Laser cooling of a trapped ion: A unified approach to the classical and quantum-mechanical regimes

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In this paper we derive the Fokker-Planck equation that describes the motion of a charged particle, trapped in a static harmonic potential and cooled by radiation pressure forces exerted by a laser beam. This equation is then discussed in both the classical and quantum regimes. The model adopted here is a simple one, in which one-dimensional motion and a two-level atom are assumed, and the time evolution of the system is described in the quasiclassical limit, in which the quantum nature of the photon exchanged between the particle and the field plays no role. We report relevant graphs for the loss of kinetic energy suffered by the particle in the cooling process and estimates of the lowest kinetic temperature that can be reached in these experiments.

I. INTRODUCTION

Optical cooling of atoms has opened new fields of research in atomic spectroscopy. By using laser beams focused onto a sample of atoms confined in a small region of space, it has been possible to cool them down to very low temperatures, which prevents atoms from escaping the confinement region. Spectroscopic studies were then performed on samples that included several millions down to a few atoms, and even single-particle spectroscopy was made possible.[1]

This field of research is so lively now that we do not even attempt to give a complete list of references of the work done in the last decade. The reader may consult several review articles that appeared recently in the literature.[2,3]

The physical processes that lead to cooling of atoms under the action of laser beams are by now well understood.[4–6] The force exerted by light onto the atoms can be classified in either of two classes: radiation pressure forces, also called scattering forces, and dipole forces, also called gradient forces. Both forces can be derived from the formula

$$\mathbf{F}=\sum_i \mathcal{P}_i \nabla E_i \quad (i=x,y,z)\ , \tag{1}$$

where $\mathcal{P}_i$ and E_i are the spatial components of the atomic dipole and the electric field, respectively, and ∇ is the del vector operator which acts on the spatial coordinates. But they are generated by quite different processes.

Radiation pressure forces stem from uneven distributions of the photons absorbed by the atom from the impinging field and the photons that are spontaneously reemitted into the vacuum. When an atom absorbs a photon from the laser field, its momentum changes by the amount $\hbar k$ in the direction along which the field propagates. The subsequent spontaneous emission occurs in a random direction $\mathbf{n}$, and it happens that the probability of emission along $\mathbf{n}$ equals the probability of emission along $-\mathbf{n}$. Thus, on the average, the emitted photon does not carry any momentum away from the atom. The resulting net momentum exchange between the field and the atom manifests itself as a force, the radiation pressure force.

The gradient force, on the other hand, originates from stimulated processes alone. In the presence of a laser field made up of several plane running waves propagating in different directions, an atom may absorb a photon from one wave and reemit it into another. In the process the atom changes its momentum by $\hbar(\mathbf{k}-\mathbf{k}')$, where $\mathbf{k}$ and $\mathbf{k}'$ are the wave vectors of the waves from which photons were absorbed and emitted, respectively. Again, the change of momentum manifests itself as a force, the gradient force.

Scattering forces saturate when the field is so large that the rate of photon absorption equals the rate of spontaneous emission, and the optical transition is bleached. At these intensities of the laser field, cooling of atoms by scattering forces is too slow to keep atoms confined in the interaction region for a sufficiently long time, and precooling of thermal atoms becomes a necessity.

Another possibility is to trap particles by means of some trapping mechanism. Then laser cooling may act on the particles for enough time to bring the atomic kinetic temperature to its ultimate limit. Charged particles (ions) can be trapped by the electric forces of a radio-frequency[1] or a Penning[7] trap.

Another way of trapping particles is through optical effects. This is much more difficult to realize, but also neutral particles can be trapped by these devices. It has been reported recently that optical trapping has been tried successfully.[8–10] Here, thermal atoms evaporated from a metal plate subject to strong laser radiation are precooled and thereby confined by several laser beams.

Another strong laser beam, focused onto a small region of space inside this optical confinement area provides the last mechanism of cooling: precooled atoms that happen to enter the region where light is strongly focused are caught and retained.

The residual motion of cooled particles plays an important role in this newly opened branch of atomic spectroscopy. On the one hand, this motion sets the ultimate limit of the kinetic temperature that can be reached in these experiments. On the other hand, the low velocities of cooled atoms are associated with long de Broglie wavelengths of their probability distribution, and quantum collective effects may emerge. Moreover, in recent experiments,[11] carried out on a few particles confined in a small region of space, sudden transitions between stable atomic configurations were observed. In these experiments, a small change in some cooling parameter (typically the laser detuning from atomic resonance) caused sudden jumps of the atomic motion from one to another pattern. These effects, which resemble ordinary phase transitions, also manifest hysteresis effects.[12]

The aim of this article is to present a detailed analysis of the laser cooling of a single particle in a radio frequency trap. We will describe the motion of the particle from the beginning of the interaction with the laser field to the final stage where the particle's residual momentum is very low, and the particle is confined in a small region of space near the center of the attracting potential. We also discuss the relevance of quantum effects in both stages of the interaction and several graphs will be shown to represent physical quantities of interest. The paper is organized as follows.

In Sec. II we will derive the equations of motion. Although most of these results are known, we have inserted this section for the sake of completeness.

In Sec. III these equations are discussed in detail. We will show there how to simplify them in order to achieve a single equation in the quasiclassical limit. Approximations will be made in two cases of interest, and in both these cases the ensuing equations will be of a Fokker-Planck type.

We will then follow the dynamical evolution of the system in both the initial and final stages of the process in Secs. IV and V, respectively. In these two sections we will discuss simplified equations that allow for a better understanding of the relevant processes that are involved.

Finally, we will discuss the obtained results and possible generalization of this work in Sec. VI.

II. KINETIC EQUATIONS

Our model consists of a two-level particle trapped in a potential well $V(r)$ generated by the radiofrequency field.

Cooling is provided by a laser beam propagating along the z direction. The electromagnetic field of the laser is characterized by the amplitude E_0 and the phase ϕ of its electric component

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0(\mathbf{r},t)\exp[i\phi(\mathbf{r},t)]+\text{c.c.} \tag{2}$$

The field is assumed to be monochromatic, with a frequency ω_L resonant or quasiresonant with the frequency ω_0 of the atomic transition.

This field is also assumed to be so large that it may be treated as classical. Thus the interaction Hamiltonian between the particle and the field (2) is a purely atomic operator:

$$H_L=-(\mathcal{P}_+\cdot\mathbf{E}_0e^{i\phi}+\text{H.c.}) . \tag{3}$$

In this equation, $\mathcal{P}_+$ represents the atomic raising operator (i.e., the operator that allows transitions from the lower level to the upper level) multiplied by the atomic dipole $\mathcal{P}$. Coupling between the field and the atomic external (motional) degrees of freedom arises because the atomic position operator $\mathbf{r}$ appears as an argument of the electric field $\mathbf{E}(\mathbf{r},t)$.

Spontaneous emission is accounted for by the interaction of the atomic system with the electromagnetic modes in the vacuum. Thus the radiative part of the Hamiltonian can be written as

$$H_R=g\sum_j[\mu_j|\mathcal{P}_+|a_j\exp(i\mathbf{k}_j\cdot\mathbf{r})+\text{H.c.}]+\sum_j\hbar ck_ja_j^\dagger a_j . \tag{4}$$

Here, g stands for a (real) coupling term which is assumed to be the same for each mode of the vacuum field and $\mathbf{k}_j$ is the wave vector of the jth mode. The term μ_j is a geometrical factor that depends on the relative orientation of the oscillating dipole and the direction of the emitted photon. The field operators $a_j^\dagger$ and a_j are, respectively, the creation and the annihilation operators of the jth mode.

Equations (3) and (4) do not contain terms oscillating at twice the field frequency: the latter have been eliminated by using the rotating-wave approximation.

The last part of the Hamiltonian contains the atomic operators for the internal and external (motional) degrees of freedom

$$H_A=\tfrac{1}{2}\hbar\omega_0\sigma_z+\frac{1}{2M}\mathbf{p}^2 , \tag{5}$$

where σ_z is the diagonal Pauli operator associated with the two-level system and $\mathbf{p}$ and M are the atomic kinetic momentum and mass, respectively. Under the action of the total Hamiltonian

$$H=H_A+H_R+H_L+V \tag{6}$$

the density operator σ of the system evolves with time according to the equation of motion

$$\frac{d\sigma}{dt}=\frac{1}{i\hbar}[H,\sigma] . \tag{7}$$

We are interested in the evolution of the atomic system, not of the whole system (i.e., the atomic system and the modes of the electromagnetic field). It is therefore convenient to introduce a reduced density operator ρ as the trace of σ over the radiation states

$$\rho=\text{Tr}_R\{\sigma\} . \tag{8}$$

Now we come to the representation of ρ. The density operator depends on both the internal and the external

degrees of freedom. Let $\mathbf{s}$ and $\mathbf{s}'$ be any valid couple of labels that identify the external (motional) configuration, and α and β two labels for the internal (electronic excitation) configuration. The choice is obvious for the latter: α and β may assume only two possible values, b for the lower state and a for the excited state.

The external degrees of freedom, on the other hand, may be represented in any of among several possible choices. We may take $\mathbf{s}=\mathbf{r}'$ and $\mathbf{s}'=\mathbf{r}''$, and the density operator has then the representation

$$\langle \mathbf{r}',\alpha|\rho|\mathbf{r}'',\beta\rangle=\rho_{\alpha\beta}(\mathbf{r}',\mathbf{r}'') . \tag{9}$$

We may also take $\mathbf{s}=\mathbf{p}'$ and $\mathbf{s}'=\mathbf{p}''$, to get

$$\langle \mathbf{p}',\alpha|\rho|\mathbf{p}'',\beta\rangle=\rho_{\alpha\beta}(\mathbf{p}',\mathbf{p}'') . \tag{10}$$

The mixed representation is also widely used, and known under the name of Wigner representation: we take $\mathbf{s}=\mathbf{p}+\mathbf{u}/2$ and $\mathbf{s}'=\mathbf{p}-\mathbf{u}/2$, where $\mathbf{p}=(\mathbf{p}'+\mathbf{p}'')/2$ and $\mathbf{u}=(\mathbf{p}'-\mathbf{p}'')$,

$$\left\langle \mathbf{p}+\frac{\mathbf{u}}{2},\alpha\left|\rho\right|\mathbf{p}-\frac{\mathbf{u}}{2},\beta\right\rangle=\rho_{\alpha\beta}(\mathbf{p},\mathbf{u}) \tag{11}$$

and the Wigner representation is then the Fourier transform of $\rho_{\alpha,\beta}(\mathbf{p},\mathbf{u})$:

$$W_{\alpha,\beta}(\mathbf{r},\mathbf{p})=\int d\mathbf{u}\frac{1}{(2\pi\hbar)^3}\rho_{\alpha,\beta}(\mathbf{p},\mathbf{u})\exp\left[\frac{i\mathbf{r}\cdot\mathbf{u}}{\hbar}\right] . \tag{12}$$

When we deal with the motion of a structureless particle, there are advantages in using the Wigner representation. Indeed, the Wigner distribution function is real (although not positive definite), its integral over the phase space $(\mathbf{r},\mathbf{p})$ is normalized to 1, and its integral over $\mathbf{r}[\mathbf{p}]$ is the diagonal element $\rho(\mathbf{p},\mathbf{p})[\rho(\mathbf{r},\mathbf{r})]$, i.e., the distribution function of $\mathbf{p}$ $[\mathbf{r}]$. Equation (12) represents a useful generalization of the Wigner representation to include internal degrees of freedom.

Before passing to an explicit evaluation of the trace in Eq. (8) we restrict the form of the electric field in (2) to a special case. We will consider the field to be of the form

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega_L t)]+\text{c.c.} , \tag{13}$$

i.e., a running wave with constant amplitude propagating along $\mathbf{k}$.

According to the discussion in Sec. I, we are not taking into consideration any deflection of the atomic motion caused by the gradient force. Indeed, a single running wave does not produce such effect. The only cooling mechanism allowed by (13) is due to (quasiresonant) scattering forces.

The off-diagonal elements of ρ, in any spatial representation, namely, $\rho_{b,a}(\mathbf{s},\mathbf{s}')$ and $\rho_{a,b}(\mathbf{s},\mathbf{s}')$, have a fast oscillation with the field frequency ω_L, superimposed to their intrinsic, much slower variation. The fast oscillation can be factorized out by redefining the off diagonal elements according to

$$\tilde{\rho}_{a,b}(\mathbf{s},\mathbf{s}')=\rho_{a,b}(\mathbf{s},\mathbf{s}')e^{i\omega_L t} , \tag{14}$$

$$\tilde{\rho}_{b,a}(\mathbf{s},\mathbf{s}')=\rho_{b,a}(\mathbf{s},\mathbf{s}')e^{-i\omega_L t} . \tag{15}$$

The kinetic equations for the four components in the Wigner representation can now be written as

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]W_{a,a}(\mathbf{r},\mathbf{p})=-\Gamma W_{a,a}(\mathbf{r},\mathbf{p})+i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{W}_{b,a}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]-e^{-i\mathbf{k}\cdot\mathbf{r}}\tilde{W}_{a,b}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]\right] , \tag{16a}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]W_{b,b}(\mathbf{r},\mathbf{p})=\Gamma\int d^2\mathbf{n}\,q(\mathbf{n})W_{a,a}(\mathbf{r},\mathbf{p}+\hbar k\,\mathbf{n})$$
$$+i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[e^{-i\mathbf{k}\cdot\mathbf{r}}\tilde{W}_{a,b}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]-e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{W}_{b,a}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]\right] , \tag{16b}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]\tilde{W}_{a,b}(\mathbf{r},\mathbf{p})=\left[i\Delta-\frac{\Gamma}{2}\right]\tilde{W}_{a,b}(\mathbf{r},\mathbf{p})-i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[W_{a,a}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]-W_{b,b}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]\right]e^{i\mathbf{k}\cdot\mathbf{r}} , \tag{16c}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]\tilde{W}_{b,a}(\mathbf{r},\mathbf{p})=-\left[i\Delta-\frac{\Gamma}{2}\right]\tilde{W}_{b,a}(\mathbf{r},\mathbf{p})+i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[W_{a,a}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]-W_{b,b}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]\right]e^{-i\mathbf{k}\cdot\mathbf{r}} . \tag{16d}$$

In Eqs. (16), Δ_0 represents the detuning of the laser frequency from the atomic resonance ω_0,

$$\Delta_0=\omega_L-\omega_0 . \tag{17}$$

Γ represents the rate of spontaneous decay from the upper atomic level,

$$\Gamma=\frac{2\pi}{\hbar}|g^2|\sum_j\mu^2(\boldsymbol{\eta}_j)\delta(\omega_j-\omega_0) \tag{18}$$

and the function $q(\mathbf{n})$ is the square of the factor μ, normalized to unity

$$q(\mathbf{n})=\frac{\mu^2(\mathbf{n})}{\int d^2\mathbf{n}\,\mu^2(\mathbf{n})} . \tag{19}$$

These are the generalized Bloch equations that include effects of the exchanged momentum on the atomic motion. In the limit of large laser wavelengths, $\mathbf{k}\rightarrow\mathbf{0}$, they reduce to the ordinary Bloch equations. These equations have been derived by several authors,[13,14] and we do not report details of this derivation here. However, a few remarks are in order.

