\mu(t^2-f^2)^{1/2})] . \tag{2.1}$$

The parameter μ in (2.1) can be interpreted as a scale of the granularity of the background medium or as a measure of its curvature when it is interpreted as a Riemannian manifold. To maintain the connection with optics and acoustics, discussion here is confined to cases in which μ is real.[9] To ensure that there is an initial outgoing spherical wave, I shall further assume that the background is locally flat, homogeneous, and isotropic, so that the amplitude A and wave front f obey

$$\lim_{r\to r_1} f=|\mathbf{r}-\mathbf{r}_1|,\quad \lim_{r\to r_1} A(\mathbf{r})f(\mathbf{r})=1 . \tag{2.2}$$

On substituting (2.1) into (1.1), one observes that (2.1) will be an exact solution of (1.1) if the wave surface satisfies the eikonal equation

$$|\nabla f|^2=n^2 , \tag{2.3}$$

energy transport is governed by

$$\nabla\cdot(A^2\nabla f)=4\pi\delta^3(\mathbf{r}-\mathbf{r}_1) , \tag{2.4}$$

and diffraction in the medium is described by the reduced wave equation

$$\nabla^2 A+\mu^2|\nabla f|^2 A=-4\pi\delta^3(\mathbf{r}-\mathbf{r}_1) . \tag{2.5}$$

Source term aside, Eqs. (2.3) and (2.4) are standard in the ray approximation. The Helmholtz relation (2.5) is special to this approach. It allows dispensing with third- and higher-order terms in the infinite wave-front series[10] that would otherwise describe the Green's function ψ. Given (2.5), the geometrical optics approximation is exact, in that the ray interpretation holds independent of frequency, and there is no constraint on the spatial interval over which $n(\mathbf{r})$ can vary significantly.[11]

III. SOLUTION OF THE SIGNALING PROBLEM

Equations (2.3)–(2.5) are analogous to the second step in a geometrical optics analysis in which one introduces a complex wave function $A\exp(if)$ and looks for a series solution to (1.1). Here, however, there are invariance properties that permit a closed-form solution of the signaling problem. To make this invariance explicit, rewrite (2.1) in the form

$$\psi=\sigma(\mathbf{r})G_+(f,t) , \tag{3.1a}$$

where the amplitude $\sigma=Af$, local homogeneity (2.2) requires $\sigma(\mathbf{r}_1)=1$, and $G_+(r,t)$ is the forward propagator of a Klein-Gordon particle with mass μ.

In the form (3.1a), we may interpret ψ as a special form of progressive wave[12] in a curved space in which the wave surface function $f(\mathbf{r})$ plays the role of radial coordinate. To make the curved-space interpretation clear, it is instructive to write (1.1) in the covariant form

$$g^{\mu\nu}\psi_{;\mu\nu}(x)=-\delta^4(x) , \tag{3.1b}$$

where x is now the four-vector position, the d'Alembertian is taken with respect to the metric

$$ds^2=n^{-1}dt^2-nd\mathbf{r}^2 , \tag{3.1c}$$

and a multiplicative factor $n^{-1}(\mathbf{r})$ has been applied to (1.1).[13] Equation (3.1b) is equivalent to the claim that there exists a family of massless fields propagating in a curved space with "conformastat" metric[14] (3.1c). Equation (3.1a) amounts to the further claim that this field can be transformed, by rescaling with a spatially dependent factor, into a massive Klein-Gordon field propagating freely in Minkowski ray space. The requirement of conformal equivalence constrains the amplitude and wave surface. In particular, in terms of σ and f, Eqs. (2.4) and (2.5) can be written in the form

$$\mu\nabla^2[\sigma i_0(\mu f)]=0 , \tag{3.2a}$$

$$\mu\nabla^2[\sigma k_0(\mu f)]=-4\pi\delta^3(\mathbf{r}-\mathbf{r}_1) , \tag{3.2b}$$

where $i_0(z)=\sinh(z)/z$ and $k_0(z)=\cosh(z)/z$ are modified spherical Bessel functions of order zero. The proposition that follows shows that the appearance of the potentials

$$\phi_1=\mu\sigma i_0(\mu f),\quad \phi_2=\mu\sigma k_0(\mu f) \tag{3.3}$$

in (3.2) allows the nonlinear equations (2.3)–(2.5), and therefore also the signaling problem, to be solved in closed form by a transformation into the language of electrostatics.

