Hyperbolic sine model: A multidisciplinary stochastic process

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The hyperbolic sine model is obtained by continuing the hierarchy of simple multiplicative stochastic processes beyond the Verhulst model or by transforming some physically important models from different disciplines to their canonical form. The Fokker-Planck equation associated with this Markov process is solved for the stationary first-order probability density function (PDF), and an exact time-dependent solution for the transition PDF is derived in terms of a new type of eigenfunction. As an example, eigenvalues and eigenfunctions are numerically evaluated for the symmetrical case. It is shown that some new models, of possible mathematical or physical importance, can be derived from the hyperbolic sine model.

INTRODUCTION

The number of stochastic processes that really have reached a multidisciplinary status is rather restricted. The Wiener process, the Ornstein-Uhlenbeck process, and, more generally, the class of Gaussian Markov processes, are well established by now, as a direct consequence of the central-limit theorem being operative in linear systems with additive noise sources.

A complete analytical description of these processes in terms of a transition PDF (probability density function) can be given, but first- and second-order moments are sufficient and usually are preferred.

The expanding interest in nonlinear phenomena resulted in a variety of nonlinear stochastic models with additive and/or multiplicative noise. Approximation techniques such as equivalent linearization or hierarchical truncation of the moment equations were used from the beginning, yielding process descriptions of varying accuracy.

Exact analytical results are sparse, and they are limited mainly to cases that can be transformed into some known stochastic process (e.g., the log-normal process,[1,2] the Hongler model[3,4]) or to cases where the Fokker-Planck equation (FPE) can be solved by reduction to some standard equation from mathematical physics (Bessel, Laguerre,[5] hypergeometric functions[6]). Integral transform methods have been used occasionally.[7,8]

The use of a particular stochastic model is confined very often to the field of research where it was encountered first. Some processes seem to have never left the context of their purely mathematical origin. Interdisciplinary diffusion of exact results appears to be a slow process. The model, which actually is well known as the Verhulst model, or the logistic growth model,[3] was studied and solved at least three times independently in Refs. 6, 9, and 10 before it finally got a multidisciplinary status in the domain of synergetics and noise-induced transitions.

The one-dimensional hyperbolic sine model, which will be presented in this paper, also results from apparently very different stochastic models used in various fields. The model, is in a sense, a common minimal version of all the original models, having in its canonical appearance only two parameters and an additive noise excitation.

Throughout this paper classical calculus will be used, implying the Stratonovich interpretation of a stochastic differential equation (SDE); white-noise sources are thus to be considered as limiting cases of real continuous noise with very short correlation times.

ORIGIN OF THE HYPERBOLIC SINE MODEL

We now elaborate on the origins of the hyperbolic sine model.

(1) The hyperbolic sine model can be arrived at by construction of a hierarchy of simple multiplicative stochastic processes. This procedure demonstrates one possible significance of the hyperbolic sine stochastic process and also shows the interrelations between a few basic multiplicative stochastic processes.

The elementary multiplicative stochastic process $\{x(t)\}$, when considered as a mere generalization of the Wiener process, is described by the SDE:

$$\dot{x}(t)=[f_0+F(t)]x(t), \quad x\in[0,+\infty] \tag{1}$$

where f_0 is a constant multiplication rate which, according to the research field, can be termed as "excess reactivity," "fertility," a "Malthusian growth parameter," i.e., $F(t)$ is a white-noise process with

$$\langle F(t)\rangle=0 \ , \tag{2}$$

$$\langle F(t)F(t+\tau)\rangle=G_F\delta(\tau) \ . \tag{3}$$

Without loss of generality, the constant white-noise spectral density G_F may for convenience be adjusted to

$$G_F=2 \tag{4}$$

by a suitable choice of the time scale. The process $\{x(t)\}$ has a log-normal transition PDF and is stochastically unstable in the probability,[1,2] except for a possible singular

$\delta(x)$ distribution when $f_0<0$.

The inversion transformation

$$x \rightarrow a/x, \quad a>0 \tag{5}$$

leaves (1) structurally invariant. Only the sign of the system parameter f_0 changes. The sign change of the noise function is irrelevant, of course.

