Localized structures in surface waves

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An amplitude equation in the form of a perturbed nonlinear Schrödinger equation is derived for parametric excitation of surface waves in an extended system. Continuous symmetries of the unperturbed system are used to identify critical modes. Dynamical equations for the latter are derived using singular perturbation theory. The existence of a stable nonpropagating kink solution is predicted. The solution connects two uniform states whose phases of oscillations differ by π, and should be observable in wide enough cells. A stable nonpropagating soliton solution is found for subcritical excitation.

I. INTRODUCTION

Localized structures, such as envelope kinks and solitary waves, have been recently observed in a number of extended physical systems. Examples include the system of parametrically excited surface waves,[1] elastic media,[2] electroconvection in nematic liquid crystals,[3] and binary mixtures under thermal gradients.[4] Besides the immediate interest that such self-sustained structures raise,[5,6] they also provide a clue for understanding spatiotemporal chaos in extended systems, for complex patterns may arise from the interactions among many such simple units.[7,8] A satisfactory theory of localized structures should address the questions of existence, stability, and dynamics, and of the effects that a perturbing field may induce. The latter can be external or due to other localized structures. In this paper we propose such a theoretical framework for the system of parametrically excited surface waves.[1,9] The approach introduced here is quite general and can be applied to other systems as well.

The system we study consists of a fluid layer in a long channel which is subjected to vertical oscillations. Parametric instability occurs when the forcing amplitude exceeds some threshold value and generates waves at the free surface of the fluid. The response frequency is commonly half the driving frequency. Recently, a nonpropagating envelope solitary wave has been observed in that system.[1] Following this observation two theoretical studies have appeared. In the first,[10] the solitary wave is attributed to a soliton solution of a nonlinear Schrödinger (NLS) type amplitude equation, derived for an unforced inviscid fluid. The second study[11] includes the effects of dissipation and periodic forcing and results in a perturbed NLS equation. An exact solution representing a stable nonpropagating solitary wave is found. In this work we extend the analysis to include dynamical aspects. We pursue an approach in which the dynamics of localized patterns are dictated by the continuous symmetries of the system. The symmetry group of the NLS equation contains four such symmetries. Consequently the evolution of any localized structure is determined by dynamical equations for four group parameters. We demonstrate this approach with the example of the solitary wave that has been considered in Refs. 10 and 11, thereby rederiving the results reported therein, and apply it to a different kind of localized structure, the envelope kink wave, predicted to appear in wide enough cells.

II. THE AMPLITUDE EQUATION

Consider a coordinate system which moves with the cell such that (x,y) is the horizontal plane and the free surface of the fluid, in the quiescent state, is at $z=0$. For an incompressible and inviscid fluid the hydrodynamic equations for the free surface take the form[12]

$$\zeta_t-\phi_z+N_1(\zeta,\phi)=0 \quad \text{at } z=0\,, \tag{1a}$$

$$\phi_t+(g-\delta\cos\omega t)\zeta-(\gamma/\rho)\nabla_\perp^2\zeta+N_2(\zeta,\phi)=0 \quad \text{at } z=0\,, \tag{1b}$$

$$\nabla^2\phi=0 \quad \text{for } -d\le z\le\zeta\,, \tag{1c}$$

where $\phi(x,y,z,t)$ is the velocity potential, $\zeta(x,y,t)$ is the (vertical) z coordinate of the free surface, N_1 and N_2 are nonlinear terms, and g, ρ, γ, δ, and ω are, respectively, gravitational acceleration, density, surface tension, amplitude, and frequency of the periodic forcing. The subscripts in (1) denote partial derivatives and ∇^2 and $\nabla_\perp^2$ are, respectively, the Laplacians in the three-dimensional space and in the (x,y) plane. To third order in ζ and ϕ the nonlinear terms read