Proposition. Let the asymptotic target in the signaling problem defined above be a set of doubly differentiable closed surfaces $\{\mathbf{S}\}$. $\{\mathbf{S}\}$ may be a single surface enclosing the source point or multiple closed surfaces to which the source is exterior. Let Γ be the electrostatic Green's function with respect to the set of surfaces $\{\mathbf{S}\}$ (now to be thought of as grounded conductors) and a unit charge at the source point $\mathbf{r}_1$. In terms of the ϕ_i in (3.3), $\Gamma=\phi_2-\phi_1$. Let $\phi^*=\phi_2-1/r$ be an otherwise arbitrary potential satisfying $\nabla^2\phi^*=0$ within the domain of influence. Define

$$\beta=1-\frac{\phi_1}{\phi_2}=\frac{|\mathbf{r}-\mathbf{r}_1|\Gamma}{1+|\mathbf{r}-\mathbf{r}_1|\phi^*} . \tag{3.4}$$

Then a family of solutions of the signaling problem is given by $\psi=\sigma(\mathbf{r})G_+(f,t)$, where $G_+(r,t)$ is the forward propagator of a free Klein-Gordon particle, and the wave surface is

$$f=\mu^{-1}\tanh^{-1}|\phi_1/\phi_2|=\mu^{-1}\tanh^{-1}(1-\beta) . \tag{3.5}$$

In (3.5), $\nabla f(\mathbf{r}_1)$ is equal to the radial unit vector $\hat{e}_{\mathbf{r}-\mathbf{r}_1}$, and $f(\mathbf{r}_1)=0$. The amplitude is

$$\sigma=\frac{\mu^{-1}\Gamma}{k_0(\mu f)-i_0(\mu f)} , \tag{3.6}$$

and the refractive index in (1.1) is

$$n=\frac{1}{2\mu}\left|\frac{\nabla\beta}{\beta(1-\beta/2)}\right| . \tag{3.7}$$

Proof. The proof follows from observing, first, that (3.4)–(3.7) imply, via (3.2) and (3.3), that the conditions (2.2)–(2.5) hold, and therefore that one has constructed a solution of (1.1) in the form (2.1). Also, it follows from (3.7) that the propagation velocity $c=1/n$ is zero on the points $\Gamma=\beta=0$, which define the target image, as required by the signaling problem.

One can also show the auxiliary potential ϕ^* only induces smooth behavior in n, so that the singularity structure of the medium is entirely determined by the geometry of the target. In particular, from (3.3) and (2.4), the differential equation

$$\mu\frac{d\mathbf{r}}{dt}=\phi_1\nabla\phi_2-\phi_2\nabla\phi_1 \tag{3.8}$$

determines the ray paths. By Gauss's theorem and Eqs. (3.2), (3.8) implies that, other than at the source, there are no singular focal points within the domain of influence of the solution (3.1). This is an important restriction from the acoustical point of view. It precludes using the functions (3.1a) specified by (3.5) and (3.6) to describe typical waveguide solutions for ψ.

Given that the Green's functions (2.1) are fully specified by the potentials (3.2) and the local homogeneity condition (2.2), the full apparatus of electrostatics is available to attack suitable boundary value problems. In particular, the construction above allows one to associate the signaling problem in cases in which the set of target points $\{\mathbf{S}\}$ form a zero-equipotential surface generated by point charges, with electrostatics problems to which the traditional method of images applies.[15] In Sec. IV, I illustrate the method by application to simple cases with spherical and planar symmetric targets, and to targets consisting of arbitrary sets of disconnected points.