To compensate for the almost sure extinction of the process $\{x(t)\}$ when $f_0<0$, a constant positive source (immigration) s_0 can be introduced in (1):

$$\dot{x}(t)=[f_0+F(t)]x(t)+s_0, \quad f_0<0, \; s_0>0 . \tag{6}$$

This also changes the nature of the boundary at $x=0$ and yields a new type of stochastic process. The inversion transformation (5) applied to (6) now gives

$$\dot{x}(t)=-f_0x(t)-(s_0/a)x^2(t)+F(t)x(t) , \tag{7}$$

which is the stochastic Verhulst model,[3] when taking $a=s_0$ and $\lambda=-f_0$:

$$\dot{x}(t)=\lambda x(t)-x^2(t)+F(t)x(t) . \tag{8}$$

Besides the mathematical artifact of an inversion transformation, there is also a more physical way to arrive at (8). The process (1) almost surely "explodes" when $f_0>0$. Stabilization can be obtained now by introducing a state-dependent (e.g., linear) feedback upon the multiplication rate f_0 (cf. the limited carrying capacity in population dynamics):

$$\dot{x}(t)=[f_0-x(t)+F(t)]x(t), \quad f_0>0 . \tag{9}$$

This again is the Verhulst model (8), with $\lambda=f_0$.

Addition of a constant source s_0 to the stochastically stable process described by (7)–(9) results in

$$\dot{x}(t)=s_0+f_0x(t)-x^2(t)+F(t)x(t), \quad s_0,f_0>0 \tag{10}$$

which is structurally invariant under the inversion (5):

$$\dot{x}(t)=a-f_0x(t)-(s_0/a)x^2(t)+F(t)x(t) . \tag{11}$$

An obvious choice for a is $a=s_0$. The processes in (10) and (11) then only differ in the sign of f_0 and have a normalizable steady-state PDF as will be shown later. Further source addition and/or inversion clearly leaves the processes described by (10) or (11) structurally invariant.

The multiplicative stochastic models (1), (8), and (10) can be transformed to additive noise models by letting

$$x=\alpha \exp(y), \quad \alpha>0 \tag{12}$$

which is indicated by the non-negative physical nature of x and by the log-normal solution for the PDF of (1).

One finds, respectively, after appropriate choice of α in (12),

$$\dot{y}(t)=f_0+F(t), \quad y\in[-\infty,+\infty] \tag{13}$$

$$\dot{y}(t)=\alpha[1-\exp(y)]+F(t) , \quad y\in[-\infty,+\infty], \alpha=\lambda>0 \tag{14}$$

$$\dot{y}(t)=\beta-\alpha\sinh(y)+F(t) , \quad y\in[-\infty,+\infty], \; \alpha=(s_0)^{1/2}>0, \; \beta=f_0 . \tag{15}$$

The three models, where (15) is the hyperbolic sine model, can be interpreted to describe Brownian motion in a particular type of force field. The mere use of mathematical transformations such as (5) or (12) was criticized in Ref. 3 as being nonphysical. The additive noise equation, however, is a canonical form to which a one-dimensional SDE can be reduced unambiguously, and which allows one to easily discover equivalent stochastic processes.

(2) The true origin of the hyperbolic sine model is in nuclear-reactor stochastic kinetics, where the following two-dimensional stochastic Markov process was considered by the author:

$$\dot{n}(t)=[f_0-1+F(t)]n(t)+\lambda c(t) , \tag{16}$$

$$\dot{c}(t)=n(t)-\lambda c(t), \quad n,c\in[0,+\infty] . \tag{17}$$

The system (16) and (17) is a simplified reactor model used for the study of delayed neutron effects.

Equation (16) again describes the neutron (n) chain reaction. The effective (or "prompt") multiplication is now reduced to f_0-1, as a fraction of the fission neutrons only becomes available as a delayed neutron source λc by the decay of delayed neutron precursors (c). The latter are produced and do decay according to (17).

The process $\{n(t), c(t)\}$ is known to be stochastically unstable in the absence of any feedback.

Transformation of (16) and (17) to the additive noise version can be done by the physically indicated substitutions

$$n=a\exp(x), \quad a>0 \tag{18}$$

$$c=b\exp(y), \quad b>0 . \tag{19}$$

This yields, for the stochastic process $\{x(t), y(t)\}$,

$$\dot{x}(t)=f_0-1+(\lambda b/a)\exp(y-x)+F(t) , \tag{20}$$

$$\dot{y}(t)=(a/b)\exp(x-y)-\lambda . \tag{21}$$

Subtracting (21) from (20), introducing z by

$$z=x-y , \tag{22}$$

and choosing $a/b=\lambda^{1/2}$ gives, for the process $\{z(t), y(t)\}$,

$$\dot{z}(t)=\beta-\alpha\sinh(z)+F(t) , \tag{23}$$

$$\dot{y}(t)=(\tfrac{1}{2}\alpha)[\exp(z)-(\tfrac{1}{2}\alpha)], \quad y,z\in[-\infty,+\infty] \tag{24}$$

with parameters

$$\alpha=2(\lambda)^{1/2} , \tag{25}$$

$$\beta=f_0+\lambda-1 . \tag{26}$$

Equation (23) again is the hyperbolic sine model, and so the z variable *separately describes a one-dimensional stable Markov process.*