Spontaneous emission coupled the upper level to the lower level, whereas interaction with the laser field may induce transitions in both directions (i.e., from the lower level to the upper one, and vice versa).

When a photon is spontaneously emitted from the upper level, the probability density of the excited state at $(\mathbf{r},\mathbf{p})$ is depleted, hence the damping term in (16a). The ground level, on the other hand, gains population at $(\mathbf{r},\mathbf{p})$ when a photon is spontaneously emitted, but this may happen only if the density of the upper level at $(\mathbf{r},\mathbf{p}+\hbar k\,\mathbf{n})$ (where $\mathbf{n}$ is the unit vector in the direction of the emitted photon) is different from zero, see Eq. (16b).

Stimulated processes occur via the atomic coherences, as in the ordinary Bloch equations. Here, the emitted or absorbed photon has a well defined momentum $\hbar\mathbf{k}$, which is transferred to the atom. Notice that the change in atomic momentum is only one half of $\hbar\mathbf{k}$. This is due to the fact that only $\mathbf{p}'$ or $\mathbf{p}''$ of Eq. (10), but not both, are changed in the process, and $\mathbf{p}$ is defined as $(\mathbf{p}'+\mathbf{p}'')/2$, see Eqs. (11) and (12).

The atomic motion affects the interaction between the field and the atom through the terms $\exp(\pm i\mathbf{k}\cdot\mathbf{r})$ in Eqs. (16). This can be seen if these terms are included in the definition of $\tilde{W}_{b,a}$ and $\tilde{W}_{a,b,}$. We define

$$\tilde{\tilde{W}}_{a,b}(\mathbf{r},\mathbf{p})=\exp(-i\mathbf{k}\cdot\mathbf{r})\tilde{W}_{a,b}(\mathbf{r},\mathbf{p})\ , \tag{20}$$

$$\tilde{\tilde{W}}_{b,a}(\mathbf{r},\mathbf{p})=\exp(i\mathbf{k}\cdot\mathbf{r})\tilde{W}_{b,a}(\mathbf{r},\mathbf{p})\ . \tag{21}$$

Because of the commutation relations

$$e^{\pm ikr}\frac{p}{M}\frac{\partial}{\partial r}-\frac{p}{M}\frac{\partial}{\partial r}e^{\pm ikr}=\mp i\frac{kp}{M}e^{\pm ikr} \tag{22}$$

Eqs. (16c) and (16d) transform into

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]\tilde{\tilde{W}}_{a,b}(\mathbf{r},\mathbf{p})=\left[i\bar{\Delta}-\frac{\Gamma}{2}\right]\tilde{\tilde{W}}_{a,b}(\mathbf{r},\mathbf{p})-i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[W_{a,a}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]-W_{b,b}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]\right]\ , \tag{23a}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial\mathbf{r}}-\frac{\partial V}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{p}}\right]\tilde{\tilde{W}}_{b,a}(\mathbf{r},\mathbf{p})=-\left[i\bar{\Delta}-\frac{\Gamma}{2}\right]\tilde{\tilde{W}}_{b,a}(\mathbf{r},\mathbf{p})-i\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\left[W_{a,a}\left[\mathbf{r},\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\right]-W_{b,b}\left[\mathbf{r},\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\right]\right]\ , \tag{23b}$$

where the detuning Δ_0 has been redefined by including the Doppler shift $\mathbf{p}\cdot\mathbf{k}/M$,

$$\bar{\Delta}=\Delta_0-\frac{\mathbf{p}\cdot\mathbf{k}}{M}\ . \tag{24}$$

Equations (16a), (16b), (23a), and (23b) represent a system of kinetic equations for the particle trapped in the potential $V(\mathbf{r})$ and cooled by a laser beam. They contain all the quantum features of the process, but in their present form they are far too complicated to be solved, even numerically. However, we can simplify them without sacrificing generality.

Even when the particle has eventually reached a stationary regime, its kinetic momentum is much larger than the momentum $\hbar\mathbf{k}$ exchanged with the field in each elementary process of absorption or emission. We can therefore pass to the quasiclassical limit of these equations. Namely, we assume that $\hbar\mathbf{k}$ is always a small quantity when compared to $\mathbf{p}$. Thus we can replace $W_{\alpha,\beta}(\mathbf{r},\mathbf{p}\pm\hbar\mathbf{k})$ by its Taylor expansion

$$W_{\alpha,\beta}(\mathbf{r},\mathbf{p}\pm\hbar\mathbf{k})=W_{\alpha,\beta}(\mathbf{r},\mathbf{p})\pm\hbar\mathbf{k}\frac{\partial W_{\alpha,\beta}(\mathbf{r},\mathbf{p})}{\partial\mathbf{p}}+\cdots\ . \tag{25}$$

The first term of the series suffices everywhere, except in the equation for $W_{b,b}(\mathbf{r},\mathbf{p})$, Eq. (16b). Here, the integral that appears in the right-hand side is zero at first order, because of the above mentioned symmetry of the geometrical factor $q(\mathbf{n})$, namely, $q(\mathbf{n})=q(-\mathbf{n})$. Thus the first-order term in the integrand function in (16b) changes sign for a space inversion, and the integral vanishes. This is a manifestation of the fact that, on the average, the spontaneously emitted photon does not carry any momentum away from the atom. We need to keep the second-order term, which will take care of the diffusion processes induced by fluctuations of the direction of the emitted photon

$$\begin{aligned}&\Gamma\int d^2\mathbf{n}\,q(\mathbf{n})W_{a,a}(\mathbf{r},\mathbf{p}+\hbar k\,\mathbf{n})\\&\quad\approx\Gamma W_{a,a}(\mathbf{r},\mathbf{p})\\&\qquad+\Gamma\left[\sum_{i,j}\hbar^2k^2\int d^2\mathbf{n}\,q(\mathbf{n})n_in_j\frac{\partial^2W_{a,a}}{\partial p_i\partial p_j}\right]\ ,\end{aligned} \tag{26}$$

where we have used the normalization of $q(\mathbf{n})$.

In order to maintain a closer parallelism with the ordinary Bloch equations, we introduce the three components

$$U(\mathbf{r},\mathbf{p})=\tilde{\tilde{W}}_{a,b}(\mathbf{r},\mathbf{p})+\tilde{\tilde{W}}_{b,a}(\mathbf{r},\mathbf{p})\ , \tag{27}$$

$$V(\mathbf{r},\mathbf{p})=i[\tilde{\tilde{W}}_{a,b}(\mathbf{r},\mathbf{p})-\tilde{\tilde{W}}_{b,a}(\mathbf{r},\mathbf{p})]\ , \tag{28}$$

$$W(\mathbf{r},\mathbf{p})=W_{a,a}(\mathbf{r},\mathbf{p})-W_{b,b}(\mathbf{r},\mathbf{p})\ , \tag{29}$$

along with the particle spatial probability distribution, which is the trace over the internal states of ρ,

$$f(\mathbf{r},\mathbf{p})=\mathrm{Tr}_A[\rho(\mathbf{r},\mathbf{p})]=W_{a,a}(\mathbf{r},\mathbf{p})+W_{b,b}(\mathbf{r},\mathbf{p})\ . \tag{30}$$

The quasiclassical equations then read

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{\partial V}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right]U=\bar{\Delta}V-\frac{\Gamma}{2}U\ , \tag{31a}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{\partial V}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right]V=-\bar{\Delta}U-\frac{\Gamma}{2}V+\Omega W+\Omega\frac{\hbar k}{2}\frac{\partial f}{\partial p}\ , \tag{31b}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{\partial V}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right]W=-\Omega V-\Gamma(f+W)-\Gamma\sum_{i,j}Q_{i,j}\frac{\partial^2(f+W)}{\partial p_i\partial p_j}\ , \tag{31c}$$

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{\partial V}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right]f=\Omega\frac{\hbar k}{2}\frac{\partial V}{\partial p}+\Gamma\sum_{i,j}Q_{i,j}\frac{\partial^2(f+W)}{\partial p_i\partial p_j}\ , \tag{31d}$$

where we have introduced the diffusion tensor

$$Q_{i,j}=(\hbar k)^2\int d^2\mathbf{n}\,q(\mathbf{n})n_in_j \tag{32}$$

and the Rabi frequency

$$\Omega=2\frac{\mathcal{P}\cdot\mathbf{E}_0}{\hbar}\ . \tag{33}$$

We have seen the origin of the diffusion term, Eq. (26). This appears in the equations for W and f. However, we are interested only in the dynamical evolution of f, and at the lowest order in $\hbar k$. Thus the quantum diffusion of W, which brings a term in $(\hbar k)^4$ in the equation for f, may be neglected in the quasiclassical limit. Equation (31c) can then be replaced by

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{p}}{M}\cdot\frac{\partial}{\partial \mathbf{r}}-\frac{\partial V}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}}\right]W=-\Omega V-\Gamma(f+W)\ . \tag{34}$$

III. FOKKER-PLANCK EQUATION

We will derive now a Fokker-Planck equation for the process, with its possible simplification in the various stages of the cooling.

The calculations will be carried out in a one-dimensional space, in the direction along which the field propagates. Moreover, we will assume that the trapping potential is a harmonic potential. In the one-dimensional space the equations have the same form as those derived in the previous section, except for the diffusion term, since now we consider only the axial projection of the momentum taken by the atom when it emits spontaneously. The generalized Bloch equations are then written as

$$\mathcal{D}\mathcal{B}=\underline{\mathcal{M}}\cdot\mathcal{B}+\mathbf{Y} \tag{35}$$

and the equation for the particle distribution function f is

$$\mathcal{D}f=\Omega\frac{\hbar k}{2}\frac{\partial V}{\partial p}+\Gamma\alpha(\hbar k)^2\frac{\partial^2(f+W)}{\partial p^2}\ , \tag{36}$$

where we have expressed $Q_{3,3}$ as $\alpha(\hbar k)^2$, α being a geometrical factor that depends on the spatial distribution of the spontaneously emitted photon.

In (35) we have introduced the vector $\mathcal{B}$, defined by

$$\mathcal{B}\equiv(U,V,W)\ , \tag{37}$$

the tensor

$$\underline{\mathcal{M}}\equiv\begin{bmatrix}-\Gamma/2 & \bar{\Delta} & 0\\ -\bar{\Delta} & -\Gamma/2 & \Omega\\ 0 & -\Omega & -\Gamma\end{bmatrix}\ , \tag{38}$$

and the vector

$$\mathbf{Y}\equiv\left[0,\Omega\frac{\hbar k}{2}\frac{\partial f}{\partial p},-\Gamma f\right]\ . \tag{39}$$

The differential operator in Eqs. (35) and (36) is

$$\mathcal{D}\equiv\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}-\chi z\frac{\partial}{\partial p}\ , \tag{40}$$

where the trapping potential has been explicitly written as the harmonic potential

$$V(z)=\chi\frac{z^2}{2}\ . \tag{41}$$

We will now find a formal solution to Eq. (35), and discuss it in the semiclassical limit we have adopted.

The right-hand side of Eq. (35) splits into two parts: the first describes the free evolution of the generalized Bloch vector, while the second represents a source (nonhomogeneous) term. The hydrodynamic derivative on the left side indicates that the internal evolution occurs following a point particle that moves classically under the action of the trapping potential. Then we can replace the differential operator $\mathcal{D}$, as given by (40), by a total time derivative if we transform the phase-space coordinates z and p accordingly.

To this end, we map the z-p plane onto the ζ-π plane by means of the transformation

$$\zeta=\mathcal{Z}(z,p,t)\ , \tag{42a}$$

$$\pi=\mathcal{P}(z,p,t)\ , \tag{42b}$$

$$\tau=t\ , \tag{42c}$$

and we require that ζ and π are constant along a classical trajectory described by a point particle in the harmonic potential. This is the same as requiring that the transformed operator $\mathcal{D}$ should not contain $\partial/\partial\zeta$ and $\partial/\partial\pi$. Using this condition we find that $\mathcal{Z}$ and $\mathcal{P}$ in (42a) and (42b) must satisfy the differential equations

$$\frac{\partial\mathcal{Z}}{\partial t}+\frac{p}{M}\frac{\partial\mathcal{Z}}{\partial z}-\chi z\frac{\partial\mathcal{Z}}{\partial p}=0\ , \tag{43a}$$

$$\frac{\partial\mathcal{P}}{\partial t}+\frac{p}{M}\frac{\partial\mathcal{P}}{\partial z}-\chi z\frac{\partial\mathcal{P}}{\partial p}=0\ , \tag{43b}$$

with the boundary conditions assigned on the $t=0$ plane

$$\mathcal{Z}(z,p,0)=z \ , \tag{44a}$$

$$\mathcal{P}(z,p,0)=p \ . \tag{44b}$$

The solutions to (43) and (44) are

$$\mathcal{Z}(z,p,t)=z\cos(\nu t)-\frac{p}{\nu M}\sin(\nu t) \ , \tag{45a}$$

$$\mathcal{P}(z,p,t)=p\cos(\nu t)+\nu M z\sin(\nu t) \ , \tag{45b}$$

where $\nu=\sqrt{(\chi/M)}$ is the angular frequency of the trapping potential.

Equations (45a) and (45b) represent the two invariants for a point particle moving along each characteristic line of the differential operator $\mathcal{D}$, namely, the lines defined by

$$z=\zeta\cos(\nu\tau)+\frac{\pi}{\nu M}\sin(\nu\tau) \ , \tag{46a}$$

$$p=\pi\cos(\nu\tau)-\nu M\zeta\sin(\nu\tau) \ . \tag{46b}$$

It should be noted that (46a) and (46b) represent the inverse transformation of (42). In the new coordinates the generalized Bloch equations read

$$\frac{d\mathcal{B}}{d\tau}=\underline{\mathcal{M}}(\zeta,\pi,\tau)\cdot\mathcal{B}+\mathbf{Y}(\zeta,\pi,\tau) \tag{47}$$

and Eq. (36) is changed into

$$\begin{aligned}\frac{\partial f}{\partial\tau}=\Omega\frac{\hbar k}{2}&\left[-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}+\cos(\nu\tau)\frac{\partial}{\partial\pi}\right]V\\&+\Gamma\alpha(\hbar k)^2\left[\left[\frac{\sin(\nu\tau)}{\nu M}\right]^2\frac{\partial^2}{\partial\zeta^2}\right.\\&\quad-2\frac{\sin(\nu\tau)\cos(\nu\tau)}{\nu M}\frac{\partial^2}{\partial\zeta\,\partial\pi}\\&\quad\left.+\cos^2(\nu\tau)\frac{\partial^2}{\partial\pi^2}\right](f+W) \ .\end{aligned} \tag{48}$$

To solve (47) and (48) we must assign $\mathcal{B}$ and f on the plane ζ,π at the initial time and then follow their evolution. However, the particular form of Eq. (47) allows for its formal solution: we can express $\mathcal{B}$ as a function of the vector $\mathbf{Y}$, i.e., of f, and substitute it in Eq. (48) to find a unique equation for f, which will result in a Fokker-Planck equation.