$$N_1=\nabla_\perp\zeta\cdot\nabla_\perp\phi-\zeta\phi_{zz}-\tfrac{1}{2}\zeta^2\phi_{zzz}+\tfrac{1}{2}\nabla_\perp\zeta^2\cdot\nabla\phi_z\,, \tag{2a}$$

$$N_2=\tfrac{1}{2}(\nabla\phi)^2+\zeta\phi_{tz}+\tfrac{1}{2}\zeta(\nabla\phi)_z^2+\tfrac{1}{2}\zeta^2\phi_{tzz}+\frac{\gamma}{2\rho}\nabla_\perp\cdot[(\nabla_\perp\zeta)^2\nabla_\perp\zeta]\,. \tag{2b}$$

Equations (1) are supplemented by the boundary conditions $\phi_z=0$ at $z=-d$, and $\phi_y=\zeta_y=0$ at $y=0,b$ where b and d are the width and depth of the cell, respectively.

The unforced linearized system corresponding to (1) has the solution $(\phi,\zeta)=\mathbf{U}_0^T(\mathbf{k})+\text{c.c.}$ where

$$\mathbf{U}_0(\mathbf{k})=\exp(i\omega_0 t+ik_x x)\cos(k_y y)\begin{bmatrix}\tau(k,z)\\-4i\sigma(k)\end{bmatrix}. \tag{3}$$

Here, $k=(k_x^2+k_y^2)^{1/2}$, $\tau(k,z)=\cosh[k(z+d)]/\cosh(kd)$, $\sigma(k)=k\tanh(kd)/4\omega_0$, and the natural frequency ω_0 is given by the dispersion relation $\omega_0^2=gk\tanh(kd)(1+\gamma k^2/\rho g)$. We shall confine ourselves in this paper to the simpler case where $k_x=0$. Thus $k=k_y=n\pi/b$ where n is an integer. The forcing frequency ω is chosen such that the $(0,n\pi/b)$ mode responds with half the driving frequency. These choices with $n=1$ correspond to the experimental conditions of Ref. 1.

The full solution to (1) is sought in the form

$$\mathbf{U}(x,t;\mathbf{k})=A(x,t)\mathbf{U}_0(k_y\hat{y})+\mathbf{V}(A(x,t)\exp(i\omega_0 t),\partial_x,y,z,t;\delta)+\text{c.c.}\ , \quad (4)$$

where $\mathbf{V}$ is $2\pi/k_y$-periodic in y, $2\pi/\omega$-periodic in t, and is assumed to have a Taylor expansion in A, A_x, $A_{xx}, \ldots$. Along with (4) we look for an amplitude equation for $A(x,t)$: $A_t=\mathcal{F}(A,\overline{A},\partial_x,t;\delta)$ where the bar denotes the complex conjugate. The functional form of $\mathcal{F}$ is dictated by the symmetries of (1).[13] Time reversal symmetry, $(t,\phi)\to(-t,-\phi)$, implies the invariance of the amplitude equation under the transformation $(t,A)\to(-t,-\overline{A})$ and, consequently, that all coefficients in the amplitude equation should be pure imaginary (see, however, the discussion below). Discrete time translation symmetry, $t\to t+2\pi n/\omega$, which implies invariance under $A\to A\exp(i\pi)$, suggests that a term proportional to $\delta\overline{A}$ should exist and that even powers of A and $\overline{A}$ should not be allowed. In addition, space reflection $(x\to -x)$ and translation $(x\to x+a)$ symmetries exclude odd space derivatives and explicit x dependence, respectively.