(3) In Refs. 11 and 3 the "genetic model" was studied as a basic example of a nonlinear system with a purely

noise-induced transition:

$$\dot{x}(t)=\alpha(\tfrac{1}{2}-x)+\beta x(1-x)+x(1-x)F(t), \quad x\in[0,1] \ . \tag{27}$$

The parameter α ($\alpha>0$) in (27) does not result from a generalization of the original genetic model ($\alpha=1$), but is due to the normalization of F according to (4).

Transformation of (27) to "canonical" additive form is now performed by

$$\exp(z)=x/(1-x) \quad \text{or} \quad x=\exp(z)/[\exp(z)+1] \ , \qquad z\in[-\infty,+\infty] \tag{28}$$

which immediately results in the hyperbolic sine model (23).

SOLUTION OF THE FOKKER-PLANCK EQUATION

The stochastic process $\{x(t)\}$ generated by the hyperbolic-sine–model SDE, with parameters rescaled for convenience and $F(t)$ according to (2)–(4),

$$\dot{x}(t)=2\beta-2\alpha\sinh(x)+F(t), \quad x\in[-\infty, \ \infty], \ \alpha\geq 0 \tag{29}$$

is a stationary Markov process, fully characterized by its transition PDF $p(x,t/x_0)$.

This PDF is obtainable as the solution of the time-dependent FPE associated with (29),[5,3]

$$\delta_t p(x,t/x_0)=\delta_x^2 p-\delta_x\{[2\beta-2\alpha\sinh(x)]p\} \ , \tag{30}$$

with initial condition

$$p(x,0/x_0)=\delta(x-x_0) \ . \tag{31}$$

As the diffusion-interval boundaries at $\pm\infty$ are GS natural or F entrance [see Ref. 3 for a comparison of the Gihman-Skorohod (GS) and Feller (F) classification schemes], no boundary conditions need to be imposed.

The stationary PDF $p_s(x)$, which basically is defined by

$$p_s(x)=\lim_{t\to\infty} p(x,t/x_0) \ , \tag{32}$$

can be obtained from (30) with $\delta_t p=0$.

One finds

$$p_s(x)=N\exp[2\beta x-2\alpha\cosh(x)] \ , \tag{33}$$

where the normalizing factor N follows from[12]

$$N^{-1}=\int_{-\infty}^{\infty}dx\,\exp[2\beta x-2\alpha\cosh(x)] =2K_{2\beta}(2\alpha), \quad \alpha>0 \tag{34}$$

with $K_{2\beta}(2\alpha)$ the modified Bessel function.

So, $p_s(x)$ is normalizable for $\alpha>0$. This proves that the processes (10) and (11) also have a normalizable steady-state PDF.

The supposed universality of the hyperbolic sine model motivates the search for an analytical time-dependent solution of (30) as well. Starting with the expansion[5,3]

$$p(x,t/x_0)=p_s(x)\sum_{n=0}^{\infty}\phi_n(x)\phi_n(x_0)\exp(-\lambda_n t) \ , \tag{35}$$

the equation for the eigenfunctions $\phi_n(x)$ is

$$\phi_n''(x)+[2\beta-2\alpha\sinh(x)]\phi_n'(x)+\lambda_n\phi_n(x)=0 \ , \tag{36}$$

which is the backward Kolmogorov eigenfunction equation.

This equation has essential singularities at $\pm\infty$ and is not reducible to some standard equation of mathematical physics. Gathering the relevant information about the solutions of (36) from Sturm-Liouville theory, from Ref. 13 (see also Elliott's theorem in Ref. 3), and from system symmetries, one has the following.

(a) The spectrum of eigenvalues λ_n forms a discrete increasing sequence of non-negative real values:

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots \lim_{n\to\infty}\lambda_n=+\infty \ .$$

(b) The set of eigenfunctions ϕ_n is complete in $C[-\infty,+\infty]$.

(c) ϕ_n has exactly n simple zeros in $[-\infty,+\infty]$.