This is made possible by the fact that the coordinates ζ,π enter Eq. (47) as parameters.

Let $\underline{\mathcal{U}}(\tau_1,\tau_0)$ be the matrix that describes the free time evolution of the vector $\mathcal{B}$, i.e., the matrix which is a solution of the equation

$$\frac{d\underline{\mathcal{U}}(\tau,\tau_0)}{d\tau}=\underline{\mathcal{M}}(\tau)\cdot\underline{\mathcal{U}}(\tau,\tau_0) \ . \tag{49}$$

This matrix is also called the fundamental matrix of the system (47). If there were only the homogeneous terms in (47), then $\mathcal{B}$ at the time τ_1 would be linked to the vector $\mathcal{B}$ at the time τ_0 by the relationship

$$\mathcal{B}(\tau_1)=\underline{\mathcal{U}}(\tau_1,\tau_0)\cdot\mathcal{B}(\tau_0) \ . \tag{50}$$

In the presence of the nonhomogeneous terms, $\mathcal{B}$ is given by

$$\mathcal{B}(\tau_1)=\underline{\mathcal{U}}(\tau_1,\tau_0)\cdot\left[\int_{\tau_0}^{\tau_1}\underline{\mathcal{U}}^{-1}(\tau',\tau_0)\cdot\mathbf{Y}(\tau')d\tau'+\mathcal{B}(\tau_0)\right] \ . \tag{51}$$

Since $\underline{\mathcal{M}}$ contains dissipative terms, $\underline{\mathcal{U}}(\tau_1,\tau_0)\to 0$ when $|\tau_1-\tau_0|\gg\Gamma^{-1}$. Hence the influence of the initial conditions on the value of $\mathcal{B}$ at time τ_1 becomes vanishingly small when τ_1 increases. Furthermore, by using the semigroup property of $\underline{\mathcal{U}}$, we can write

$$\underline{\mathcal{U}}(\tau_1,\tau_0)\cdot\underline{\mathcal{U}}^{-1}(\tau',\tau_0)=\underline{\mathcal{U}}^{-1}(\tau',\tau_1) \ . \tag{52}$$

Hence, for $|\tau_1-\tau_0|$ large enough in comparison with Γ^{-1},

$$\mathcal{B}(\tau_1)\approx\int_{\tau_0}^{\tau_1}\underline{\mathcal{U}}^{-1}(\tau',\tau_1)\cdot\mathbf{Y}(\tau')d\tau' \ . \tag{53}$$

The lower limit of integration may also be taken as $-\infty$.

The fundamental matrix $\underline{\mathcal{U}}$ cannot be given an analytical expression when the matrix $\underline{\mathcal{M}}$ depends on time, or, more precisely, when for $\tau'\neq\tau''$ the commutator $[\underline{\mathcal{M}}(\tau'),\underline{\mathcal{M}}(\tau'')]$ is different from zero. It can, however, be evaluated numerically, by integration of the homogeneous system. In fact, it can be shown that each column vector $\mathbf{v}_j\equiv((\underline{\mathcal{U}})_{1,j},(\underline{\mathcal{U}})_{2,j},(\underline{\mathcal{U}})_{3,j})$ $(j=1,2,3)$ of the matrix $\underline{\mathcal{U}}(\tau_1,\tau_0)$ is the solution of the homogeneous system that has initial conditions

$$[\underline{\mathcal{U}}(\tau_0,\tau_0)]_{i,j}=\delta_{i,j} \ . \tag{54}$$

To find $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ that appears in (53), one must integrate the homogeneous system from τ_1 to τ' (i.e., backward in time), evaluate $\underline{\mathcal{U}}(\tau',\tau_1)$, and take its inverse matrix. In this process, large numerical errors may easily occur since $\underline{\mathcal{U}}(\tau',\tau_1)$ grows exponentially with time. It is advantageous to use a general property of these matrices,[15] according to which the transpose of $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ is the fundamental matrix of the system adjoint to (49). This has been done in the calculations described below.

The form (53) in which the generalized Bloch vector is expressed in terms of $\mathbf{Y}$ and hence of the distribution function f, is particularly useful in the discussion of the quantum effects on the motion of the particle. We distinguish two cases. In one case, the particle's kinetic momentum p changes appreciably during one lifetime of the atomic transition. This is likely to happen in the initial stage of the process, in which the particles are injected into the region of space where the trapping potential is active. The particles then have enough kinetic energy to move around relatively fast, spanning most of the region where trapping is active. As we will see later, in this stage the quantum effects are negligible.

In the other case, the momentum of the particle is small and does not change appreciably in an atomic lifetime. This occurs at the end of the cooling process, when the particle is confined in a tiny region of space around the center of the attracting potential, and its momentum is close to its limit value. In this case, quantum effects are relevant.

The kinetic momentum or its changes are large or small depending on the effective detuning $\bar{\Delta}$ associated with it. According to (25) and (46b), the change $\delta\bar{\Delta}$ of

the frequency detuning in an atomic lifetime is given by

$$\delta\bar{\Delta}=k\left[\frac{\pi}{M}+\nu\zeta\right]\frac{\nu}{\Gamma} . \tag{55}$$

We first discuss the form of the equation during the initial stage of the cooling.

At the startup of the process, the particle occupies regions of phase space with ζ and π large enough to make $\delta\bar{\Delta}\gg\Gamma$. Since the cooling mechanism works through resonant absorption of photons, it is effective only for a short time interval in which $|\delta\bar{\Delta}|\leq\Gamma$. When the particle crosses the region of phase space in which the latter condition is satisfied, it is kicked off by the laser photons. In the rest of the phase space, cooling action is much less effective, or may even turn into a slight heating. Equation (53) shows that the generalized Bloch vector $\mathcal{B}$ at time τ_1 depends on the distribution function f at earlier times, the link being provided by the fundamental matrix $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$. In Fig. 1, we show the graphs for some matrix elements of $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$, see Appendix A for a detailed description of the parameters used in the numerical calculations.

We see that the matrix elements oscillate wildly in the first stage of the cooling. This is due to the large values taken by $\bar{\Delta}$. Although $\bar{\Delta}$ may change its value in the time interval of a few lifetimes in which $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ is worth being calculated (is appreciably different from zero), the oscillatory character does not change. It is therefore plausible to assume $\bar{\Delta}$ to be constant during the time interval in which we integrate the system. This allows us to give an analytical expression for $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$:

$$\underline{\mathcal{U}}^{-1}(\tau',\tau_1)=\exp[\underline{\mathcal{M}}(\tau_1)(\tau_1-\tau')] , \tag{56}$$

where the matrix $\underline{\mathcal{M}}$ is evaluated at the final time τ_1.

The large oscillations of the matrix elements of $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ at times prior to τ_1 allows us to neglect changes in $\mathbf{Y}$ that might have occurred in a few lifetimes preceding τ_1. The history of $\mathbf{Y}$ is made less influential by the erratic oscillations of $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$. We are then led to the conclusion that $\mathcal{B}$ can be expressed by

$$\mathcal{B}(\tau_1)=\left[\int_{-\infty}^{\tau_1}\exp[\underline{\mathcal{M}}(\tau_1)(\tau_1-\tau')]d\tau'\right]\cdot\mathbf{Y}(\tau_1) . \tag{57}$$

Although the proof is too lengthy to be included here, it can be shown that the matrix $\underline{\mathcal{M}}(\tau_1)$ has eigenvalues whose real part are always negative and this in turn implies that at the lower limit of integration $\exp[\underline{\mathcal{M}}(\tau_1)(\tau_1-\tau')]$ vanishes and the integral in (57) converges. Thus $\mathcal{B}$ follows adiabatically $\mathbf{Y}$; the linear operator that transforms the vector $\mathbf{Y}$ into the vector $\mathcal{B}$ can be evaluated analytically, and we find

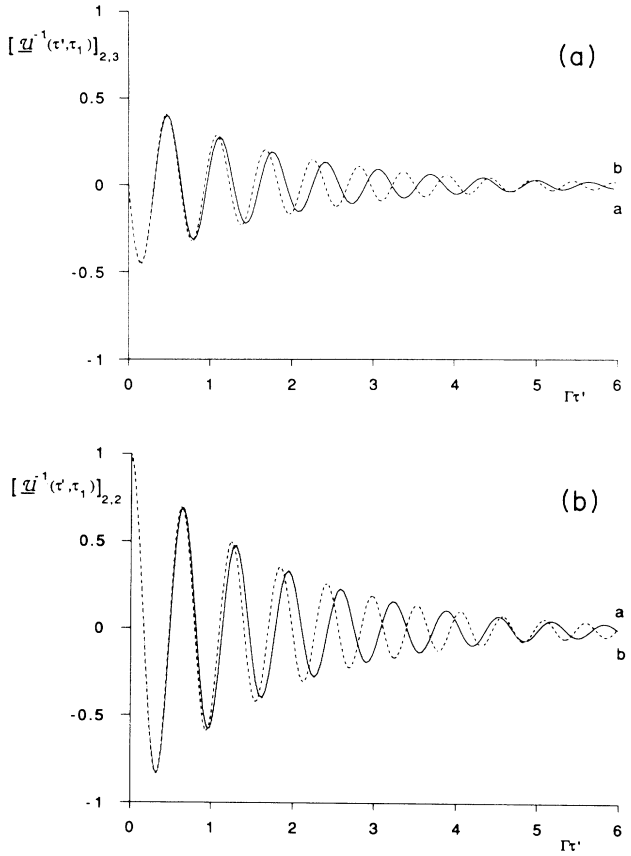

FIG. 1. (a) The matrix element $[\underline{\mathcal{U}}^{-1}(\tau',\tau_1)]_{2,3}$ as a function of $\Gamma\tau'$ at a fixed value of τ_1. The values of π and ζ at $\tau'=\tau_1$ are 20 and 0.5 in our units (see Appendix A.) The solid line a represents the behavior as given by full integration of the equation of motion for $\underline{\mathcal{U}}$, Eq. (49), while the dashed line b represents the behavior given by Eq. (56). (b) The matrix element $[\underline{\mathcal{U}}^{-1}(\tau',\tau_1)]_{2,2}$ as a function of $\Gamma\tau'$ at a fixed value of τ_1. The parameters are the same as in (a).

$$\int_{-\infty}^{\tau_1}\exp[\underline{\mathcal{M}}(\tau_1)(\tau_1-\tau')]d\tau'=-[\underline{\mathcal{M}}(\tau_1)]^{-1}\equiv\underline{\mathcal{S}}(\tau_1) . \tag{58}$$

Equation (57) can then replace the vector $\mathcal{B}$ in the equation for the Wigner distribution function f, Eq. (48). To the lowest order in $(\hbar k)$ (i.e., in the quasiclassical limit discussed in Sec. II) we find that f satisfies the partial differential equation of the Fokker-Planck type:

$$\frac{\partial f}{\partial\tau}=\Gamma\Omega\frac{\hbar k}{2}\left[\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}-\cos(\nu\tau)\frac{\partial}{\partial\pi}\right](\underline{\mathcal{S}})_{2,3}f+\Gamma\alpha(\hbar k)^2\left[\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}\right]^2\right][1-\Gamma(\underline{\mathcal{S}})_{3,3}]f$$

$$+\left[\frac{\hbar k\Omega}{2}\right]^2\left[\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}\right]^2\right](\underline{\mathcal{S}})_{2,2}f , \tag{59}$$

where $(\underline{\mathcal{S}})_{2,2}$ and $(\underline{\mathcal{S}})_{2,3}$ and $(\underline{\mathcal{S}})_{3,3}$ are elements of the inverse matrix of $-\underline{\mathcal{M}}$, see Eq. (58), and are given explicitly by

$$(\underline{\mathcal{S}})_{2,2}=\frac{2\Gamma}{4\bar{\Delta}^2+\Gamma^2+2\Omega^2}\ , \tag{60a}$$

$$(\underline{\mathcal{S}})_{2,3}=\frac{2\Omega}{4\bar{\Delta}^2+\Gamma^2+2\Omega^2}\ , \tag{60b}$$

$$(\underline{\mathcal{S}})_{3,3}=\frac{\Gamma^2+4\bar{\Delta}^2}{\Gamma(4\bar{\Delta}^2+\Gamma^2+2\Omega^2)}\ . \tag{60c}$$

Using these formulas we can write the Fokker-Planck equation in a compact form,

$$\begin{aligned}\frac{\partial f}{\partial\tau}=&\Gamma\Omega\frac{\hbar k}{2}\left[\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}-\cos(\nu\tau)\frac{\partial}{\partial\pi}\right](\underline{\mathcal{S}})_{2,3}f\\&+\Gamma(\hbar k)^2(\alpha+\tfrac{1}{4})\\&\times\left[\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]^2\right]\\&\times[1-\Gamma(\underline{\mathcal{S}})_{3,3}]f\ .\end{aligned} \tag{61}$$

In the last stage of the cooling process, on the other hand, the kinetic momentum of the particle is very small, and the particle fluctuates around some point which is very close to the center of the attracting potential. Fluctuations are caused by quantum diffusion, which plays an important role in this stage of the process.

When the particle has reached this stage, Eq. (57) does not apply any more: the Bloch vector $\mathcal{B}$ at time τ_1 does depend on the values it assumed at times prior to τ_1, and $\mathcal{B}$ does not follow adiabatically the vector $\mathbf{Y}$. This is because the evolution matrix $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ does not oscillate in time any more, so that correlations of $\mathcal{B}$ at different times may arise. In spite of these difficulties, we can still determine a Fokker-Planck equation for the process at the lowest order in the parameter $\hbar k$, which is the smallest amount of momentum exchanged by the field and the particle in each elementary process.

Let us first derive the equation for f to the first order in $\hbar k$, by writing the source vector $\mathbf{Y}(\tau_1)$ in terms of $f(\tau)$ to zero order in $\hbar k$. We find

$$\mathbf{Y}(\tau_1)\approx(0,0,-\Gamma f(\tau))$$

since $f(\tau_1)$ differs from $f(\tau)$ by terms in $\hbar k$. Inserting this expression into (53) and the result into (48), the equation for f reads

$$\frac{\partial f}{\partial\tau}=-\hbar k\mathcal{A}(\tau)f(\tau) \tag{62}$$

where $\mathcal{A}(\tau)$ is the operator

$$\begin{aligned}\mathcal{A}(\tau)\equiv&\frac{\Gamma\Omega}{2}\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]\\&\times\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,3}d\tau_1\ .\end{aligned}$$

Next we determine the diffusive terms in the equation for f by keeping the vector $\mathbf{Y}(\tau_1)$ up to the first order in $\hbar k$,

$$Y_2(\tau_1)\approx\Omega\frac{\hbar k}{2}\left[\cos(\nu\tau_1)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau_1)}{\nu M}\frac{\partial}{\partial\xi}\right]f(\tau)\ , \tag{63a}$$

$$\begin{aligned}Y_3(\tau_1)\equiv&-\Gamma f(\tau_1)\\\approx&-\Gamma\left[1+\hbar k\int_{\tau_1}^{\tau}\mathcal{A}(\tau_2)d\tau_2\right]f(\tau)\ .\end{aligned} \tag{63b}$$

The expression of Y_3 comes directly from Eq. (62) and in both (63a) and (63b) we have replaced $f(\tau')$ by $f(\tau)$ because, as remarked above, this introduces corrections of higher order in $\hbar k$.