Using (4) in (1) we find the solution

$$\mathbf{V}=-\frac{1}{2k_y}\cos(k_y y)\exp(i\omega_0 t)A_{xx}\begin{bmatrix}\partial\tau/\partial k_y\\ -4i\partial\sigma/\partial k_y\end{bmatrix}-\frac{\delta\sigma}{4\omega_0}\cos(k_y y)\times A\begin{bmatrix}3\tau\exp(3i\omega t)\\ 16i\sigma\exp(-3i\omega t)-4i\sigma\exp(3i\omega t)\end{bmatrix}+\text{c.c.}+O(A^2)\ , \quad (5)$$

where A satisfies

$$iA_t=-i\eta A-\delta\sigma(k_y)\overline{A}\exp(-i\nu t)+\alpha(k_y)A_{xx}+\beta(k_y)|A|^2A+c\ , \quad (6)$$

where c represents higher-order terms. In (6), $\nu\equiv 2\omega_0-\omega$ is a detuning parameter and $\alpha\equiv(1/2k_y)\partial\omega_0/\partial k_y$ is positive. We do not display here the explicit form of the coefficient β, but instead, present in Fig. 1 a graph of $\beta(k_y)$. The damping term $i\eta A$ has been introduced to account for dissipation in the real system (mainly due to friction at the walls of the cell). We assume that η is small and of $O(\delta)$. Equation (6) is equivalent to the perturbed NLS equation that has been derived in Ref. 11. The dissipation may have the additional effect of introducing small imaginary parts into the coefficients in (6). We shall discuss the implications of this effect in Sec. V. Introducing the scaling $x\to[(1/k_y)\partial\omega_0/\partial k_y]^{-1/2}x$ and $A\to|\beta|^{-1/2}A$, we can rewrite Eq. (6) in the form of a perturbed NLS equation:

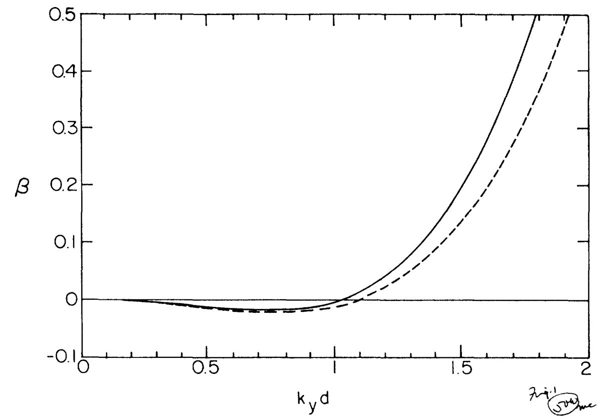

FIG. 1. A graph of the coefficient β in Eq. (6) as a function of $k_y d$ where the depth d of the cell has been taken to be 2 cm. The solid and the dashed curves correspond, respectively, to zero surface tension and to surface tension of water.

$$iA_t-\tfrac{1}{2}A_{xx}-\text{sgn}(\beta)|A|^2A=-i\eta A-\delta\sigma\overline{A}\exp(-i\nu t)\equiv\delta P\ . \quad (7)$$

In the following two sections we shall distinguish between two cases corresponding to β being positive or negative. The range of applicability of each case is implied by Fig. 1.

III. THE SURFACE-WAVE SOLITON

For β positive, the unforced $(\delta=0)$ amplitude equation (7) admits a soliton solution $A_s=\exp(-it/2)\text{sech}(x)$. The most general one-soliton solution of the unforced system is obtained by acting on A_s with the symmetry group of the NLS equation. This group corresponds to the following four transformations.

Dilatation: $x\to\lambda x$, $t\to\lambda^2 t$, $A\to\lambda A$.

Galilean transformation: $x\to x+vt$, $A\to A\exp(ivx+iv^2t/2)$.

Rotation [i.e. U(1)]: $A\to A\exp(i\psi_0)$.

Space-translation: $x\to x+\chi_0$.