(d) Lemma 4.2 of Ref. 13 asserts the existence of a single solution $y(x)$, bounded in $[0,\infty]$ and satisfying the condition (zero-probability flux boundary condition)

$$\lim_{x\to\infty} p_s(x)y'(x)=0 \ . \tag{37}$$

The same is true for the interval $[-\infty,0]$ and the other entrance boundary at $x=-\infty$.

(e) The symmetries of the SDE (29) induce the following property of the eigenfunctions:

$$\phi_n(x,\beta)=\pm\phi_n(-x,-\beta) \ . \tag{38}$$

Summarizing, it is concluded that for each eigenvalue λ_n there exists one single eigenfunction $\phi_n(x)$. The eigenfunctions are bounded in $[-\infty,+\infty]$ and have the symmetry

$$\phi_n(x,\beta)=(-1)^n\phi_n(-x,-\beta) \ . \tag{39}$$

The solution of (36) can now proceed as follows.

The coefficient of the first derivative in the equation is periodic with imaginary period $2\pi i$. Characteristic periodic solutions similar to Mathieu functions[14] may exist and can be found by substituting a series solution

$$\phi_n(x)=\sum_k a_k q_k(x) \ , \tag{40}$$

where $q_k(x)$ is some suitable nucleus satisfying

$$q_k(x+2\pi i)=q_k(x) \ . \tag{41}$$

A direct analogy with the case of Mathieu's (modified) equation where $q_k(x)=\exp(kx)$ [or $=\cosh(kx)$, $\sinh(kx)$] does not work out here. A *bounded* $2\pi i$-periodic function of x, however, is

$$\tanh(x/2)=[\exp(x)-1]/[\exp(x)+1] \ , \qquad |\tanh(x/2)|\leq 1 \tag{42}$$

resulting in a nucleus with desirable properties:

$$q_k(x)=[\tanh(x/2)]^k, \quad |q_k(x)|\le 1 \quad \text{for } k\ge 0 . \tag{43}$$

The series solution (40) acquires the form

$$\phi_n(x)=\sum_{k=0}^{\infty} a_k[\tanh(x/2)]^k , \tag{44}$$

which upon substitution in (36) yields a five-term, fourth-order recurrence formula for the coefficients a_k:

$$(k+1)(k+2)a_{k+2}+4\beta(k+1)a_{k+1}+(4\lambda-8\alpha k-2k^2)a_k-4\beta(k-1)a_{k-1}+(k-1)(k-2)a_{k-2}=0, \quad k\ge 3 \tag{45}$$

together with the starting relations (or recurrence boundary conditions)

$$a_2+2\beta a_1+2\lambda a_0=0 , \tag{46}$$

$$3a_3+4\beta a_2+(2\lambda-4\alpha-1)a_1=0 , \tag{47}$$

$$3a_4+3\beta a_3+(\lambda-4\alpha-2)a_2-\beta a_1=0 . \tag{48}$$

It is clear from (45)–(48) that, provided λ is known, a pair of (a_0,a_1) values entirely determines the solution. So, apparently two independent solutions are possible, in general. However, the symmetry condition (39) translates into an additional relation for the coefficients,

$$a_k(\beta)=(-1)^{k+n}a_k(-\beta) , \tag{49}$$

stating, e.g., that odd coefficients of even eigenfunctions should be odd functions of β. Specifically, for (a_0,a_1) if a_0 is odd in β, then a_1 should be even and vice versa. This essentially destroys the independence and leaves for each eigenvalue only one solution, which is determined up to an arbitrary normalizing constant.

Higher-order recurrences have not been very popular in mathematical physics, despite an increased understanding and an enlarged availability of computational tools and methods.[15–17]

The problem one is faced with, basically, is twofold.

(1) Calculation of the eigenvalues. Essentially, this can be done by expressing the compatibility of the homogeneous relations [(45)–(48)] by demanding an infinite (Hill) determinant to be zero. The infinite number of roots λ_n is the set of eigenvalues. Simplifying computational algorithms, as alternatives to infinite determinant calculations, are fortunately available. See, e.g., Ref. 15.