IV. EXAMPLES

A. Spherical symmetry

Let the target be a spherical shell of radius R, with the source at the center. Then the analog electrostatic Green's function is

$$\Gamma=\frac{1}{r}-\frac{1}{R} . \tag{4.1}$$

Given spherical symmetry, the auxiliary potential is a constant, and we may write

$$\phi^*=1/R' , \tag{4.2}$$

where R' is an arbitrary length. From (3.4),

$$\beta=\frac{1-r/R}{1+r/R'} , \tag{4.3}$$

and the wave surface [cf. (3.5)] is

$$f=(2\mu)^{-1}\ln\left[\frac{1+r(1/R'+\mu)}{1+r(1/R'-\mu)}\right] . \tag{4.4}$$

In (4.4), local homogeneity (2.2) has been used to set

$$\mu=\frac{1}{R}+\frac{1}{R'} . \tag{4.5}$$

From (3.7) and (4.3), the refractive index is

$$n=\frac{1}{[1+r(1/R'+\mu)][1+r(1/R'-\mu)]} . \tag{4.6}$$

From (3.6), (4.1), and (4.4), the amplitude is

$$\sigma=i_0^{-1}(\mu f) . \tag{4.7}$$

There are two classes of solutions, distinguished by the relative positions of the targets: (i) $(\mu R')^{-1}\geq 1$ and (ii) $(\mu R')^{-1}<1$. In case (i), the image lies at infinity, and the wave front acquires infinite radius at the finite time

$$t^*=\frac{1}{\mu}\coth^{-1}\left[\frac{1}{\mu R'}\right] . \tag{4.8}$$

In case (ii), the wave front asymptotically approaches the image at the horizon $r=R$.

In the absence of the auxiliary potential ϕ^*, the refractive index profile (4.6) becomes

$$n=\frac{1}{1-\mu^2r^2} . \tag{4.9}$$

Taking the metric form $ds^2=dt^2-n^2d\mathbf{r}^2$, (4.9) yields the constant negative curvature space defined by

$$ds^2=dt^2-\frac{1}{(1-\mu^2r^2)^2}d\mathbf{r}^2 . \tag{4.10}$$

Constant curvature spaces have been used as background in studying massless quantized fields[16] and in exploring the transition from quantum to classical mechanics. To make contact with the literature, I will show that the Green's function associated with the metric (4.9) has a quantum-mechanical analog with essentially the same Fourier structure, on a constant negative curvature space. It is well known that in the geometrical optics interpretation the rays for a solution of (1.1) follow geodesics in the space defined by the underlying metric.[17] Given spherical symmetry, we can therefore interpret f as the radial geodesic coordinate ρ in ray space, and write the Fourier representation of the Green's function

$$\psi=i_0^{-1}(\mu f)G_+(f,t) , \tag{4.11}$$

following from (4.4) with $1/R'=0$, as

$$G(\rho,\omega)=\frac{1}{2\pi\rho i_0(\mu\rho)}\exp[i\rho(\omega^2-\mu^2)^{1/2}] . \tag{4.12}$$

If we replace the quadratic energy dependence ω^2 in (4.12) by the linear form $2mE$ appropriate to a first-order diffusion process, we have the Schrödinger Green's function

$$G_S(\rho,E)=\frac{1}{2\pi\rho i_0(\mu\rho)}\exp[i\rho(2mE-\mu^2)^{1/2}] \ . \quad (4.13)$$

Allowing for a difference in normalization, (4.13) is identical to the Green's function used by Gutzwiller[18] in comparing the quantum energy spectrum in a space of constant negative curvature to the length spectrum of classical periodic orbits in a chaotic dynamical system. Using the same procedure, one may write down an explicit Schrödinger Green's function in ray space in every case in which (3.5) can be inverted to give r in terms of f.