(Notice that in our case time translations are equivalent to rotations.) Applying these symmetry transformations to $A_s(x,t)$ we obtain

$$A_s(x,t;\Lambda)=\lambda\,\text{sech}[\lambda(x+\chi_0+vt)]\times\exp[i(v^2-\lambda^2)t/2+iv(x+\chi_0)+i\psi_0]\ , \quad (8)$$

where $\Lambda\equiv\{\lambda,v,\psi_0,\chi_0\}$ is the set of group parameters. In the presence of a perturbation $(\delta\neq 0)$ the group parameters Λ become slow dynamical variables. One can therefore look for a solution to (7) in the form

$$A=A_s(x,t;\mathbf{\Lambda}(T))+\delta R(z,T)$$
$$\equiv[A_0+\delta R_0(z,T)]\exp(i\psi)\ , \tag{9}$$

where R is a correction term, $T\equiv\delta t$, $z\equiv x+\chi$, $\chi\equiv\chi_0+vt$, and $\psi\equiv\psi_0-(v^2+\lambda^2)t/2$.

Substituting (9) in (7) we obtain to leading order in δ

$$\mathfrak{L}R_0=\exp(-i\psi)\{P-i\partial_T[A_0\exp(i\psi)]\}\equiv W\ , \tag{10}$$

where

$$\mathfrak{L}\equiv(v+i\partial_z)^2/2+\lambda^2/2-2|A_0|^2-A_0^2\mathfrak{C}$$

and $\mathfrak{C}$ is the conjugation operator $\mathfrak{C}B=\bar{B}$. Dynamical equations for the group parameters $\mathbf{\Lambda}$ are obtained by demanding that W in (10) is orthogonal to the null space of the adjoint of $\mathfrak{L}$, where the inner product is defined to be $(F_1,F_2)=\mathrm{Re}\int_{-\infty}^{\infty}dz\,F_1\bar{F}_2$. Realizing that $\mathfrak{L}$ is self-adjoint we need to find the zero modes of $\mathfrak{L}$ itself. These are, respectively, the rotation and translation modes, iA_0 and $\partial_z A_0$:

$$\mathfrak{L}iA_0=0,\quad \mathfrak{L}\partial_z A_0=0\ . \tag{11}$$

Equations (11) imply only two solvability conditions. The critical modes associated with the dilatation and Galilean symmetries are zero modes of $(i\mathfrak{L})^2$ rather than of $i\mathfrak{L}$ (or $\mathfrak{L}$):

$$i\mathfrak{L}(-\lambda^{-1}\partial_\lambda A_0)=iA_0\ ,$$
$$i\mathfrak{L}(v^{-1}\partial_v A_0-\lambda^{-1}\partial_\lambda A_0)=v^{-1}\partial_z A_0\ . \tag{12}$$

The freedom that (11) leaves can be used to simplify the dynamical equations for χ and ψ. These can be written as $\dot{\chi}=v+\delta\partial_T\chi$ and $\dot{\psi}=-(v^2+\lambda^2)/2+\delta\partial_T\psi$, where the overdot denotes total time derivative. Setting $\partial_T\chi=\partial_T\psi=0$ would not affect the equations for λ and v and would merely modify R accordingly through (10). This stems from the observation that the terms in W [see (10)] that contain $\partial_T\chi$ and $\partial_T\psi$ belong to the image of $\mathfrak{L}$. The conditions that W is orthogonal to the zero modes iA_0 and $\partial_z A_0$ lead to the equations

$$\partial_T\lambda=\mathrm{Im}\int_{-\infty}^{\infty}dz\,\exp(-i\psi-ivz)P\lambda\,\mathrm{sech}(\lambda z)\ , \tag{13a}$$

$$\partial_T v=-\mathrm{Re}\int_{-\infty}^{\infty}dz\,\exp(-i\psi-ivz)P\lambda\,\mathrm{sech}(\lambda z)\tanh(\lambda z)\ . \tag{13b}$$

We stress that we have not used yet the specific form of the perturbation P. The analysis until now is quite general and applicable to any system which is described by a perturbed NLS equation.