Substituting (63) into (53) we obtain the Block vector $\mathcal{B}$ to the first order in $\hbar k$,

$$\begin{aligned}\mathcal{B}_i(\tau)=&-\Gamma\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{i,3}d\tau_1 f(\tau)-\hbar k\Gamma\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{i,3}\int_{\tau_1}^{\tau}\mathcal{A}(\tau_2)d\tau_2 d\tau_1 f(\tau)\\&+\frac{\Omega\hbar k}{2}\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{i,2}\left[\cos(\nu\tau_1)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau_1)}{\nu M}\frac{\partial}{\partial\xi}\right]d\tau_1 f(\tau)\ .\end{aligned} \tag{64}$$

Although we managed to find an expression of $\mathcal{B}_i(\tau)$ in terms of the distribution function f at the same time τ, the effects of the past history of f manifest themselves in the second term in (64), whereas the first and third terms are a natural generalization of the adiabatic following approximation discussed above.

Replacing the components of the generalized Bloch vector in (48) by their expressions (64), we find the Fokker-Planck equation for the process in the quasiclassical limit

$$\begin{aligned}\frac{\partial f}{\partial\tau}=&-\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]Ff+\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]Df\\&+\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\xi}\right]\left[\cos(\nu\tau)\frac{\partial}{\partial\xi}+M\nu\sin(\nu\tau)\frac{\partial}{\partial\pi}\right]\Phi f\ ,\end{aligned} \tag{65}$$

where the coefficients are given by

$$F\equiv\frac{\hbar k\Omega\Gamma}{2}\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,3}d\tau_1\ , \tag{66a}$$

$$D \equiv \Gamma\alpha(\hbar k)^2 \left[1-\Gamma\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{3,3}d\tau_1\right] + \left[\frac{\hbar k\Omega}{2}\right]^2 \int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,2}\cos[\nu(\tau-\tau_1)]d\tau_1$$
$$- \left[\frac{\hbar k\Omega\Gamma}{2}\right]^2 \int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,3}\int_{\tau_1}^{\tau}\cos[\nu(\tau-\tau_2)]\int_{-\infty}^{\tau_2}[\underline{\mathcal{U}}^{-1}(\tau_3,\tau_2)]_{2,3}d\tau_3 d\tau_2 d\tau_1 , \tag{66b}$$

$$\Phi \equiv \left[\frac{\hbar k\Omega}{2}\right]^2 \frac{1}{M\nu}\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,2}\sin[\nu(\tau-\tau_1)]d\tau_1$$
$$- \left[\frac{\hbar k\Omega\Gamma}{2}\right]^2 \frac{1}{M\nu}\int_{-\infty}^{\tau}[\underline{\mathcal{U}}^{-1}(\tau_1,\tau)]_{2,3}\int_{\tau_1}^{\tau}\sin[\nu(\tau-\tau_2)]\int_{-\infty}^{\tau_2}[\underline{\mathcal{U}}^{-1}(\tau_3,\tau_2)]_{2,3}d\tau_3 d\tau_2 d\tau_1 . \tag{66c}$$

The Fokker-Planck equation (65) will be worked out in Sec. V, where approximate analytical expression for the matrix elements $\underline{\mathcal{U}}^{-1}(\tau',\tau_1)$ will be reported for two cases of physical interest.

It should be noticed that we could have derived the basic equations in both classical and quasiclassical cases by using the "old" variables z and p instead of the "new" ones ζ and π. However, it turns out that the description of the process in the initial stage is much simpler if carried out in the variables ζ and π as we will show in Sec. IV. Moreover, the new variables have allowed us to treat both stages of the process in an unified manner, pointing out the physical meaning of the various approximations made in this paper.

Generalization to include several particles or three-dimensional motion should also be possible in the new coordinates.

IV. INITIAL STAGE OF THE COOLING PROCESS

The Fokker-Planck equation (59) derived in Sec. III conserves the total probability $\int\int f\,d\pi\,d\zeta$ during the motion of the particle in the trap. This can be seen by rewriting the equation as a continuity equation,

$$\frac{\partial f}{\partial\tau}+\frac{\partial\mathcal{I}_\zeta}{\partial\zeta}+\frac{\partial\mathcal{I}_\pi}{\partial\pi}=0 , \tag{67}$$

where the probability current $\mathcal{I}\equiv(\mathcal{I}_\zeta,\mathcal{I}_\pi)$ in the phase space has the components

$$\mathcal{I}_\zeta=\frac{\sin(\nu\tau)}{\nu M}\left[\tilde{\mathcal{D}}f+\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}\right]Df\right] , \tag{68}$$

$$\mathcal{I}_\pi=-\cos(\nu\tau)\left[\tilde{\mathcal{D}}f+\left[\cos(\nu\tau)\frac{\partial}{\partial\pi}-\frac{\sin(\nu\tau)}{\nu M}\frac{\partial}{\partial\zeta}\right]Df\right] , \tag{69}$$

and $\tilde{\mathcal{D}}$ and D are the drift and diffusion coefficients

$$\tilde{\mathcal{D}}=-\frac{\hbar k}{2}\Omega\Gamma(\underline{\mathcal{S}})_{2,3} , \tag{70}$$

$$D=\Gamma(\hbar k)^2(\alpha+\tfrac{1}{4})[1-\Gamma(\underline{\mathcal{S}})_{3,3}] . \tag{71}$$

When the particle is first injected into the trap, it starts an oscillatory motion under the action of the harmonic potential. The motion would be periodic if there were no perturbation, namely, if there were no cooling mechanism. For a periodic motion, ζ and π would be constant in time.

Notice that ζ and π play an interchangeable role in this treatment. Indeed the same orbit in the harmonic potential can be characterized by different couples of values for ζ and π, the only difference being the phase of the particle along the same trajectory. We will see a consequence of this fact later in this section, when we discuss the motion of the particle subject to laser cooling. Suffice it to say now that we can always take the initial conditions in such a way that $\zeta=0$ and π equal to the maximum kinetic momentum of the particle in its first orbit in the harmonic potential.

The light pressure force acts on the particle as a small perturbation. Its action is so small that we can consider ζ and π to be constant in a time of the order of the period of the harmonic trap. This allows us to evaluate the effects of drift and diffusion on the particle motion by averaging over one harmonic period. Thus we define the averaged drift coefficients

$$\tilde{\mathcal{D}}_\zeta(\zeta,\pi)\equiv\frac{1}{T}\int_{-T/2}^{T/2}\frac{\sin(\nu\tau)}{\nu M}\tilde{\mathcal{D}}d\tau$$
$$=-\frac{\Omega^2\Gamma\hbar k}{T\nu M}\int_{-T/2}^{T/2}\frac{\sin(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau , \tag{72a}$$

$$\tilde{\mathcal{D}}_\pi(\zeta,\pi)\equiv-\frac{1}{T}\int_{-T/2}^{T/2}\cos(\nu\tau)\tilde{\mathcal{D}}d\tau$$
$$=\frac{\Omega^2\Gamma\hbar k}{T}\int_{-T/2}^{T/2}\frac{\cos(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau , \tag{72b}$$

and the averaged diffusion coefficients

$$D_{\zeta,\zeta}(\zeta,\pi)\equiv\frac{1}{T}\int_{-T/2}^{T/2}\frac{\sin^2(\nu\tau)}{\nu^2M^2}D\,d\tau$$
$$=\frac{2\Omega^2\Gamma(\hbar k)^2(\alpha+\frac{1}{4})}{T\nu^2M^2}\int_{-T/2}^{T/2}\frac{\sin^2(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau , \tag{73a}$$

$$D_{\zeta,\pi}(\zeta,\pi)\equiv-\frac{1}{T}\int_{-T/2}^{T/2}\frac{2\cos(\nu\tau)\sin(\nu\tau)}{\nu M}D\,d\tau$$
$$=\frac{-4\Omega^2\Gamma(\hbar k)^2(\alpha+\frac{1}{4})}{T\nu M}$$
$$\times\int_{-T/2}^{T/2}\frac{\sin(\nu\tau)\cos(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau , \tag{73b}$$

$$D_{\pi,\pi}(\zeta,\pi)\equiv\frac{1}{T}\int_{-T/2}^{T/2}\cos^2(\nu\tau)D\ d\tau$$
$$=\frac{2\Omega^2\Gamma(\hbar k)^2(\alpha+\frac{1}{4})}{T}\int_{-T/2}^{T/2}\frac{\cos^2(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau\ . \tag{73c}$$

In Eqs. (72) and (73), T is the period of the harmonic trap and ζ and π are kept constant in the integrand functions. The function $\mathcal{H}(\zeta,\pi,\tau)$ is defined as

$$\mathcal{H}(\zeta,\pi,\tau)=4\bar{\Delta}^2+\Gamma^2+2\Omega^2$$
$$=4\left[\Delta_0-\frac{k}{M}[\pi\cos(\nu\tau)-M\nu\zeta\sin(\nu\tau)]\right]^2$$
$$+\Gamma^2+2\Omega^2\ . \tag{74}$$

From Eqs. (72) and (73) we can get an estimate of the point where quantum-mechanical effects sets in. Let us assume $\zeta=0$ and $\pi=\pi_0$ at the initial time, i.e., the particle is injected at the center of the harmonic trap with a velocity $v=\pi_0/M$. Due to the cooling effects, π varies with time following the law

$$\delta\pi=\tilde{\mathcal{D}}_\pi\delta t\ , \tag{75}$$

where δt is assumed much larger than T, so that we can use the expression (72b) for $\tilde{\mathcal{D}}_\pi$ in (75). The spread $(\Delta\pi)^2$, in the same time interval δt, increases according to

$$\delta(\Delta\pi)^2=D_{\pi,\pi}\delta t\ . \tag{76}$$

The ratio of these two equations gives the incremental spread when the particle's momentum π varies by $\delta\pi$

$$\frac{\delta(\Delta\pi)^2}{\delta\pi}=\frac{D_{\pi,\pi}}{\tilde{\mathcal{D}}_\pi}\ . \tag{77}$$

In Fig. 2 the ratio $D_{\pi,\pi}/\tilde{\mathcal{D}}_\pi$ is plotted versus π. This ratio, as expected, is very small everywhere except in a narrow region around $\pi=0$. At $\pi=0$ it diverges. The spread of the particle's momentum when π is small is given by

$$[(\Delta\pi)^2]^{1/2}=\left[\int_{\pi_0}^{\pi}\frac{D_{\pi,\pi}}{\tilde{\mathcal{D}}_\pi}d\pi'\right]^{1/2}\ , \tag{78}$$

in which the momentum initial spread has been neglected. When $(\Delta\pi)^2$ is of the order of π^2, quantum-mechanical effects cannot be neglected any longer.

Figures 3(a) and 3(b) show the graph of $[(\Delta\pi)^2]^{1/2}$ versus π and its crossing point with the line $[(\Delta\pi)^2]^{1/2}=\pi$. With the numerical values used in our calculations (see Appendix A) we see that quantum effects can be neglected as far as π is much larger than π_c. We will now discuss the equations of motion in this range of validity, deferring the discussion of the quantum effects until Sec. V.

In the initial stage of the cooling process, the motion of the particle under the combined action of the harmonic potential and the radiation pressure, is described by a

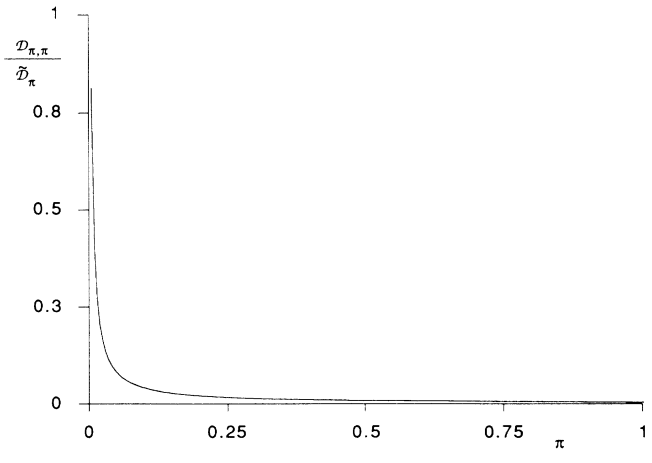

FIG. 2. The ratio of the particle momentum spread to the particle drift, as a function of the particle's kinetic momentum. ζ is taken equal to zero, the laser detuning Δ_0 is -2.5Γ, and the Rabi frequency Ω is 0.5Γ.

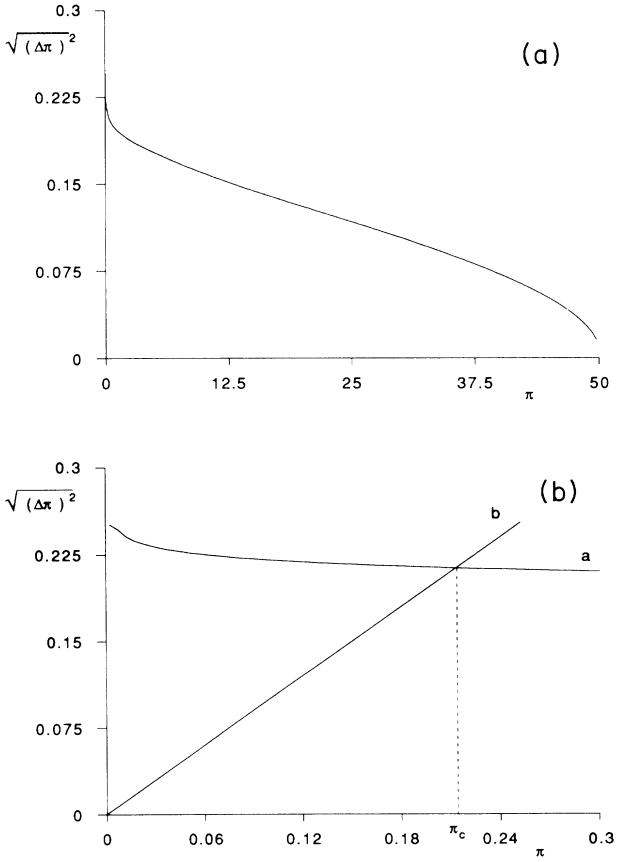

FIG. 3. (a) The particle's momentum spread vs the particle's kinetic momentum. The parameters are the same as in Fig. 2. (b) Magnification of (a) in the region of small particle momentum (graph *a*). The straight line *b* crosses the curve in a point π_c where the particle's momentum spread equals the particle's kinetic momentum, defining a threshold for quantum effects to set in.

linear partial differential equation of the first order for the Wigner distribution function. This is to be distinguished from a Liouville equation for a classical distribution function, however, since in the latter case the distribution function keeps a constant value along the trajectories in the phase space (which are the characteristic lines of the Liouville equation), whereas the Wigner distribution function does not.