(2) Calculation of the coefficients. Even when an eigenvalue λ_n is known, the straightforward use of the recurrence relation is hampered by the existence of m different solutions for a recurrence of order m. Due to the finite precision of the computation, the so-called "dominant solution" emerges when proceeding in a forward way. Regressive calculation with the Miller algorithm, starting at sufficiently large N with

$$a_N=1,\ a_{N+1}=a_{N+2}=\cdots=a_{N+m-1}=0 , \tag{50}$$

however, generates the "minimal solution," which usual-

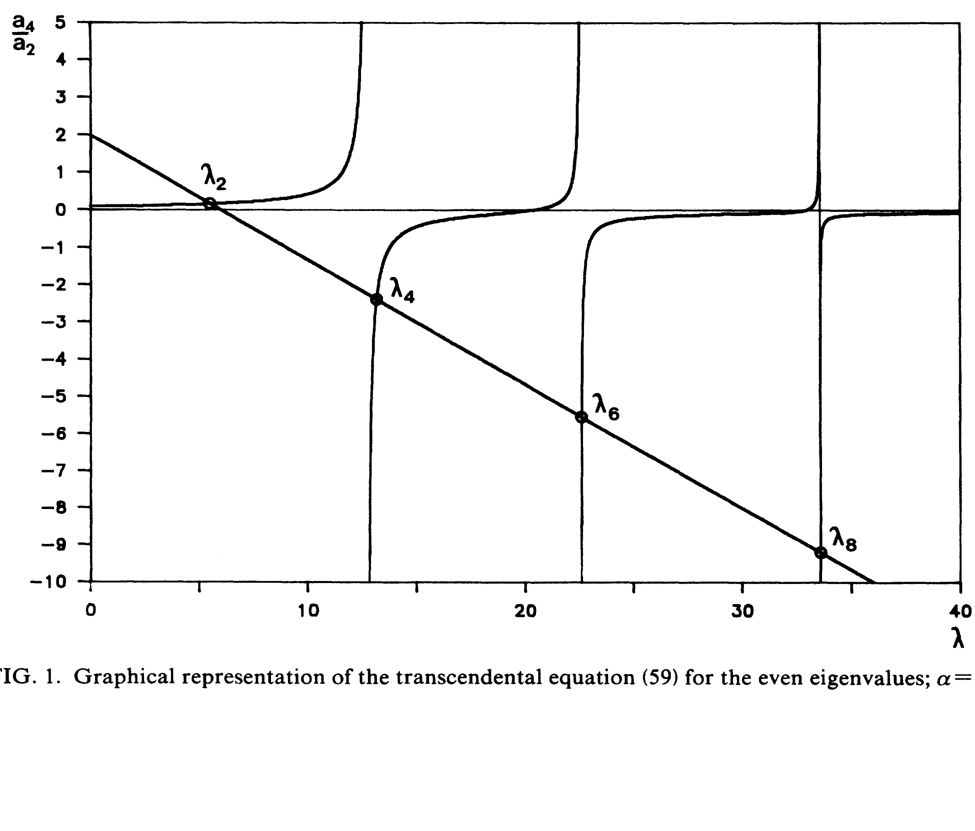

FIG. 1. Graphical representation of the transcendental equation (59) for the even eigenvalues; $\alpha=1$.

ly is the convergent series one is looking for in physical applications. See Ref. 16 for an in-depth analysis of this computational problem.

In this paper the feasibility of the above approach will be illustrated only for the highly symmetrical case with $\beta=0$, where the method becomes particularly simple and elegant. Rephrasing the relevant equations for $\beta=0$, one has the following.

(i) Eigenfunction equation (36):

$$\phi_n''(x)-2\alpha\sinh(x)\phi_n'(x)+\lambda_n\phi_n(x)=0 . \tag{51}$$

(ii) Series solution (44):

$$\phi(x)=\sum_{k=0}^{\infty} a_k[\tanh(x/2)]^k . \tag{52}$$

(iii) Recurrence for the a_k, (45), and starting relations (46)–(48):

$$(k+1)(k+2)a_{k+2}+(4\lambda-8\alpha k-2k^2)a_k +(k-1)(k-2)a_{k-2}=0,\quad k\geq 3 \tag{53}$$

$$a_2+2\lambda a_0=0 , \tag{54}$$

$$3a_3+(2\lambda-4\alpha-1)a_1=0 , \tag{55}$$

$$3a_4+(\lambda-4\alpha-2)a_2=0 . \tag{56}$$

(iv) Symmetry of the eigenfunctions (39) or (49):

$$\phi_n(x)=(-1)^n\phi_n(-x) , \tag{57}$$

or all $a_{2k+1}=0$ for even eigenfunctions, which now become even functions of x, and all $a_{2k}=0$ for odd eigenfunctions, which are now odd in x.

Recurrence (53) is still fourth order, but eventually is reducible to a pair of second-order recurrences, one for the even eigenfunctions and one for the odd. Eigenvalues λ_n and coefficients a_k can be obtained simultaneously as follows.