For P given by (7) we find the following dynamical equations:

$$\dot{\chi}=v\ , \tag{14a}$$
$$\dot{v}=-2\pi\delta\sigma v^2\lambda^{-1}\sin(2\phi)\mathrm{csch}(\pi v\lambda^{-1})\ , \tag{14b}$$
$$\dot{\phi}=(\nu-v^2-\lambda^2)/2\ , \tag{14c}$$
$$\dot{\lambda}=-2\eta\lambda+2\pi\delta\sigma v\sin(2\phi)\mathrm{csch}(\pi v\lambda^{-1})\ , \tag{14d}$$

where $\phi\equiv\psi+\nu t/2$. Fixed-point solutions of (14) exist for $\nu>0$ and satisfy $\lambda_*^2=\nu$, $v_*=0$ and $\sin(2\phi_*)=\eta/(\delta\sigma)$. There are two such solutions: $2\phi_*=\arcsin(\eta/\delta\sigma)$ and $2\phi_*=\pi-\arcsin(\eta/\delta\sigma)$. They correspond to *nonpropagating* solitons whose amplitudes and phases are fixed but whose positions in space remain arbitrary. Stability analysis of (14) results in the condition $\cos(2\phi_*)>0$ for stability. Thus one soliton is stable while the other is unstable. Strictly speaking the stable soliton is only marginally stable since the perturbed system (7) is still translationally invariant. Any perturbation which breaks this invariance may induce motion. These results are in accordance with the experimental observations:[1] in the absence of translational symmetry breaking perturbation the observed soliton is stationary, however, when the cell is tilted or two solitons are created, soliton dynamics sets in.

The next step in the analysis is to solve (10) for R_0. We do this for the case where the group parameters attain their fixed-point values $\mathbf{\Lambda}_*$. Equation (10) then reads $\mathfrak{L}R_0=-\sigma\cos(2\phi_*)A_0$. Using (12) we find $R_0=\sigma\lambda_*^{-1}\cos(2\phi_*)\partial_{\lambda*}A_0$. Inserting this expression in (9) we get the first-order approximation to the soliton solution:

$$A(x,t;\mathbf{\Lambda}_*)=[A_0(x;\mathbf{\Lambda}_*)+\delta\sigma\lambda_*^{-1}\cos(2\phi_*)\partial_{\lambda_*}A_0(x;\mathbf{\Lambda}_*)]$$
$$\times\exp(i\phi_*-i\nu t/2)\ . \tag{15}$$

Equation (15), being the first term in a Taylor expansion of the soliton amplitude around $\delta=0$, suggests that an *exact* solution to (7) exists in the form

$$A(x,t;\mathbf{\Lambda}_*)=\lambda_\infty\mathrm{sech}[\lambda_\infty(x+\chi_0)]\exp(i\phi_*-i\nu t/2)\ , \tag{16}$$

where $\lambda_\infty=[\nu+2\delta\sigma\cos(2\phi_*)]^{1/2}$. Indeed, (16) is an exact solution of (7) for all ν values greater than $-2\delta\sigma\cos(2\phi_*)$. An equivalent solution has been found in Ref. 11. The two soliton solutions in the range of positive ν values have different amplitudes: the amplitude of the stable soliton $[\cos(2\phi_*)>0]$ is larger than that of the unstable one $[\cos(2\phi_*)<0]$. In the range $0>\nu>-2\delta\sigma\cos(2\phi_*)$ the uniform quiescent state is unstable. Consequently the soliton solution is expected to be unstable as well. We recall that in the above analysis β [in (7)] is positive. This and the condition $\nu>0$ ($\nu<0$) imply a subcritical (supercritical) bifurcation.[14] We conclude that a stable (large amplitude) soliton solution exists only in the case where the onset of surface waves is subcritical.

IV. THE SURFACE-WAVE KINK

We now proceed to the case of β negative. As β is decreased below zero the NLS soliton solution gives place to a stable traveling wave solution (Benjamin-Feir instability[15]). For $\nu<0$ the unperturbed ($\delta=0$) NLS equation has, in addition, a kink solution of the form $A_k(x,t)=\sqrt{-\nu/2}\tanh(\sqrt{-\nu/2}x)\exp(-i\nu t/2)$. The values of A_k as $|x|\to\infty$ are related by the discrete symmetry, $A\to-A$, of (7). The kink solution is therefore equivalent to a topological defect.