A linear, first-order differential equation, such as (59) after cancellation of the second order derivatives, may be solved by the method of the characteristics. Starting at the initial time $\tau=0$ with the distribution function $f_0(\zeta,\pi)$, we can follow the time evolution of f by solving the system of total differential equations

$$\frac{d\zeta}{d\tau}=-\frac{\Omega^2\Gamma\hbar k}{\nu M\mathcal{H}(\zeta,\pi,\tau)}\sin(\nu\tau)\ , \tag{79a}$$

$$\frac{d\pi}{d\tau}=\frac{\Omega^2\Gamma\hbar k}{\mathcal{H}(\zeta,\pi,\tau)}\cos(\nu\tau)\ , \tag{79b}$$

$$\frac{df}{d\tau}=\frac{\Omega^2\Gamma\hbar k}{\mathcal{H}(\zeta,\pi,\tau)}\frac{k}{M}f\ , \tag{79c}$$

where, in (79c), $\mathcal{H}(\zeta,\pi,\tau)$ is defined by

$$\mathcal{H}^{-1}(\zeta,\pi,\tau)=\frac{\partial}{\partial\bar{\Delta}}\mathcal{H}^{-1}(\zeta,\pi,\tau) \tag{80}$$

and $\mathcal{H}(\zeta,\pi,\tau)$ is defined in (74).

Along the characteristic line $\zeta(\tau),\pi(\tau)$ which is the solution of (79a) and (79b) with initial conditions ζ_0,π_0, the distribution function f varies according to (79c), with $f=f_0(\zeta_0,\pi_0)$ at the initial time. Total probability $\int\int f\,d\pi\,d\zeta$ is conserved because of the continuity equation (63), with the probability current $\mathcal{J}$ given by (67) and (68), with $D=0$.

As noted earlier in this section, radiation pressure acts on a time scale that is much longer than the period of the harmonic potential. Hence we can take the average over the trap period of the right-hand side (rhs) in equations (79a) and (79b). These equations then become

$$\frac{d\zeta}{d\tau}=\tilde{\mathcal{D}}_\zeta(\zeta,\pi)\ , \tag{81a}$$

$$\frac{d\pi}{d\tau}=\tilde{\mathcal{D}}_\pi(\zeta,\pi)\ , \tag{81b}$$

with $\tilde{\mathcal{D}}_\zeta$ and $\tilde{\mathcal{D}}_\pi$ defined in (72a) and (72b).

To maintain probability conservation, Eq. (79c) must also be replaced by

$$\frac{df}{d\tau}=-\left[\frac{\partial\tilde{\mathcal{D}}_\zeta}{\partial\zeta}+\frac{\partial\tilde{\mathcal{D}}_\pi}{\partial\pi}\right]. \tag{81c}$$

Equations (81a) and (81b) can be interpreted as the classical equations of motion of the particle's center of mass. They are written for the slowly varying coordinates π and ζ, rather than the usual coordinates x and p. Numerical solution of (81a) and (81b) is much faster since the rhs of these equations do not contain terms oscillating at the harmonic frequency.

As noted above, a single orbit in the phase space can be characterized by infinitely many couples of coordinates π and ζ, for which

$$\chi\zeta^2+\frac{\pi^2}{M}=2E=\text{const}\ . \tag{82}$$

In Eq. (82), E is the energy of that orbit.

As a consequence of this fact, $\tilde{\mathcal{D}}_\zeta$ and $\tilde{\mathcal{D}}_\pi$ in (81a) and (81b) are not independent of each other. This can be shown by first noting that

$$\int_{-T/2}^{T/2}\frac{\zeta\nu M\cos(\nu\tau)+\pi\sin(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau$$
$$=\frac{1}{\nu}\int_{-T/2}^{T/2}\frac{d[\pi\cos(\nu\tau)-\zeta\nu M\sin(\nu\tau)]}{\mathcal{H}(\zeta,\pi,\tau)}=0\ . \tag{83}$$

Hence

$$\frac{1}{\zeta}\tilde{\mathcal{D}}_\zeta=\frac{1}{\pi}\tilde{\mathcal{D}}_\pi\equiv\tilde{\mathcal{D}}_T\ . \tag{84}$$

The functions $\tilde{\mathcal{D}}_\zeta$, $\tilde{\mathcal{D}}_\pi$, and $\tilde{\mathcal{D}}_T$ are evaluated in Appendix B.

Because of (84), the path in the phase space (ζ,π) followed by the particle can be found by solving the differential equation

$$\frac{d\pi}{d\zeta}=\frac{\tilde{\mathcal{D}}_\pi}{\tilde{\mathcal{D}}_\zeta}=\frac{\pi}{\zeta}\ , \tag{85}$$

which has the solution

$$\frac{\zeta}{\pi}=\text{const}\ . \tag{86}$$

The particle's coordinates (ζ,π), therefore, under the action of the radiation pressure, change while keeping their ratio constant.

The orbit in the phase space is a straight line passing through the center of the attracting potential. Cooling is achieved when the particle moves towards the origin, heating when it moves outwards.

We can choose any of these straight lines to describe the particle's motion. They just differ by the phase of the harmonic oscillations, i.e., by the choice of the initial time taken as $\tau=0$.

Our choice is $\zeta=0$ at the initial time. This allows us to derive the rate of change of the particle's energy

$$\frac{dE}{d\tau}=\frac{\pi\tilde{\mathcal{D}}_\pi}{M}=2\tilde{\mathcal{D}}_T E\ . \tag{87}$$

We would have arrived at the same equation also by choosing $\pi=0$ as the initial condition. This is obvious from the fact that $\tilde{\mathcal{D}}_T$ depends on E rather than on ζ and π separately, as shown in Appendix B.

In Fig. 4 we show $\tilde{\mathcal{D}}_\pi(\zeta=0,\pi)$ versus π. The detuning Δ_0 is chosen negative. We see that $\dot{\pi}$ and π have opposite signs everywhere. Thus $|\pi|$ decreases in time, i.e., the particle gets cooled. Changing the sign of Δ_0 results in a change of sign of $\tilde{\mathcal{D}}_\pi$, and the particle's kinetic momentum $|\pi|$ increases at all times, i.e., the particle gets heated.

From Fig. 4 it is apparent that the maximum rate of momentum transfer from the field to the particle is achieved when $\pi\approx\Delta_0/(k/M)$. This fact has a simple ex-

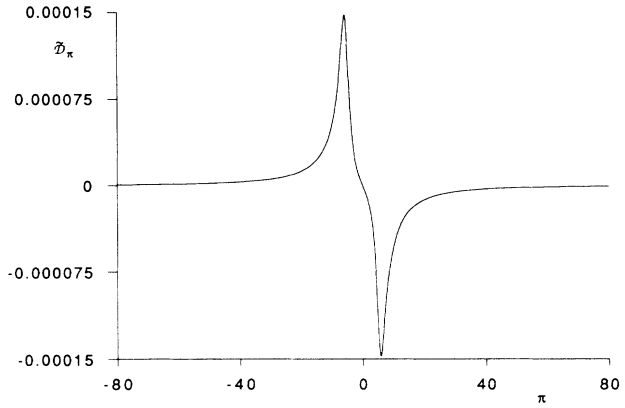

FIG. 4. The drift coefficient $\tilde{\mathcal{D}}_\pi$ vs π in the classical regime. The parameters are the same as in Fig. 2, i.e., $\zeta=0$, $\Delta_0=-2.5\Gamma$, and $\Omega=0.5\Gamma$.

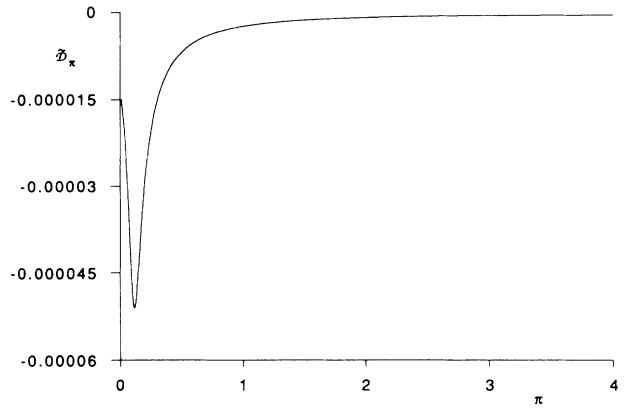

FIG. 6. Energy damping rate vs energy in the classical regime. We see that the damping rate gets its largest value when the Doppler shift due to the particle velocity compensates for the laser frequency detuning. Parameters are $\zeta=0$, $\Delta_0=-2.5\Gamma$, and $\Omega=0.5\Gamma$.

planation. The kinetic momentum p oscillates between $-\pi$ and π in one period of the harmonic motion. The peak at $\pi=\Delta_0/(k/M)$ arises because the particle spends the largest fraction of the period in a condition of resonance with the field, i.e., in a condition in which the Doppler shift compensates for the frequency detuning of the laser field. At resonance, momentum transfer is maximum.

The resonance peak in $\tilde{\mathcal{D}}_\pi$ is broadened when the laser intensity grows, as shown in Fig. 5.

In Fig. 6 we show the function $2\tilde{\mathcal{D}}_T(E)$ versus E, for Δ_0 negative. The function $2\tilde{\mathcal{D}}_T(E)$ is the instantaneous rate of change of the particle's total energy. Here, too, the resonance peak is evident. The rate at which energy decreases (when the laser detuning is negative) is not constant. As a result, the energy does not decay exponentially with time. Rather, it falls off to zero with an approximately linear behavior with time, as shown in Fig. 7. Increasing the laser power does not modify appreciably this feature.

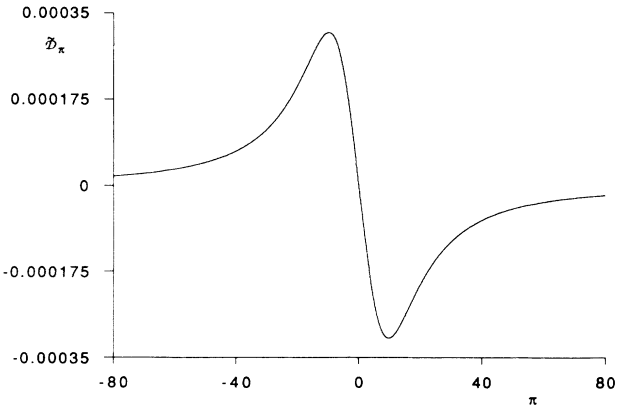

FIG. 5. Same as in Fig. 4 but in the strong-field regime (Rabi frequency $\Omega=7.5\Gamma$).

When the detuning is negative, i.e., when the radiation pressure force acts on the particle as a cooling force, all the dynamical variables in this approximation goes to zero when the time goes to infinity. But of course this approximation breaks down well before this ultimate limit is reached. The last stage of the cooling must be treated quantum mechanically, as discussed above. Section V is devoted just to these problems.

V. QUANTUM EFFECTS

The Fokker-Planck equation derived in Sec. III [Eq. (65)] describes the quantum behavior of the ion in the last

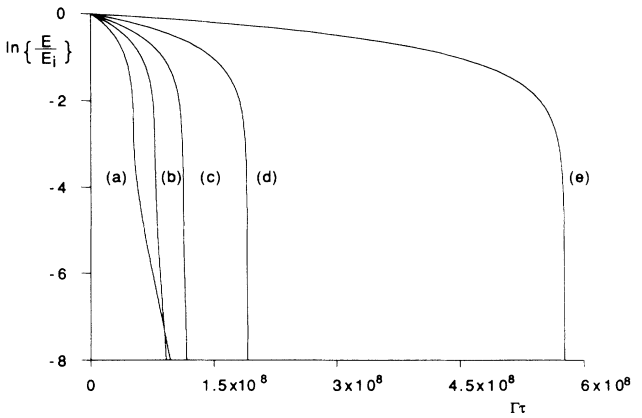

FIG. 7. Cooling of the particle motion in the classical regime: the logarithmic plot of the particle energy vs $\Gamma\tau$. We see that cooling occurs over a period of time of the order of several millions atomic lifetimes. In this graph the energy would go to zero for any negative value of the laser detuning, regardless of the initial kinetic parameters of the particle. Here, $\Omega=0.5\Gamma$ and $\Delta_0=-5\Gamma$ (graph a), $\Delta_0=-3.5\Gamma$ (graph b), $\Delta_0=-2.5\Gamma$ (graph c), $\Delta_0=-1.5\Gamma$ (graph d), and $\Delta_0=-0.5\Gamma$ (graph e).

stage of the cooling process, where the kinetic momentum of the particle is so small that diffusion effects play a major role. However, the coefficients (66) in Eq. (65) have a complicated dependence on the variables ξ, π,, and τ to make Eq. (66) impossible to be solved analytically, and even difficult to be solved numerically.

Useful limit cases can be readily derived from Eq. (65), and the form into which it has been cast makes such derivation simple enough. We discuss in this section two limit cases, which have been worked out by other authors. For the second case we also describe a method of solution that deals directly with the physical quantities that are related to the data measured in the experiments.

A. The limit case of a heavy ion

This limit case has been worked out in the literature[13] in an attempt to describe the particle's motion in all stages of cooling, keeping the quantum features, such as diffusion, in it. Not surprisingly, the main approximation made in this limit case resembles the one we made in Sec. IV to find the classical motion in the early stages (large kinetic momentum) of the cooling process.

The heavy-ion limit case is defined as the case in which the particle's mass M is so large that the forces acting on it are unable to modify appreciably its kinetic momentum in a time of the order of a few lifetimes. If this is the case, the absorption and reemission of a photon from and to the electromagnetic field occurs while the Doppler shift, associated with the particle's velocity, is practically unchanged, i.e., its change is much smaller than the natural width of the optical transition.

Of the two forces acting on the particle, i.e., the radiation force and the trapping force, the latter is, by several orders of magnitude, the largest one. Thus we need to consider only variations in the Doppler shift induced by the harmonic (trapping) force. Two conditions need to be satisfied.