The ratio of two successive coefficients,

$$R_k=a_k/a_{k-2} ,$$

can be computed from an infinite continued fraction that is equivalent to (53):

$$R_k=(k-1)(k-2)[(2k^2+8\alpha k-4\lambda) -(k+1)(k+2)R_{k+2}]^{-1} , \tag{58}$$

starting with $R_N=0$ at sufficiently large N and working backward.

For even eigenfunctions ϕ_{2m}, one starts at a large even N and one arrives at R_4, the value of which is dependent upon the λ value used during the calculation. Using, now, the even starting relation (56), it follows that

$$a_4/a_2=R_4(\lambda)=(4\alpha+2-\lambda)/3 , \tag{59}$$

which is a transcendental equation for the even eigenvalues λ_{2m}, as graphically represented in Fig. 1.

Solving (59) iteratively simultaneously yields the sequences $R_4,R_6,\ldots,R_{2k}$. Together with R_2 from (54),

$$R_2=a_2/a_0=-2\lambda , \tag{60}$$

TABLE I. Lower eigenvalues λ_k for the symmetrical case ($\beta=0$).

k	$\alpha=0.1$	$\alpha=1.0$	$\alpha=5.0$
0	0.0	0.0	0.0
1	0.431 236 19	2.396 525 95	10.468 404 93
2	1.228 446 18	5.480 983 57	21.823 448 28
3	2.276 557 50	9.105 704 11	33.979 604 94
4	3.546 019 48	13.202 430 34	46.874 846 14
5	5.018 331 68	17.727 686 66	60.461 131 60
6	6.680 401 00	22.650 446 86	74.701 106 07
7	8.522 313 22	27.947 047 54	89.562 302 76
8	10.536 204 42	33.598 626 42	105.018 303 3

all ratios up to R_2 are determined, and choosing $a_0=1$ (or another suitable normalization value) generates the coefficients. Only ratios R_k with $k \ll N$, which are sufficiently stabilized when increasing N, should be considered for this step. This implies that the value of N is dictated by the number of stable coefficients one needs for a prescribed precision of the eigenfunctions.

Similarly, for odd eigenfunctions ϕ_{2m+1} one finds

$$a_3/a_1=R_3(\lambda)=(4\alpha+1-2\lambda)/3 , \tag{61}$$

which generates the λ_{2m+1} and the corresponding sequences $R_3,R_5,\ldots,R_{2k+1}$. Here, $a_1=1$ is one possible choice. Finally, the lowest even eigenfunction, ϕ_0, corresponding to $\lambda_0=0$ is a constant.

Table I contains the first few eigenvalues for three α values.

Figures 2 and 3 show some eigenfunctions $\psi_n(x)$ of the FPE, which are related to the eigenfunctions of (51) by

$$\psi_n(x)=p_s(x)\phi_n(x) ,$$

with $p_s(x)$ according to (33) and (34) with $\beta=0$.

DERIVED MODELS

The hyperbolic sine model was shown to be the canonical form of different known multiplicative stochastic models.

Inversely, an infinity of nonlinear transformations can be applied to the hyperbolic sine model to generate new stochastic processes. However, only a few of them can be expected to have a possible physical or mathematical significance.

(a) The hypberbolic sine model (29) is transformed back to the "stochastic Verhulst model with constant source" by choosing the variable

$$y=\exp(x),\quad y\in[0,+\infty] . \tag{62}$$

One obtains the SDE:

$$\dot y(t)=\alpha(1-y^2)+2\beta y+yF(t),\quad \alpha\geq 0 . \tag{63}$$

The stationary PDF of this process follows from (33) and

$$|p_s(y)dy|=|p_s(x)dx| . \tag{64}$$

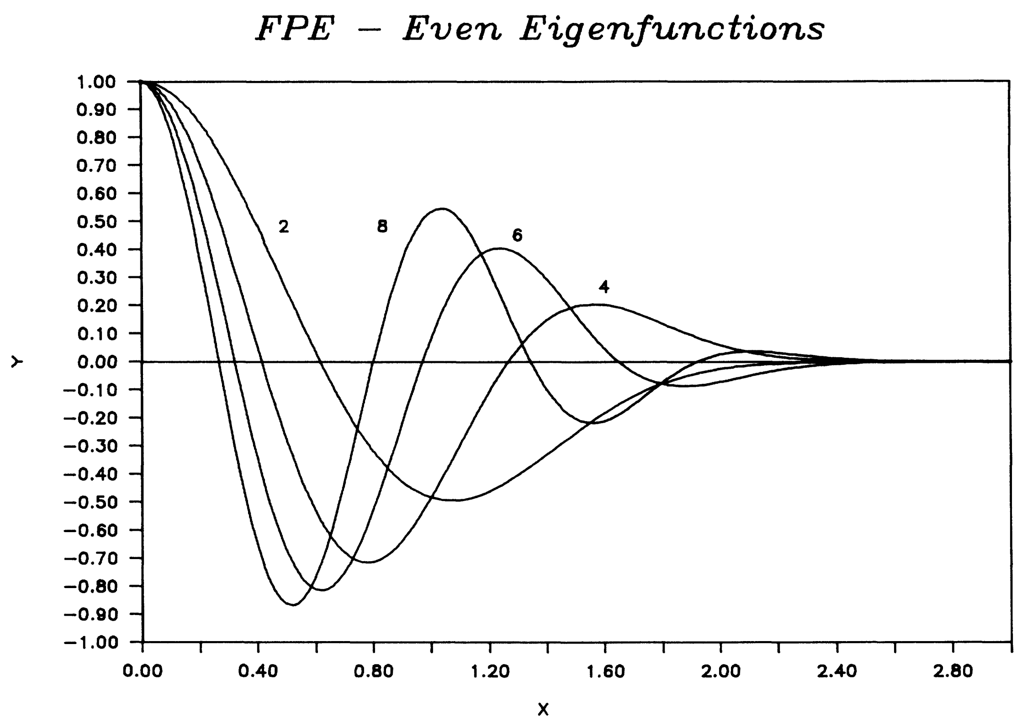

FIG. 2. Even eigenfunctions ψ_k of the FPE for $k=2,4,6,8$; symmetrical case $\beta=0$; $\alpha=1$.

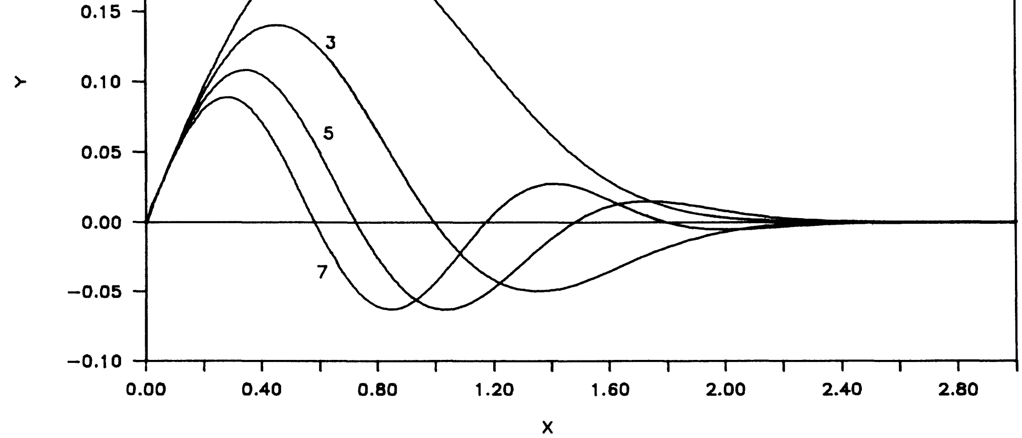

FIG. 3. Odd eigenfunctions ψ_k of the FPE for $k=1,3,5,7$; symmetrical case $\beta=0$; $\alpha=1$.

One finds

$$p_s(y)=Ny^{2\beta-1}\exp[-\alpha(y+1/y)],\quad y\in[0,+\infty]\ . \tag{65}$$

The inverse process,

$$z=1/y=\exp(-x)\ , \tag{66}$$

also diffuses in $[0,+\infty]$ and clearly has the same PDF's, with a sign change of β. In the symmetrical case $\beta=0$, one observes that the process $\{y(t)\}$ and its inverse *have exactly the same PDF's.*

(b) The central role of $\tanh(x/2)$ in the time-dependent solution of the FPE suggests that one consider as a new variable

$$y=\tanh(x/2),\quad y\in[-1,+1] \tag{67}$$

which obeys the SDE,

$$\dot y(t)=\beta(1-y^2)-2\alpha y+\tfrac{1}{2}(1-y^2)F(t)\ , \tag{68}$$

describing a stochastic process with quadratic (*Ito*) drift and diffusion. Inversion now results in

$$z=1/y=\coth(x/2),\quad |z|\geq 1 \tag{69}$$

$$\dot z(t)=\beta(1-z^2)+2\alpha z+\tfrac{1}{2}(1-z^2)F(t)\ . \tag{70}$$

The PDF's for z are the same functions as for y, but with the sign of α reversed, and defined over a different diffusion interval.