Acting with the NLS symmetry group on A_k we write a solution in the form

$$A_k(x,t;\mathbf{\Lambda}(T))=[\kappa\tanh(\kappa z)\exp(ivz)+\delta R(z,T)]\exp(i\psi)\ , \qquad (17)$$

where $T=\delta t$, $z=x+\chi$, and $\chi=\chi_0+vt$ remain as before, $\psi=\psi_0-(v^2-2\kappa^2)t/2$ and $\kappa\equiv\lambda\sqrt{-\nu/2}$. Similar considerations to those used in the soliton case lead to the dynamical equations

$$\dot{\chi}=v\ , \qquad (18a)$$

$$\dot{v}=-2\pi\delta\sigma v^2\kappa^{-1}\sin(2\phi)\mathrm{csch}(\pi v\kappa^{-1})\ , \qquad (18b)$$

$$\dot{\phi}=(\nu-v^2+2\kappa^2)/2\ , \qquad (18c)$$

$$\dot{\kappa}=-2\eta\kappa+2\delta\sigma\kappa\sin(2\phi)I(v,\kappa)\ , \qquad (18d)$$

where $\phi=\psi+vt/2$ is as before and $I(v,\kappa)=\lim_{\epsilon\to 0}\epsilon^2/(\epsilon^2+4v^2\kappa^{-2})$. Fixed-point solutions of (18) should satisfy $\kappa_*=\sqrt{-\nu/2}$, $v_*=0$, and $\sin(2\phi_*)=\eta/(\delta\sigma)$. Again, there are two such solutions: $2\phi_*=\arcsin(\eta/\delta\sigma)$ and $2\phi_*=\pi-\arcsin(\eta/\delta\sigma)$. The condition for stability is now $\cos(2\phi_*)<0$, implying that only one kink solution is stable.

The correction term for the kink solution, calculated for the fixed-point parameter values, is found to be $R=2\sigma\cos(2\phi_*)\partial_\nu A_k(x,0;\mathbf{\Lambda}_*)$. This form suggests the existence of the *exact* solution

$$A(x,t;\mathbf{\Lambda}_*)=\kappa_\infty\tanh(\kappa_\infty x)\exp(i\phi_*-i\nu t/2)\ , \qquad (19)$$

where $\kappa_\infty=[-\nu/2-\delta\sigma\cos(2\phi_*)]^{1/2}$. As in the soliton case, the amplitude of the stable kink solution $[\cos(2\phi_*)<0]$ is larger than that of the unstable solution.

Another implication of (19) is that a kink solution exists for positive ν values as well, provided $\nu<-2\delta\sigma\cos(2\phi_*)$. This solution is expected to be stable since the two uniform states it connects are stable. We may conclude that a stable kink solution exists in the regions $\nu<0$, $\cos(2\phi_*)<0$, and $\nu>0$, $\nu/2+\delta\sigma\cos(2\phi_*)<0$.

V. DISCUSSION

The kink solution found in Sec. IV connects two states whose phases of oscillations differ by π and exists for negative β values. As implied by Fig. 1, this condition for existence translates into $n\pi d/b<K_c$ where $K_c\approx 1$. The exact value of K_c depends on the surface tension γ of the fluid. The stable kink solution should therefore be observable in cells whose lateral aspect ratio b/d is large enough.

As already noted, one may expect to find small imaginary parts in the coefficients α and β in (6) due to dissipation. In fact, such components have been recently evaluated experimentally[16] and theoretically.[17] We can study their effect by absorbing them in the perturbation P. The outcome of such a calculation is the appearance of unstable propagating localized structures in addition to the stationary ones discussed above. The effect of any other perturbation can be studied in a similar manner. In particular, one may study in this way the dynamics of many interacting localized structures.[8]

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