(i) The change of the Doppler shift $\delta\bar{\Delta}$ in a time Γ^{-1} must be much smaller than $\bar{\Delta}$ itself, i.e.,

$$\Gamma \gg \nu \ . \tag{88a}$$

(ii) The Doppler shift must change, in a time Γ^{-1}, at most by a fraction of Γ, what yields

$$\Gamma^2 \gg \frac{kp}{M}\nu \tag{88b}$$

in the assumption that $\bar{\Delta}$ is made up mostly by the particle's velocity p/M. When both conditions (88) are satisfied, the free, damped evolution of the vector $\mathcal{B}$, as given by Eqs. (50) and (49), can be evaluated at a fixed value of $\bar{\Delta}$, what makes $\underline{\mathcal{M}}$ independent of time. The fundamental matrix $\underline{\mathcal{U}}^{-1}(\tau_1,\tau)$ is then found by integration

$$\underline{\mathcal{U}}^{-1}(\tau_1,\tau)=\exp[\underline{\mathcal{M}}(\tau)(\tau-\tau_1)] \ . \tag{89}$$

Since the matrix $\underline{\mathcal{U}}^{-1}(\tau_1,\tau)$ relaxes to zero in a few lifetimes Γ^{-1}, and, as assumed in (88a), Γ is much larger than ν, we can expand the slowly oscillating functions $\cos(\nu\theta)$ and $\sin(\nu\theta)$, in the expression (66) for the Fokker-Planck coefficients, in power series of $\nu\theta$, and retain only the first nonvanishing terms. The integrals in (66) can be evaluated by using the relationship

$$\int_{-\infty}^{\tau} \exp[\underline{\mathcal{M}}(\tau)(\tau-\tau_1)](\tau-\tau_1)^m d\tau_1 = (-1)^{m+1} m![\underline{\mathcal{M}}^{-1}(\tau)]^{m+1} \ . \tag{90}$$

We find the following expression for the coefficients in the Fokker-Planck equation: the drift coefficient is given by

$$F=-\frac{\hbar k\,\Omega\Gamma}{2}[\underline{\mathcal{M}}^{-1}(\tau)]_{2,3} \tag{91}$$

and the diffusion coefficients are

$$D=\Gamma\alpha(\hbar k)^2\{1+\Gamma[\underline{\mathcal{M}}^{-1}(\tau)]_{3,3}\} - \left[\frac{\hbar k\,\Omega}{2}\right]^2[\underline{\mathcal{M}}^{-1}(\tau)]_{2,2} + \left[\frac{\hbar k\,\Omega\Gamma}{2}\right]^2[\underline{\mathcal{M}}^{-2}(\tau)]_{2,3}[\underline{\mathcal{M}}^{-1}(\tau)]_{2,3} \ , \tag{92a}$$

$$\Phi=\left[\frac{\hbar k\,\Omega}{2}\right]^2\frac{1}{M}[\underline{\mathcal{M}}^{-2}(\tau)]_{2,2} - \left[\frac{\hbar k\,\Omega\Gamma}{2}\right]^2\frac{1}{M}[\underline{\mathcal{M}}^{-3}(\tau)]_{2,3}[\underline{\mathcal{M}}^{-1}(\tau)]_{2,3} \ . \tag{92b}$$

When we transform back to the "old" variables z, p, and t, we see that $\underline{\mathcal{M}}$ and its inverse depends only on p. The Fokker-Planck equation reads

$$\left[\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}-M\nu^2 z\frac{\partial}{\partial p}\right]f = -\frac{\partial}{\partial p}Ff+\frac{\partial^2}{\partial p^2}Df+\frac{\partial^2}{\partial p\,\partial z}\Phi f \ . \tag{93}$$

This equation has been found and discussed by several authors[4,14,16] for the free ion ($\nu=0$). The case of a trapped ion ($\nu\neq 0$) has been treated in Refs. 13 and 17. We have derived it here for the sake of completeness. The reader may consult these articles for greater detail.

Although Eq. (93) might serve as an investigation tool for the behavior of the trapped ion in all stages of the process, its utility is limited by two factors.

(i) In the early stages of cooling, when the kinetic momentum is still very large, Eq. (93) introduces unnecessary features such as quantum diffusion, which play a marginal role in the process at that stage, as pointed out in Sec. IV.

(ii) When the cooling process has reached its ultimate limit, Eq. (93) can be replaced by another equation, which does not suffer from any limitations, except that it is valid only near the equilibrium point.

B. Quantum effects in the last stage of cooling

As shown in Sec. IV, the particle, in the regime where the classical equations of motion do apply, is pushed towards the center of the attracting potential, losing its kinetic energy by scattering photons of the impinging laser

beam. According to the classical laws, the particle will eventually sit at the center of the trap with zero kinetic energy; and this point in the phase space is stable under all conditions.

But of course classical mechanics breaks down well before such configuration is reached. The breakpoint occurs for values of the kinetic momentum that attain to the Lamb-Dicke limit,[18–20] i.e., to the limit case in which the residual Doppler effect induces a shift of the resonance condition, which may be considered small when compared to the natural linewidth.

Within our theoretical framework, this means that the fundamental matrix $\underline{\mathcal{U}}$ will follow a law of the type

$$\frac{\partial}{\partial\tau}\underline{\mathcal{U}}(\tau,\tau_0)=[\underline{\mathcal{M}}_0+\underline{\mathcal{L}}g(\tau)]\cdot\underline{\mathcal{U}}(\tau,\tau_0)\ , \tag{94}$$

which is a convenient way of rewriting Eq. (49) with $\underline{\mathcal{M}}$ replaced by the sum of two terms, the first of which is the evolution matrix at equilibrium ($\xi=0$, $\pi=0$), and the second a small perturbation. In (94), $\underline{\mathcal{L}}$ is the off diagonal matrix,

$$\underline{\mathcal{L}}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\ , \tag{95}$$

and $g(\tau)$ is the residual Doppler shift,

$$g(\tau)=\frac{k}{M}[\pi\cos(\nu\tau)-\nu M\zeta\sin(\nu\tau)]\ . \tag{96}$$

Ordinary techniques in perturbation theory allow us to write the first order solution to (94) as

$$\underline{\mathcal{U}}_1(\tau,\tau_0)=\underline{\mathcal{U}}_0(\tau,\tau_0)\cdot\left[\underline{1}+\int_{\tau_0}^{\tau}d\tau' g(\tau')\underline{\mathcal{U}}_0^{-1}(\tau',\tau_0)\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{U}}_0(\tau',\tau_0)\right]\ , \tag{97}$$

where the zeroth order solution $\underline{\mathcal{U}}_0(\tau,\tau_0)$ is

$$\underline{\mathcal{U}}_0(\tau,\tau_0)=\exp[\underline{\mathcal{M}}_0(\tau-\tau_0)]\ . \tag{98}$$

This is all we need to find the drift term in the Fokker-Planck equation. Replacing $(\underline{\mathcal{U}}^{-1})_{2,3}$ in (66a) by the corresponding element found in the inverse matrix $\underline{\mathcal{U}}^{-1}(\tau,\tau_0)$, we find

$$F=-\frac{\hbar k\Omega\Gamma}{2}(\underline{\mathcal{M}}_0^{-1})_{2,3}+\frac{\hbar k\Omega\Gamma}{2}\frac{k}{M}[\pi\cos(\nu\tau)-\nu M\zeta\sin(\nu\tau)]\{\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{M}}_0^{-1}\}_{2,3}$$
$$+\frac{\hbar k\Omega\Gamma}{2}\frac{k}{M}[\pi\sin(\nu\tau)+\nu M\zeta\cos(\nu\tau)]\{\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{M}}_0^{-1}\}_{2,3} \tag{99}$$

where we have used the relationship

$$\int_{-\infty}^{\tau}d\tau'\underline{\mathcal{U}}_0^{-1}(\tau',\tau)\exp[i\nu(\tau-\tau')]=-(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}\ . \tag{100}$$

Details of these calculations can be found in Appendix C. We evaluate the diffusion coefficients D and Φ to the lowest order in the expansion (97), namely, we replace $\underline{\mathcal{U}}$ by $\underline{\mathcal{U}}_0$ in (66b) and (66c). This is both convenient and plausible, since higher-order corrections in $g(\tau)$ are small in the Lamb-Dicke limit and can be neglected in our context. This gives

$$D=\Gamma\alpha(\hbar k)^2[+\Gamma(\underline{\mathcal{M}}_0^{-1})_{3,3}]$$
$$-\left[\frac{\hbar k\Omega}{2}\right]^2\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,2}$$
$$-\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{\nu}\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,3}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ , \tag{101a}$$

$$\Phi=-\left[\frac{\hbar k\Omega}{2}\right]^2\frac{1}{M\nu}\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,2}$$
$$+\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{M\nu^2}\{\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,3}-(\underline{\mathcal{M}}_0^{-1})_{2,3}\}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ . \tag{101b}$$

Details of this derivation are in Appendix C. We emphasize that, in the present formulation of the problem, it would be possible to evaluate these terms to any order in $g(\tau)$. However, in view of our approximations, there is no need to go beyond the lowest order.

The drift term, Eq. (99), takes a particularly simple form in the old variables z and p

$$F=F_0-bp-M\nu_c^2 z\ , \tag{102}$$

where the constants F_0, b, and ν_c^2 are given by

$$F_0=-\frac{\hbar k\Omega\Gamma}{2}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ , \tag{103a}$$

$$b=-\frac{\hbar k\Omega\Gamma}{2}\frac{k}{M}\{\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot(\underline{\mathcal{M}}_0^{-1})\}_{2,3}\ , \tag{103b}$$

$$\nu_c^2=-\frac{\hbar k\Omega\Gamma}{2}\frac{k\nu}{M}\{\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot(\underline{\mathcal{M}}_0^{-1})\}_{2,3}\ . \tag{103c}$$

The (constant) diffusion coefficients D and Φ pass unaltered into the Fokker-Planck equation written in the old variables z and p. This equation now reads

$$\left[\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial z}-M(\nu^2+\nu_c^2)\left[z-\frac{F_0}{M(\nu^2+\nu_c^2)}\right]\frac{\partial}{\partial p}\right]f$$
$$=\frac{\partial}{\partial p}bpf+\frac{\partial^2}{\partial p^2}Df+\frac{\partial^2}{\partial p\,\partial z}\Phi f\ . \tag{104}$$

The physical meaning of Eq. (102) is apparent. The radiation force and the trapping force are intertwined by

the nonlinear response of the system to their combined action. In the linearization process, effective restoring forces emerge: a friction term $-bp$ and an additional trapping term $-M\nu_c^2 z$. The constant term F_0 merely displaces the equilibrium position from $z=0$, as given by the classical theory, to the new position $z=F_0/M\nu_1^2$, where

$$\nu_1^2=\nu^2+\nu_c^2 \ . \tag{105}$$

The friction constant b is positive (i.e., the term $-bp$ is frictional indeed) only if $\Delta_0<0$. On the other side of the resonance condition, i.e., $\Delta_0>0$, the constant b is negative, and the term $-bp$ pulls the particle out of equilibrium, as expected on physical grounds.

Equation (104) has been derived in Ref. 20. It describes an Ornstein-Uhlenbeck process[21] which generalizes the Brownian motion to include the effects of a potential force.

Although Eq. (104) can be solved analytically by standard methods (see, for instance, Ref. 22), we prefer to deal with quantities that are directly related to the parameters measured in actual experiments.

From the known, exact solution of (104) we only borrow the fact that the stationary state has a Gaussian distribution function in both space and momentum variables. We assume, accordingly, that the distribution function f at equilibrium is fully defined by the knowledge of the average values $\langle z\rangle$ and $\langle p\rangle$, along with the quantities that measure the relative spread, $\langle pz-\langle p\rangle\langle z\rangle\rangle$, $\langle(z-\langle z\rangle)^2\rangle$ and $\langle(p-\langle p\rangle)^2\rangle$. To simplify our notations, we define a new spatial coordinate q, whose null point is displaced from the origin by the amount $F_0/M\nu_1^2$,

$$q=z-\frac{F_0}{M\nu_1^2} \ . \tag{106}$$

Thus, at equilibrium, both $\langle q\rangle$ and $\langle p\rangle$ get a null value.

We make the ansatz that the distribution function be described at all times by the above quantities. By introducing the variables

$$\delta q^2(t)=\langle(q-\langle q\rangle)^2\rangle \ , \tag{107a}$$

$$\delta p^2(t)=\langle(p-\langle p\rangle)^2\rangle \ , \tag{107b}$$

$$\delta r(t)=\langle(pq-\langle p\rangle\langle q\rangle)\rangle \ , \tag{107c}$$

we arrive at the following set of equations:

$$\frac{d}{dt}\langle q\rangle=\frac{\langle p\rangle}{M} \ , \tag{108a}$$

$$\frac{d}{dt}\langle p\rangle=-M\nu_1^2\langle q\rangle-b\langle p\rangle \ , \tag{108b}$$

$$\frac{d}{dt}\delta q^2=\frac{2\delta r}{M} \ , \tag{109a}$$

$$\frac{d}{dt}\delta p^2=-2M\nu_1^2\delta r-2b\delta p^2+2D \ , \tag{109b}$$

$$\frac{d}{dt}\delta r=\frac{\delta p^2}{M}-M\nu_1^2\delta q^2-b\delta r+\Phi \ , \tag{109c}$$

see Appendix C.

According to our assumption, these equations form a closed set amenable to an analytical solution. They are completely equivalent to the Fokker-Planck equation from which they have been derived.

The two averages $\langle q\rangle$ and $\langle p\rangle$ evolve with time with characteristic rates given by the solution of the secular equation

$$\begin{vmatrix} -b-\lambda & -M\nu_1^2 \\ 1/M & -\lambda \end{vmatrix}=0 \tag{110}$$

which yields

$$\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4\nu_1^2}}{2} \ . \tag{111}$$

These rates are both negative (i.e., the system approaches equilibrium) if $b>0$, as expected on physical grounds. Overdamping occurs if $b^2>4\nu_1^2$, otherwise $\langle q\rangle$ and $\langle p\rangle$ approach their stationary value with a damped oscillation of angular frequency $[\nu_1^2-(b^2/4)]^{1/2}$. The three other quantities in Eqs. (109) approach their stationary value

$$(\delta q^2)_{\rm eq}=\frac{1}{M\nu_1^2}\left[\frac{D}{Mb}+\Phi\right] \ , \tag{112a}$$

$$(\delta p^2)_{\rm eq}=\frac{D}{b} \ , \tag{112b}$$

$$(\delta r)_{\rm eq}=0 \ , \tag{112c}$$

which is a solution of (109) with the time derivatives set equal to zero. The characteristic rates of approach to equilibrium are the roots of the secular equation associated to (109). These are

$$\mu_1=2\lambda_1 \ , \tag{113a}$$

$$\mu_2=2\lambda_2 \ , \tag{113b}$$

$$\mu_3=-b \ . \tag{113c}$$

Again, the sign of b determines whether the particle reaches equilibrium or not. As before, a positive value of b ensures that the equilibrium condition is eventually reached. The general solution to Eqs. (108) and (109) is reported in Appendix C. In Fig. 8 we show the graphs of the average value of total energy

$$\langle E\rangle=\tfrac{1}{2}M\nu_1^2\langle q^2\rangle+\frac{\langle p^2\rangle}{2M} \tag{114}$$

as a function of time in the last stage of cooling. Since the term Φ in (112a) can be neglected, as shown in Appendix C, we see from (114) that equipartition of energy is satisfied at equilibrium:

$$\langle E\rangle_{\rm eq}=\tfrac{1}{2}M\nu_1^2(\delta q^2)_{\rm eq}+\frac{(\delta p^2)_{\rm eq}}{2M}$$
$$\cong\frac{D}{2Mb}+\frac{D}{2Mb}=\frac{D}{Mb} \ . \tag{115}$$

(At equilibrium, both $\langle q\rangle$ and $\langle p\rangle$ are zero.)