The Green's function (4.13) differs from its free-particle counterpart in two respects: The signal strength factor $i_0^{-1}(\mu\rho)$, which maintains the local conservation of energy flow and the term $-\mu^2$ in the phase, which corresponds to a phase factor $\exp(-it\mu^2/2m)$ in the quantum propagator $G_S(\rho,t)$, and here scales the inhomogeneity of the fluid medium. That there is a relation between wave propagation in an optically inhomogeneous medium and the quantum mechanics of constant negative curvature spaces has been pointed out by Balasz and Voros[19] in a $(2+1)$-dimensional context.

B. Planar image: Green's function in Rindler space

Place a point source on the z axis at a distance z_0 from the plane $z=0$, which we take to represent the receiver and bound the domain of influence. The appropriate electrostatic analog is a point charge located at the source point and an infinite grounded conducting plate at $z=0$. We wish to obtain the exact field for which any wave surface enclosing the source is projected onto the $z=0$ plane.

Following the discussion above, make the boundary of the domain of influence the $z=0$ plane by placing a unit charge at $(0,0,z_0)$ and an image of opposite sign at $(0,0,-z_0)$. Then the electrostatic Green's function is

$$\Gamma=1/r_1-1/r_2 \ , \quad (4.14)$$

where the distances from sources to field point are

$$r_i=[(z-z_i)^2+R^2]^{1/2} \ , \qquad i=1,2, \quad z_1=z_0, \quad z_2=-z_0 \ . \quad (4.15)$$

Set the auxiliary potential $\phi^*=0$. Given (4.14), the domain of influence is the half-space $z\geq 0$. From (3.5) and (4.14), the wave surface is given by

$$f=\frac{1}{2\mu}\ln\frac{r_2+r_1}{r_2-r_1} \ , \quad (4.16)$$

and the amplitude is

$$\sigma=\frac{2(zz_0)^{1/2}}{r_1r_2}f \ . \quad (4.17)$$

Differentiating (4.16), one obtains the linear propagation velocity

$$c=1/n=2\mu z=z/z_0 \ , \quad (4.18)$$

with a simple zero which insures the image is projected onto the $z=0$ plane. From (4.18), we may write the underlying metric as $ds^2=2\mu z\,dt^2-(2\mu z)^{-1}d\mathbf{r}^2$. On making a conformal transformation by an overall factor $2\mu z$, one obtains the metric $ds^2=(2\mu z)^2dt^2-d\mathbf{r}^2$, which characterizes a Rindler reference frame moving with constant acceleration 2μ. The Rindler Green's function[20] differs from the ψ defined in the forward light cone by (4.16) and (4.17) by a multiplicative factor $\sqrt{n}$.

The geometry of the wave surface (4.16) is as follows. The wave fronts are defined at time t by

$$\frac{r_1}{r_2}=\frac{1-e^{-2\mu t}}{1+e^{-2\mu t}} \ . \quad (4.19)$$

In terms of cylindrical coordinates (R,z), (4.19) yields

$$R^2+[z-z_0\cosh(2\mu t)]^2=z_0^2\sinh^2(2\mu t)\equiv(r_0)^2 \ . \quad (4.20)$$

The wave front defined by (4.20) is a sphere of radius $r_0(t)$, exponentially expanding and translating with time. At time t, the center of the wave lies on the positive z axis at $z=e^{\mu t}z_0$; it moves vertically in the positive-z direction with exponential velocity in such a way that the wave front approaches, but never contacts, the $z=0$ boundary. Consistent with the presence of a background uniform field, (4.20) shows that an observer co-moving with the wave center experiences an acceleration of 2μ along the positive-z axis. The wave has a mirror image in the half-space $z<0$. The rays in this model are surface arcs on vertical sections of the one-parameter family of toroids

$$z^2+(R-z_0\tan\alpha)^2=z_0^2\sec^2\alpha, \quad -\pi/2\leq\alpha\leq\pi/2 \ , \quad (4.21)$$

whose centers of symmetry lie in the $z=0$ plane, to which all rays are asymptotically normal.[21]

C. Off-center spherical image

The solution defined by (4.16)–(4.18) is a special case of a more general class of Green's functions which are axially symmetric and associated with wave fronts that generate conformal maps from ordinary space to ray space. In particular, the inverse relation to (4.16),

$$\tanh(\mu f)=\frac{r_1}{r_2} \ , \quad (4.22)$$

conformally maps the surfaces $f=$constant into the spheres (4.20). In the from (4.16), f itself is the electrostatic point-to-plane Green's function in a two-dimensional Cartesian space with coordinates (R,z).