The stationary PDF for z, e.g., is

$$p_s(z)=2N|z+1|^{2\beta-1}|z-1|^{-2\beta-1}\times\exp[-2\alpha(z^2+1)/(z^2-1)],\quad |z|\geq 1\ . \tag{71}$$

This distribution is bimodal, having one lobe at each side of the inaccessible interval $[-1,+1]$.

The process is continuously "tunneling" between the two lobes via $\pm\infty$, with a finite transition probability. Physical applications are not obvious.

(c) Translational invariance of the hyperbolic sine model is trivial:

$$\dot x(t)=2\beta-2\alpha\sinh(x-\gamma)+F(t) \tag{72}$$

is reduced to (29) by $x-\gamma\rightarrow x$.

This, however, defines a one-parameter family of nonlinear transformations, leaving (68) structurally invariant. To show this, let x represent the original hyperbolic-sine–model variable as in (29), and

$$y=\tanh(x/2),\quad |y|\leq 1 \tag{73}$$

$$z=\tanh[(x-\gamma)/2],\quad |z|\leq 1\ . \tag{74}$$

Then,

$$z=(y-\mu)/(1-\mu y)\ , \tag{75}$$

with

$$\mu=\tanh(\gamma/2),\quad |\mu|\leq 1\ . \tag{76}$$

While y describes the process (68), z has the SDE:

$$\dot z(t)=-[\beta+\alpha\sinh(\gamma)]z^2-2\alpha\cosh(\gamma)z+[\beta-\alpha\sinh(\gamma)]+\tfrac{1}{2}(1-z^2)F(t)\ . \tag{77}$$

The transformation (75) and (76) thus leaves the diffusion coefficient absolutely invariant—it maps the diffusion interval onto itself and it leaves the drift structurally invariant as it only changes the coefficients of this quadratic term.

As an example of possible application, it follows that a SDE of the type

$$\dot z(t)=-az^2-2bz+c+\tfrac{1}{2}(1-z^2)F(t),\quad |z|\leq 1,\ b>0 \tag{78}$$

can be reduced to the standard type (68), which only has two parameters and which is equivalent to the hyperbolic sine model, by taking

$$\beta=(a+c)/2\ , \tag{79}$$

$$\tanh(\gamma)=(a-c)/2b\ , \tag{80}$$

$$\alpha=b/\cosh(\gamma)\ , \tag{81}$$

$$\mu=\tanh(\gamma/2)\ , \tag{82}$$

$$y=(z+\mu)/(1+\mu z)\ . \tag{83}$$

The condition for this reduction to the possible follows from (80):

$$|a-c|<2b\ . \tag{84}$$

CONCLUSIONS

The hyperbolic sine stochastic model arises as the canonical form of several nonlinear multiplicative stochastic processes.

The simple appearance of the model and its associated FPE inspires an exact-solution method for the transition PDF in terms of a new type of eigenfunction which is expressible as a power series of $\tanh(x/2)$.

The problem whether nuclei more suitable for normalization or for second-order moment calculation exist is still unresolved. The solution method is believed to be generalizable to other similar equations, as recurrence calculations are no longer problematic, and as they offer an interesting alternative to "crude" numerical solutions.

A few models derived from the hyperbolic sine process were presented without regard for their physical content. There is some hope (or some synergetic belief) that in view of their particular symmetries and mathematical properties, these models may fit some physical process as well. Transformation properties of the models considered can indeed be related to the fact that the variable in the hyperbolic sine model (29) is representing a (logarithmic) ratio of two dependent physical quantities, e.g., neutron density and delayed neutron precursor density in (22) and (23), or two complementary chemical concentrations or allele frequencies in (28).

Physical symmetry dictates the observation of x/y or of y/x to be equivalent.

Finally, a remark related to noise-induced transitions is in order.

Whereas the genetic model (27) exhibits bimodality of its stationary PDF above some critical noise level [corresponding to $\alpha \leq 0.5$ in (27)],[3] no similar phenomenon is displayed in the stationary PDF of the hyperbolic sine process, the bimodality being masked by the transformation (28). This example shows that noise-induced transitions—or "parameter-induced transitions" if the noise intensity is kept normalized—may appear as a pure consequence of the accidental physical scale of observation one is dealing with in a particular application.

The question whether "critical dynamics,"[3] as determined by the eigenfunctions and the invariant spectrum of eigenvalues of the FPE, do constitute a more robust characteristic of noise-induced transitions is actually being investigated.

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