The calculation of the limit value for the energy, Eq.

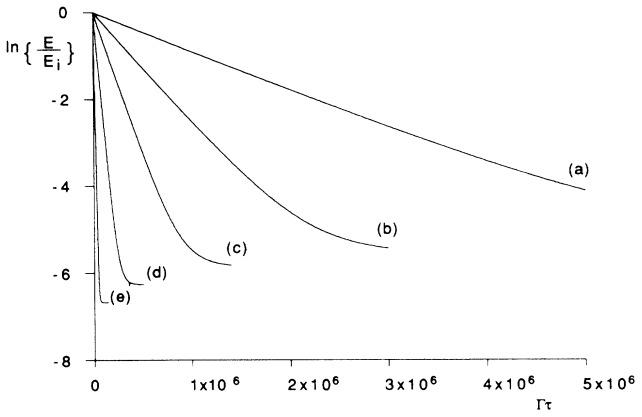

FIG. 8. Cooling of the particle in the quantum regime for the same values of the Rabi frequency as in Fig. 7. $\Omega=0.5\Gamma$ and $\Delta_0=-5\Gamma$ (graph a), $\Delta_0=-3.5\Gamma$ (graph b), $\Delta_0=-2.5\Gamma$ (graph c), $\Delta_0=-1.5\Gamma$ (graph d), and $\Delta_0=-0.5\Gamma$ (graph e). The particle energy tends to a constant value dependent on the frequency detuning. This value sets the ultimate limit of kinetic temperature that can be reached in these experiments.

(115), or of the associated kinetic temperature $T_{eq}=\langle E\rangle_{eq}/K_B$, where K_B is the Boltzmann's constant, is immediate from Eqs. (101)–(103). We find $T_{eq}=5.6\times10^{-4}$ K for $\Delta_0=-(2.5\Gamma)$ and $T_{eq}=2.3\times10^{-4}$ K for $\Delta_0=-(0.5\Gamma)$. The values of the parameters have been reported in Appendix A. The same predictions for the limit temperature have also been reported in Refs. 19 and 20. We remark that the values taken by q and p in the last stage of the cooling process are well within the range of validity for the Lamb-Dicke limit.

VI. CONCLUSIONS

We have derived and discussed the Fokker-Planck equation for a particle in a Paul trap, undergoing a strong interaction with a quasiresonant laser field that cools the particle down to very low temperature, while the trapping potential prevents the particle from escaping the interaction region. The model that we have chosen to work on is possibly the simplest one: the Paul trap has been assumed to consist of its static component alone, leaving out the radiofrequency component that is also present. Thus the heating of the particle associated with the presence of the radiofrequency terms is not accounted for in this model. Heating can be drawn only from the interaction with the laser field, if particular off resonance conditions are met. The ensuing static potential has then been stripped of its anharmonic terms, resulting in a harmonic force acting upon the particle. One-dimensional motion and single-particle cooling was also assumed. The particle is supposed to interact with the field as a two-level, quasiresonant system. Moreover, the motion is described in the quasiclassical limit, in which the very quantum nature of the photon exchanged by the field and the atom is neglected, while retaining, when necessary, the quantum features of the atomic motion. In spite of all these—presumably oversimplified—assumptions, this model presents some interesting features that deserve consideration. First of all, the model unifies the classical and the quantum cases, since it is possible to obtain useful information from it about both stages of the cooling process. Sections IV and V contain details of the derivation, and present the cooling rates for the two stages. Second, the calculations are simple enough to be carried out analytically for the most part. This has the added advantage to yield simple, yet meaningful, formulas for the relevant quantities in the process, in both the classical and quasiclassical limit cases. As a third fact, this model can be generalized to treat three-level systems with one metastable state, for which the coherence times may be much longer than the trap period, thus yielding novel features not found in the present model. Another feasible generalization is a treatment of the heating determined by nonstatic components of the radiofrequency trapping field. The method, adopted in this article, of adiabatic elimination of the periodic motion of the particle should prove fruitful in the analysis of the above mentioned cases.

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APPENDIX A

Here we give the numerical values of the parameters that have been used to perform the calculations reported in this article. In an atomic transition of an isolated atom, with the ground level as the lower level, the coherence damping rate γ_2 equals one half of the upper level damping rate Γ. On a time scale in which $\gamma_2=1$, the angular frequency of the trapping potential has been set equal to 0.15. In these units, the natural line width of the atomic transition Γ equals 2, and the detuning Δ_0 of the field from the atomic transition may take values up to 50. The Rabi frequency Ω is of the same order of magnitude, ranging up to several tens.

Calculations have been made for the Ba ion, which has a mass of 137 atomic units. The coherence time of the Ba^+ transition $6\,^2S_{1/2}$–$6\,^2P_{1/2}$ is 2.5×10^{-8} sec, so that time lengths are scaled by the same amount. The field wave vector k at resonance with the Ba^+ transition is $1.26\times10^5\ cm^{-1}$.

The linear size of the trap is of the order of 10^{-1} cm, but the confinement region may be as small as 10^{-6} cm. To avoid using very small numbers, we have set the length unit to 10^{-3} cm. Thus $k=126$ and ζ ranges from 100 (at the initial stage of the process) down to 10^{-1} (final stage of cooling).

Having redefined the units of time, mass, and length, the kinetic momentum must be scaled accordingly. We find that, in these units, the kinetic momentum ranges from 10 to 10^{-1}. Thus the kinetic momentum of a particle with $\pi=1$ is 6.5×10^{-20} g cm sec^{-1} and the corresponding kinetic temperature is 0.1 K.

APPENDIX B

The integrals that appear in Sec. IV,

$$\tilde{\mathcal{D}}_{\xi}(\zeta,\pi)=-\frac{\Omega^2\Gamma\hbar k}{T\nu M}\int_{-T/2}^{t/2}\frac{\sin(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau \equiv-\frac{\Omega^2\Gamma\hbar k}{T\nu M}I_s(\zeta,\pi)\ , \tag{B1}$$

$$\tilde{\mathcal{D}}_{\pi}(\zeta,\pi)=\frac{\Omega^2\Gamma\hbar k}{T}\int_{-T/2}^{t/2}\frac{\cos(\nu\tau)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau \equiv\frac{\Omega^2\Gamma\hbar k}{T}I_c(\zeta,\pi)\ , \tag{B2}$$

can be evaluated analytically by using the Cauchy theorem.

The function $\mathcal{H}(\zeta,\pi,\tau)$ is first factorized into two terms

$$\mathcal{H}(\zeta,\pi,\tau)=(a+b\cos x+c\sin x)\times(a^*+b\cos x+c\sin x)\ , \tag{B3}$$

with

$$a=2\Delta_0+i\omega\ , \tag{B4}$$

$$b=-\frac{2k\pi}{M}\ , \tag{B5}$$

$$c=2k\nu\zeta\ , \tag{B6}$$

$$\omega=(2\Omega^2+\Gamma^2)^{1/2}\ , \tag{B7}$$

$$x=\nu\tau\ . \tag{B8}$$

By means of the transformation $t=\tan(x/2)$ the integrals become

$$I_s=\frac{4}{\nu}\int_{-\infty}^{\infty}\frac{t}{|\mathcal{P}(t)|^2}dt\ , \tag{B9}$$

$$I_c=\frac{2}{\nu}\int_{-\infty}^{\infty}\frac{1-t^2}{|\mathcal{P}(t)|^2}dt\ , \tag{B10}$$

where

$$\mathcal{P}(t)=(a-b)t^2+2ct=(a+b)\ . \tag{B11}$$

We denote by α and β the two (complex) roots of $|\mathcal{P}(t)|^2$ that lie in the lower half-plane of the complex domain. Then, by the Cauchy theorem, I_s and I_c can be expressed in terms of their residues in α and β. We find

$$I_s=-2\pi i(c_{\alpha}^{(s)}+c_{\beta}^{(s)})\frac{4}{\nu|a-b|^2}\ , \tag{B12}$$

$$I_c=-2\pi i(c_{\alpha}^{(c)}+c_{\beta}^{(c)})\frac{2}{\nu|a-b|^2}\ , \tag{B13}$$

where

$$c_{\alpha}^{(s)}=\frac{\alpha}{(\alpha-\alpha^*)(\alpha-\beta^*)(\alpha-\beta)}\ , \tag{B14}$$

$$c_{\beta}^{(s)}=\frac{\beta}{(\beta-\alpha^*)(\beta-\beta^*)(\beta-\alpha)}\ , \tag{B15}$$

$$c_{\alpha}^{(c)}=\frac{1-\alpha^2}{(\alpha-\alpha^*)(\alpha-\beta^*)(\alpha-\beta)}\ , \tag{B16}$$

$$c_{\beta}^{(c)}=\frac{1-\beta^2}{(\beta-\alpha^*)(\beta-\beta^*)(\beta-\alpha)}\ . \tag{B17}$$

Formulas (B12)–(B17) allow for an easy evaluation of the two integrals (B1) and (B2).

The function $\tilde{\mathcal{D}}_T$, as given by (84), depends on ζ and π only through the total energy E of the particle's motion in the harmonic potential,

$$E=\frac{\pi^2}{2M}+\frac{M\nu^2\zeta^2}{2}\ . \tag{B18}$$

Because of the relationship (84), we need only to show that $\mathcal{D}_{\pi}/\pi$ is invariant if we change ζ and π keeping the energy (B18) constant.

Changing ζ and π with the constraint to keep the energy E constant is possible only if the transformed coordinates ζ' and π' are related to ζ and π by the transformation

$$\zeta'=\zeta\cos\theta+\frac{\pi}{\nu M}\sin\theta\ , \tag{B19}$$

$$\pi'=\pi\cos\theta-\nu M\zeta\sin\theta\ , \tag{B20}$$

with an arbitrary angle θ.

Replacing ζ and π by ζ' and π' and using (B19) and (B20), we find

$$\tilde{\mathcal{D}}_{\pi}(\zeta',\pi')=\frac{\Omega^2\Gamma\hbar k}{T}\int_{-T/2}^{T/2}\frac{\cos(\nu\tau)}{\mathcal{H}(\zeta',\pi',\tau)}d\tau =\frac{\Omega^2\Gamma\hbar k}{T}\int_{-T/2+\theta}^{T/2+\theta}\frac{\cos(\nu\tau)\cos(\theta)+\sin(\nu\tau)\sin(\theta)}{\mathcal{H}(\zeta,\pi,\tau)}d\tau\ . \tag{B21}$$

Since the integrand function is periodic in τ, we can integrate over the interval $[-T/2,T/2]$ even the second term in (B21). We obtain

$$\tilde{\mathcal{D}}_{\pi}(\zeta',\pi')=\cos(\theta)\tilde{\mathcal{D}}_{\pi}(\zeta,\pi)-\nu M\sin(\theta)\tilde{\mathcal{D}}_{\zeta}(\zeta,\pi) \tag{B22}$$

and, replacing $\tilde{\mathcal{D}}_{\zeta}(\zeta,\pi)$ by $\tilde{\mathcal{D}}_{\pi}(\zeta,\pi)\zeta/\pi$ [see Eq. (84)], we get

$$\tilde{\mathcal{D}}_{\pi}(\zeta',\pi')=\left[\cos(\theta)-\frac{\nu M\zeta}{\pi}\sin(\theta)\right]\tilde{\mathcal{D}}_{\pi}(\zeta,\pi) \tag{B23}$$

or, by (B20),

$$\tilde{\mathcal{D}}_{\pi}(\zeta',\pi')=\frac{\pi'}{\pi}\tilde{\mathcal{D}}_{\pi}(\zeta,\pi)\ . \tag{B24}$$

APPENDIX C

In this appendix we complete Section V by reporting details of the calculations concerning the Fokker-Planck equation in the Lamb-Dicke limit.

In the final stages of cooling the expression (66a) of the radiative force F can be approximated by replacing the matrix element $(\underline{\mathcal{U}}^{-1})_{2,3}$ with the first-order term in its perturbation expansion, Eq. (97),

$$F=\frac{\hbar k\Omega\Gamma}{2}\int_{-\infty}^{\tau}d\tau_1\left[\underline{\mathcal{U}}_0^{-1}(\tau_1,\tau)\cdot\left[\underline{1}+\int_{\tau_1}^{\tau}d\tau_2 g(\tau_2)\underline{\mathcal{U}}_0(\tau_1,\tau_2)\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{U}}_0^{-1}(\tau_1,\tau_2)\right]\right]_{2,3} \tag{C1}$$

with $\underline{\mathcal{U}}_0(\tau,\tau_0)=\exp[\underline{\mathcal{M}}_0(\tau-\tau_0)]$.