For a more general conformal map that extends the planar and spherical symmetry examples, assume a spherical target with the source off center. Place the source at the origin, and let the target have radius R_0 and center $(R=0, z=z_0)$. The method of images suggests introducing the potentials

$$\phi_2=1/r \ , \quad (4.23a)$$

$$\phi_1=\frac{q}{[R^2+(z-z^*)^2]^{1/2}} \ , \quad (4.23b)$$

with the image charge q at $z^* = (1-q^2)z_0$, and $R_0 = qz_0$.

For definiteness, set the image charge $q < 1$, so that the source is external to the target sphere. Also, set the auxiliary potential $\phi^* = 0$. The geometry of the medium is then entirely set by the electrostatic Green's function. Since $\Gamma = \phi_2 - \phi_1$ from (4.23) has the same singularity structure as the electrostatic Green's function (4.14), we expect the index of refraction that emerges here to have the spatial form of (4.9), which also has $\phi^* = 0$.

The wave surface is now given by inserting (4.23) into (3.5), and the amplitude is

$$\sigma = \sqrt{n}\,\frac{f\phi_1}{\mu r} . \tag{4.24}$$

On inserting (4.23) into (3.4) and (3.7), we derive the refractive index

$$n = \frac{1}{\mu^2(\phi_1^{-2} - \phi_2^{-2})} = \frac{z_0^2 - R_0^2}{R^2 + (z - z_0)^2 - R_0^2} , \tag{4.25}$$

where $\mu = \phi_1(0) = q/z^*$.[22] As anticipated, (4.25) has the same structure as (4.9).

The wave surface geometry following from (4.23)–(4.25) is as follows. For $q < 1$ the wave front given by (3.5) and (4.23) expands to infinite radius in finite time. It then acquires negative curvature and converges asymptotically on the image point at $R = 0, z = z_0$. This is the qualitative behavior of the wave fronts of the Maxwell fish eye in geometrical optics.[23]

D. Multiple point images

Targets consisting of arbitrary disconnected points are generated most easily by taking the $\mu = 0$ limit of (1.1), so that a conformal transformation of the metric and a rescaling of ψ results in a massless field in Minkowski ray space. Then (3.1) takes the form

$$\psi = \sigma\delta[t^2 - f^2(\mathbf{r})] , \tag{4.26}$$

and Eqs. (3.2) become

$$\nabla^2\sigma = 0, \quad \nabla^2(\sigma/f) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}_1) , \tag{4.27}$$

so that σ and σ/f play the roles of the potentials ϕ_1 and ϕ_2. Let ϕ_2 be generated by a single charge at the origin, and let ϕ_1 be generated by charges at arbitrary image points $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$. Then the wave fronts are the zero-equipotential surfaces generated by a positive charge of magnitude t at the source point $\mathbf{r}_1$, and negative charges at the $\mathbf{s}_i$, of magnitudes $-q_i$ such that, by (2.2),

$$\sum_{i=1}^{n} \frac{q_i}{|\mathbf{r}_1 - \mathbf{s}_i|} = 1 . \tag{4.28}$$

The wave surfaces are multiple bubbles which separate from the initial spherical pulse and converge monotonically on the target points $\mathbf{s}_1$, which are the only singularities of $n(\mathbf{r})$.