In order to evaluate the integrals in (C1) we use the property

$$\underline{\mathcal{U}}_0^{-1}(\tau_1,\tau)\cdot\underline{\mathcal{U}}_0(\tau_1,\tau_2)=\underline{\mathcal{U}}_0(\tau,\tau_2) \tag{C2}$$

then we make the following change of integration variables:

$$\eta_1=\tau-\tau_2\ , \tag{C3}$$

$$\eta_2=\tau_2-\tau_1\ . \tag{C4}$$

By using the relation (58), Eq. (C1) becomes

$$F=\frac{\hbar k\Omega\Gamma}{2}\left[-(\underline{\mathcal{M}}_0^{-1})_{2,3}+\int_0^{\infty}d\eta_1\int_0^{\infty}d\eta_2 g(\tau-\eta_1)\times[\exp(\underline{\mathcal{M}}_0\eta_1)\cdot\underline{\mathcal{L}}\cdot\exp(\underline{\mathcal{M}}_0\eta_2)]_{2,3}\right]\ . \tag{C5}$$

Next we rewrite the Doppler shift $g(\tau-\eta_1)$, by means of trigonometric formulas, in the form

$$g(\tau-\eta_1)=\frac{k}{M}\{[\pi\cos(\nu\tau)-\nu M\zeta\sin(\nu\tau)]\cos(\nu\eta_1)+[\pi\sin(\nu\tau)+\nu M\zeta\cos(\nu\tau)]\sin(\nu\eta_1)\}\ . \tag{C6}$$

Inserting this expression in (C5) and using Eq. (100), we find

$$F=F_0-b[\pi\cos(\nu\tau)-\nu M\zeta\sin(\nu\tau)]-M\nu_c^2\left[\zeta\cos(\nu\tau)+\frac{\pi}{M\nu}\sin(\nu\tau)\right]\ , \tag{C7}$$

where

$$F_0=-\frac{\hbar k\Omega\Gamma}{2}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ , \tag{C8}$$

$$b=-\frac{\hbar k\Omega\Gamma}{2}\frac{k}{M}\{\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{M}}_0^{-1}\}_{2,3}\ , \tag{C9}$$

$$\nu_c^2=-\frac{\hbar k\Omega\Gamma}{2}\frac{k\nu}{M}\{\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]\cdot\underline{\mathcal{L}}\cdot\underline{\mathcal{M}}_0^{-1}\}_{2,3}\ . \tag{C10}$$

The diffusion coefficients D and Φ are then calculated to zero order in the perturbative parameter $g(\tau)$, namely, substituting $\underline{\mathcal{U}}$ with $\underline{\mathcal{U}}_0$ in (66b) and (66c). From (58) and (100) the expressions of D and Φ are found to be

$$D=\Gamma\alpha(\hbar k)^2[1+\Gamma(\underline{\mathcal{M}}_0^{-1})_{3,3}]-\left[\frac{\hbar k\Omega}{2}\right]^2\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,2}-\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{\nu}\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,3}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ , \tag{C11}$$

$$\Phi=-\left[\frac{\hbar k\Omega}{2}\right]^2\frac{1}{M\nu}\mathrm{Im}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,2}+\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{M\nu^2}\{\mathrm{Re}[(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}]_{2,3}-(\underline{\mathcal{M}}_0^{-1})_{2,3}\}(\underline{\mathcal{M}}_0^{-1})_{2,3}\ . \tag{C12}$$

To evaluate these coefficients in terms of the physical quantities we only need to know the matrices $\underline{\mathcal{M}}_0^{-1}$ and $(\underline{\mathcal{M}}_0+i\nu\underline{1})^{-1}$. For convenience we introduce the coherence damping rate $\gamma_2=\Gamma/2$, and write $\underline{\mathcal{M}}_0$ and its inverse $\underline{\mathcal{M}}_0^{-1}$ as

$$\underline{\mathcal{M}}_0=\begin{bmatrix}-\gamma_2 & \Delta_0 & 0\\ -\Delta_0 & -\gamma_2 & \Omega\\ 0 & -\Omega & -\Gamma\end{bmatrix}\ , \tag{C13}$$

$$\underline{\mathcal{M}}_0^{-1}=\frac{-1}{\gamma_2(\gamma_2\Gamma+\Omega^2)+\Gamma\Delta_0^2}\times\begin{bmatrix}\gamma_2\Gamma+\Omega^2 & \Gamma\Delta_0 & \Omega\Delta_0\\ -\Gamma\Delta_0 & \gamma_2\Gamma & \gamma_2\Omega\\ \Omega\Delta_0 & -\gamma_2\Omega & \gamma_2^2+\Delta_0^2\end{bmatrix}\ . \tag{C14}$$

Then, $(\underline{M}_0+i\nu\underline{1})^{-1}$ can be obtained from $\underline{\mathcal{M}}_0^{-1}$ by adding $i\nu$ to the diagonal elements of $\underline{\mathcal{M}}_0$. Defining

$$\mathcal{G}(\nu)\equiv(\gamma_2-i\nu)[(\gamma_2-i\nu)(\Gamma-i\nu)+\Omega^2]+(\Gamma-i\nu)\Delta_0^2 \tag{C15}$$

we get, after some lengthy algebraic calculations, the expressions

$$F_0=\frac{\hbar k\Gamma\gamma_2\Omega^2}{2\mathcal{G}(0)}\ , \tag{C16}$$

$$b=-\frac{\hbar\Gamma\Delta_0 k^2\Omega^2}{2M}\frac{(2\gamma_2\Gamma-\nu^2)\mathrm{Re}[\mathcal{G}(\nu)]-\nu(\Gamma+2\gamma_2)\mathrm{Im}[\mathcal{G}(\nu)]}{\mathcal{G}(0)|\mathcal{G}(\nu)|^2}\ , \tag{C17}$$

$$\nu_c^2=\frac{\hbar\Gamma\Delta_0\nu k^2\Omega^2}{2M}\frac{\nu(\Gamma+2\gamma_2)\mathrm{Re}[\mathcal{G}(\nu)]+(2\gamma_2\Gamma-\nu^2)\mathrm{Im}[\mathcal{G}(\nu)]}{\mathcal{G}(0)|\mathcal{G}(\nu)|^2}\ , \tag{C18}$$

$$D=\Gamma\alpha(\hbar k)^2\frac{\gamma_2\Omega^2}{\mathcal{G}(0)}+\left[\frac{\hbar k\Omega}{2}\right]^2\frac{(\Gamma\gamma_2-\nu^2)\mathrm{Re}[\mathcal{G}(\nu)]-\nu(\Gamma+\gamma_2)\mathrm{Im}[\mathcal{G}(\nu)]}{|\mathcal{G}(\nu)|^2}$$
$$+\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{\nu}\frac{\Omega\gamma_2(\Omega\nu\,\mathrm{Re}[\mathcal{G}(\nu)]+\Omega\gamma_2\mathrm{Im}[\mathcal{G}(\nu)])}{\mathcal{G}(0)|\mathcal{G}(\nu)|^2}\ , \tag{C19}$$

$$\Phi=-\left[\frac{\hbar k\Omega}{2}\right]^2\frac{1}{M\nu}\frac{\nu(\Gamma+\gamma_2)\mathrm{Re}[\mathcal{G}(\nu)]+(\Gamma\gamma_2-\nu^2)\mathrm{Im}[\mathcal{G}(\nu)]}{|\mathcal{G}(\nu)|^2}$$
$$+\left[\frac{\hbar k\Omega\Gamma}{2}\right]^2\frac{1}{M\nu^2}\frac{\Omega\gamma_2}{\mathcal{G}(0)}\left[\frac{-\Omega\gamma_2}{\mathcal{G}(0)}+\frac{\Omega\gamma_2\mathrm{Re}[\mathcal{G}(\nu)]-\nu\Omega\,\mathrm{Im}[\mathcal{G}(\nu)]}{|\mathcal{G}(\nu)|^2}\right]\ . \tag{C20}$$

In the "old" variables z and p the Fokker-Planck equation takes a compact form. Defining a renormalized oscillation frequency $\nu_1\equiv(\nu^2+\nu_c^2)^{1/2}$ and a new spatial coordinate $q\equiv z-F_0/(M\nu_1^2)$ the equation for $f(z,p)$ becomes

$$\left[\frac{\partial}{\partial t}+\frac{p}{M}\frac{\partial}{\partial q}-M\nu_1^2 q\frac{\partial}{\partial p}\right]f$$
$$=\frac{\partial}{\partial p}bpf+\frac{\partial^2}{\partial p^2}Df+\frac{\partial^2}{\partial p\,\partial q}\Phi f\ . \tag{C21}$$

The general solution of Eq. (C21) is completely defined by the transition probability $P(q,p,t|q_0,p_0,0)$, i.e., by the solution with δ-like initial condition

$$f(q,p,0)=\delta(q-q_0)\delta(p-p_0)\ . \tag{C22}$$

Next we remark that one of the properties of an Ornstein-Uhlenbeck process is that the distribution function f must be Gaussian at every time t if it is at $t=t_0$. Hence $P(q,p,t|q_0,p_0,0)$ will be a Gaussian distribution and it will be completely defined by its first and second moments. We can then solve Eq. (C21) by solving the equations for the average values of q and p and for the relative spreads.

These equations can be derived directly from (C21). To get the equation for $\langle q\rangle$, we multiply (C21) by q and integrate it by parts over the whole phase space. Using the fact that $f(q,p,t)$ and its derivatives vanish at $|q|=\infty$ and $|p|=\infty$ (f describes a bound particle), we obtain

$$\frac{d}{dt}\langle q\rangle=\frac{\langle p\rangle}{M}\ . \tag{C23}$$

In a similar way, we get

$$\frac{d}{dt}\langle p\rangle=-M\nu_1^2\langle q\rangle-b\langle p\rangle\ , \tag{C24}$$

$$\frac{d}{dt}\delta q^2=\frac{2\delta r}{M}\ , \tag{C25}$$

$$\frac{d}{dt}\delta p^2=-2M\nu_1^2\delta r-2b\delta p^2+2D\ , \tag{C26}$$

$$\frac{d}{dt}\delta r=\frac{\delta p^2}{M}-M\nu_1^2\delta q^2-b\delta r+\Phi\ , \tag{C27}$$

where we have defined $\delta q^2(t)\equiv\langle(q-\langle q\rangle)^2\rangle$, $\delta p^2(t)\equiv\langle(p-\langle p\rangle)^2\rangle$, and $\delta r(t)\equiv\langle(pq-\langle p\rangle\langle q\rangle)\rangle$.

Solving the secular equation associated to (C23)–(C27) we find that the characteristic rates of change of the average values $\langle q\rangle$ and $\langle p\rangle$ are given by

$$\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4\nu_1^2}}{2}\ . \tag{C28}$$

The spreads $\delta q^2(t)$, $\delta p^2(t)$, and $\delta r(t)$ evolve on the same time scale with the rates

$$\mu_1=2\lambda_1\ , \tag{C29}$$

$$\mu_2=2\lambda_2\ , \tag{C30}$$

$$\mu_3=\lambda_1+\lambda_2=-b\ . \tag{C31}$$

The general solution of Eqs. (C23)–(C27) can then be found by using standard methods. Assuming (C22) as the initial condition, we obtain the following expression for the average values

$$\langle q\rangle=\frac{q_0}{\lambda_2-\lambda_1}(\lambda_2 e^{\lambda_1 t}-\lambda_1 e^{\lambda_2 t})+\frac{p_0}{M(\lambda_2-\lambda_1)}(e^{\lambda_2 t}-e^{\lambda_1 t})\ , \tag{C32}$$

$$\langle p\rangle=\frac{p_0}{\lambda_2-\lambda_1}(\lambda_2 e^{\lambda_2 t}-\lambda_1 e^{\lambda_1 t})+\frac{M\nu_1^2 q_0}{\lambda_2-\lambda_1}(e^{\lambda_1 t}-e^{\lambda_2 t})\ , \tag{C33}$$

and for the relative spreads

$$\delta q^2=\frac{1}{M^2\nu_1^2}\left[\frac{D}{b}+M\Phi\right]+\frac{1}{M^2(\lambda_2-\lambda_1)^2}\left[2\left[2\frac{D}{b}+M\Phi\right]e^{-bt}+\frac{D-\lambda_2 M\Phi}{\lambda_1}e^{2\lambda_1 t}+\frac{D-\lambda_1 M\Phi}{\lambda_2}e^{2\lambda_2 t}\right], \tag{C34}$$

$$\delta p^2=\frac{D}{b}+\frac{\nu_1^2}{(\lambda_2-\lambda_1)^2}\left[2\left[\frac{2D}{b}+M\Phi\right]e^{-bt}+\left[\frac{D}{\lambda_2}-M\Phi\right]e^{2\lambda_1 t}+\left[\frac{D}{\lambda_1}-M\Phi\right]e^{2\lambda_2 t}\right], \tag{C35}$$

$$\delta r=\frac{1}{M(\lambda_2-\lambda_1)^2}[-(2D+Mb\Phi)e^{-bt}+(D-\lambda_2 M\Phi)e^{2\lambda_1 t}+(D-\lambda_1 M\Phi)e^{2\lambda_2 t}]. \tag{C36}$$

The transition probability $P(q,p,t|q_0,p_0,0)$ is then given by

$$P(q,p,t|q_0,p_0,0)=\frac{1}{2\pi B}\exp\left[-\frac{1}{2B^2}[\delta p^2(q-\langle q\rangle)^2+\delta q^2(p-\langle p\rangle)^2-2\delta r(p-\langle p\rangle)(q-\langle q\rangle)]\right], \tag{C37}$$

where

$$B\equiv[\delta q^2\delta p^2-(\delta r)^2]^{1/2}. \tag{C38}$$

In the cooling regime, i.e., when $\Delta_0<0$ so that $b>0$, $P(q,p,t|q_0,p_0,0)$ goes towards a stationary value, given by the moments (C32)–(C36) with $t=\infty$, which are independent of q_0 and p_0. Therefore the stationary Wigner distribution $f_{\rm eq}(q,p)$, does not depend on the initial conditions and reads

$$f_{\rm eq}(q,p)=\frac{1}{2\pi}\left[\frac{M^2b^2\nu_1^2}{D(D+Mb\Phi)}\right]^{1/2}\times\exp\left[-\frac{b(M\nu_1 q)^2}{2(D+Mb\Phi)}-\frac{bp^2}{2D}\right]. \tag{C39}$$

The equilibrium distribution $f_{\rm eq}$ is characterized by the values of the residual spreads, around the average values $\langle q\rangle_{\rm eq}=0$ and $\langle q\rangle_{\rm eq}=0$, which define the lowest energy of the ion

$$\langle E\rangle_{\rm eq}=\tfrac{1}{2}M\nu_1^2(\delta q^2)_{\rm eq}+\frac{(\delta p^2)_{\rm eq}}{2M}=\frac{D}{Mb}+\frac{\Phi}{2}. \tag{C40}$$

The term $\Phi/2$ is of the order of $(\hbar k)^2$, i.e., of the same order of the contributions coming from terms neglected in the derivation of the Fokker-Planck equation (65). For consistency this term must be neglected even in the present context. Then the final energy of the ion is given by $D/(Mb)$ and can be expressed as:

$$\langle E\rangle_{\rm eq}=-\frac{\hbar}{2\Gamma\Delta_0}(4\alpha\gamma_2\Gamma\{[\gamma_2^2\Gamma+\gamma_2\Omega^2+\Gamma\Delta_0^2-\nu^2(\Gamma+2\gamma_2)]^2+\nu^2(2\gamma_2\Gamma+\Omega^2+\gamma_2^2+\Delta_0^2-\nu^2)^2\}+\Gamma\gamma_2\{(\gamma_2^2\Gamma+\gamma_2\Omega^2+\Gamma\Delta_0^2)^2+\Gamma\Omega^2[\Delta_0^2(\Gamma-\gamma_2)-(\gamma_2^2+\nu^2)(\Gamma+\gamma_2)]\}+\nu^2(\gamma_2^2\Gamma+\gamma_2\Omega^2+\Gamma\Delta_0^2)(\gamma_2\Gamma^2+\gamma_2^3+\Gamma\Omega^2+\gamma_2\Delta_0^2+\gamma_2\nu^2))\times\frac{1}{2\Gamma\gamma_2(\gamma_2^2\Gamma+\gamma_2\Omega^2+\Gamma\Delta_0^2)+\nu^2[\Gamma\Omega^2+\gamma_2(\Omega^2+2\gamma_2^2+2\Delta_0^2)]}. \tag{C41}$$

The same expression for the energy at the stationary point has been obtained in Refs. 19 and 20.

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