E. Image at infinity: Two parallel bounding surfaces

In the examples above the refractive index n is singular on the boundary of the domain of influence. I note in passing that there is a class of examples in which n is nonsingular on a part of the boundary, which is then not an asymptotic image. The simplest examples are those in which one enforces a Neumann condition in which the normal component of $\nabla\phi$ is zero on the boundary. For one such case, set $\phi_1 = \text{const}$, and introduce an infinite string of equally spaced, identical image charges, so that

$$\phi_2 = \sum_{j=-\infty}^{\infty} \frac{1}{[R^2 - (z - 2z_0 j)^2]^{1/2}} - \frac{1}{2z_0 j} , \tag{4.29}$$

and $\nabla\phi_2$ is transverse along the two planes $z = \pm z_0$. The wave fronts are then confined within the infinite slab $-z_0 < z < z_0$, and the target surface lies at infinite range.

V. DISCUSSION

Presented here is a simple method for specifying a Green's function that describes scalar wave propagation in a bounded three-dimensional medium. The background medium may be interpreted as an acoustic or optical fluid, or a Riemannian space. Two extensions suggest themselves. First, the signaling problem on which the discussion relies requires a smooth mapping from one spatial dimension to three that does not allow singular focal points within the domain of influence. This precludes using the present approach to describe in closed form the waveguide structure of typical problems in fluid acoustics. A possible remedy is to allow multiple sources sequenced in time and distributed appropriately in space. It can be shown that sources of this sort can be created by allowing additional singularities in the scalar potential ϕ_2, which carries the initial singularities of the Green's functions discussed here.

Second, considered as a boundary value problem, the signaling problem used as a starting point here is not well posed. The source-target geometry is uniquely determined by the associated electrostatic Green's function. But the structure of the background medium also depends on an auxiliary harmonic function constrained only by the absence of singularities within the domain of influence. It would be interesting to specify the model further, for example, by formulating it in terms of two interacting scalar fields, one of which carries signals while the other encodes the geometry of the background. The dynamics of nonlinear models of this sort, with interaction Lagrangian $\phi_1^2\phi_2^2$, have been analyzed in part, in the time domain in Yang-Mills theory,[24] and in the context of a perturbation from a flat space-time vacuum in quantum cosmology.[25]

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APPENDIX: ONE-DIMENSIONAL SIGNALING

Consider, as in the (3+1)-dimensional problems in the main text, the (1+1)-dimensional wave equation for an inhomogeneous medium

$$n^2(x)\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}=-\delta(x)\delta(t) \ . \tag{A1}$$

There are two ways to derive the analog of the (3+1)-dimensional results in the text. For illustrative purposes I first follow (2.1), and assume the solution of the source-target problem takes the form

$$\psi(x,t)=\sigma(x)G_+(f(x),t) \ , \tag{A2}$$

where $G_+(x,t)$ is the Green's function with the appropriate initial conditions satisfying

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+\mu^2\right]G_+=-\delta(x)\delta(t) \ , \tag{A3}$$

and local homogeneity requires $\sigma(0)=|f'(0)|=1$.

Substituting (A2) in (A1), one obtains the three conditions

$$(f')^2=n^2 \ , \tag{A4}$$

$$f''+2\sigma'f'/\sigma=0 \ , \tag{A5}$$

$$\sigma''+\mu^2(f')^2\sigma=0 \ , \tag{A6}$$

which are the (1+1)-dimensional forms of (2.3)–(2.5).

The solution is immediate if one follows (3.2) and introduces one-dimensional potentials $\phi_1=\sigma\cosh(\mu f)$ and $\phi_2=\sigma\sinh(\mu f)$ that satisfy $d^2\phi_i/dx^2=0$ by virtue of (A5) and (A6). However, it is instructive to interpret (A4)–(A6) as a Hamiltonian system with generalized coordinates

$$q=n^{-1/2}, \quad p=dq/dx \ . \tag{A7}$$

With only one spatial dimension, (A4) is effectively linear, and (A4)–(A6) may be solved exactly in terms of the variables (A7). For convenience, set the scale $\mu=1$. From (A4) and (A5) and the local homogeneity condition, the amplitude $\sigma=q$. Then in p-q space (A4) and (A6) give the equation of motion of a particle of unit mass moving in an attractive inverse-cube field with total energy

$$\tfrac{1}{2}p^2-\tfrac{1}{2}q^{-2}=E \ . \tag{A8}$$

On integration, (A8) yields the refractive index

$$n(x)=q^{-2}=\{[1-x/x_-(E)][1-x/x_+(E)]\}^{-1} \ , \tag{A9}$$

where the target points are

$$x_\pm(E)=\pm\frac{1}{2E}+\left[\frac{1}{2E}\left[\frac{1}{2E}+1\right]\right]^{1/2} \ , \tag{A10}$$

and a solution symmetric to (A9), with $x_\pm\to -x_\pm$, has been suppressed.

There are two cases: (i) $x_\pm(E)>0$, $E>0$, when there is a single target at $x=x_-(E)$, and the wave moves rightward from $x=0$; and (ii) $x_-(E)>0>x_+(E)$, $-\frac{1}{2}\le E<0$, in which case two pulses moving in opposite directions emerge from $x=0$.

From (A4) and (A9), the waveform is

$$f(x)=\tfrac{1}{2}\Theta(x)\ln\left[\frac{1-x/x_+}{1-x/x_-}\right], \quad E>0 \tag{A11a}$$

$$f(x)=\tfrac{1}{2}\left|\ln\left[\frac{1-x/x_+}{1-x/x_-}\right]\right|, \quad -\tfrac{1}{2}\le E<0 \ , \tag{A11b}$$

where $\Theta(x)$ is the unit step function.

A second way to proceed takes advantage of the conformal flatness of 2-space to introduce the eikonal relation (A4) via a coordinate transformation. As above let $q=n^{-1/2}$. Under the coordinate transformation $dx=q^2dy$, the metric underlying (A1) can be written

$$ds^2=n^{-1}(dt^2-dy^2) \ . \tag{A12}$$

Under the further conformal transformation $\psi\to q\psi'$, the wave equation takes the form

$$\frac{\partial^2\psi'}{\partial t^2}-\frac{\partial^2\psi'}{\partial y^2}+k^2(x)\psi'=-\delta(y)\delta(t) \ , \tag{A13}$$

where

$$k^2=-q^3(x)d^2q/dx^2 \ . \tag{A14}$$

On imposing translational invariance, we set $k^2=1$ and recover the equation of motion leading to (A8).

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[1]L. Brekhovskikh, *Waves in Layered Media,* 2nd ed. (Academic, New York, 1980).

[2]See, e.g., R. Dashen, J. Math. Phys. **20**, 894 (1979); M. Beran *et al.*, J. Acoust. Soc. Am. **71**, 1124 (1982); J. Keller and D. Ahluwalia, in *Wave Propagation and Underwater Acoustics* (Springer, New York, 1977); S. Flatte, Proc. IEEE **71**, 1267 (1983).

[3]R. Gilbert, Y. Xu, and D. Wood, Wave Motion **10**, 285 (1988).

[4]J. L. Synge, Q. Appl. Math. **39**, 292 (1981).

[5]D. Clement and C. Rogers, Mathematika **30**, 94 (1983).

[6]R. Bellman and R. Vasudevan, *Wave Propagation* (Reidel, Dordrecht, 1986), pp. 22 and 23. Lewis has used arguments related to Synge's to show that a two-flow helical structure underlies the one-dimensional wave equation and can be used to derive the WKBJ series for the heterogeneous string. J. T. Lewis, Q. Appl. Math. **38**, 461 (1981).

[7]G. Bluman and S. Kumei, J. Math. Phys. **28**, 307 (1987).

[8]For an explicit derivation of the scalar Green's function in derivative form, see P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 1, Sec. 7.3. The usual initial conditions $\psi(\mathbf{r},0)=0$ and $\partial\psi(\mathbf{r},0)/\partial t=\delta^3(\mathbf{r})$ define the fundamental solution of (1.1). The standard treatment of the Cauchy problem for the wave operator with constant coefficients also applies here. See, e.g., I Gel'fand and G. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I.

[9]In what may be more familiar field-theoretic terms, attention is restricted here to the curved-space analog, in the form (2.1), of the retarded Klein-Gordon propagator Δ_R.

[10]G. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974), Chap. 7.

[11]Compare the discusion of the approximations underlying the

usual ray theory treatment in, e.g., C. Boyles, *Acoustic Waveguides* (Wiley, New York, 1984), Chap. 5.

[12]F. Friedlander, *The Wave Equation in a Curved Space-Time* (Cambridge, University Press, Cambridge, England, 1975), Sec. 3.6.

[13]In the discussion that follows, unless otherwise noted, D'Alembertians on a curved space are taken with respect to the metric $ds^2=n^{-1}dt^2-nd\mathbf{r}^2$. In some instances, explicitly noted, the metric is related to the one above by a multiplicative factor. Provided causality is respected, the conformal transformation $ds^2 \rightarrow \rho^2(\mathbf{r})ds^2$ is equivalent to rescaling the Green's function by a factor $[\rho(\mathbf{r})\rho(\mathbf{r}_1)]^{-1}$ where, as in the text, $\mathbf{r}_1$ is the source point. See, e.g., Friedlander, Ref. 12, Sec. 4.6. The analytic, Hadamard structure of the massless field described by (3.1b) is well known. See, e.g., B. De Witt and R. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).

[14]J. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1964), p. 339.

[15]Recall that the method of images in electrostatics applies when a closed conducting surface under the influence of a point charge satisfies an equation of the form $\sum_{m=0}^{\infty} q_m/r_m=0$.

[16]See, e.g., C. Fronsdal, Phys. Rev. D **12**, 3819 (1975); M. Gutzwiller, Helv. Phys. Acta **29**, 313 (1956).

[17]M. Kline and I. Kay, *Electromagnetic Theory and Geometrical Optics* (Wiley, New York, 1965), Chap. 5, Appendix A. The use of the wave surface as a geodesic coordinate was suggested by an old and beautiful differential geometry argument that shows that all simple progressive-wave solutions of the ordinary wave equation have wave surfaces that are cyclides of Dupin. See F. Friedlander, Proc. Cambridge, Philos. Soc. **43**, 360 (1946).

[18]M. Gutzwiller, Phys. Scr. T **9**, 184 (1985).

[19]N. Balasz and A. Voros, Phys. Rep. **143**, 109 (1983).

[20]D. Boulware, Phys. Rev. D **11**, 1404 (1975).

[21]The geometry (4.20) and (4.21) may be derived independently by Fourier transforming (1.1) in (x,y,t) and constructing the one-dimensional Green's function in z as a bilinear in the eigenfunctions of the inverse-cube field, by following Boulware (Ref. 20) and expressing the Rindler function in terms of its counterpart in Minkowski space, or by using classical techniques and solving (1.1) in a comoving coordinate system. For the latter, see P. Morse and K. Ingard, in *Akustik I*, Vol. 11 of *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1961); D. Blokhintsev, *The Acoustics of an Inhomogeneous Moving Medium*, translated by R. Beyer and D. Mintzer (Research Analysis Group, Brown University, Providence, RI, 1952), Chap. 3.

[22]The refractive-index profiles (4.9) and (4.25) belong to the family $n=1/|-\kappa+r^2|$ in geometrical optics. R. K. Luneburg, *Mathematical Methods of Optics* (University of California Press, Berkeley, 1964), Sec. 28. This family is related by a Legendre transformation to the profiles $n^2=\kappa+1/r$, where the associated rays follow the trajectories of particles in an attractive Coulomb field $-1/2r$, with energy $\frac{1}{2}\kappa$.

[23]O. Stavroudis, *The Optics of Rays, Wavefronts and Caustics* (Academic, New York, 1972), p. 193.

[24]S. Chang, Phys. Rev. D **29**, 259 (1984); see also W.-H. Steeb *et al.*, Phys. Scr. **37**, 328 (1987) for a discussion of closely related classical and quantum Hamiltonian systems.

[25]J. Narlikar and P. Padmanabhan, *Gravity, Gauge Theories, and Quantum Cosmology* (Reidel, Dordrecht, 1986), Chap. 13.