Influence of noise on delayed bifurcations

H. Zeghlache and Paul Mandel
Optique Nonlinéaire Théorique, Université Libre de Bruxelles, Campus Plaine Code Postal 231, 1050 Bruxelles, Belgium

C. Van den Broeck
Department of Physics, Limburgs Universitair Centrum, 3610 Diepenbeek, Belgium
(Received 27 January 1989)

We study the influence of noise on the delay experienced by a steady bifurcation point involving a zero or constant solution when the control parameter is swept in time starting in the vicinity of the stable constant solution. This study is realized by a direct analysis of the moment equations rather than via a Fokker-Planck equation with an emphasis on analytical results. We treat separately multiplicative and additive noise sources and the noise can be white or colored. Furthermore, we analyze the case where the noise is added during the sweep but is absent during the preparation of the system and compare it with the case where noise is already present during the preparation of the system. In general, noise reduces the delay even to the extent that the bifurcation point can occur before the static bifurcation point. However, when the initial condition is noisy the delay may increase beyond the deterministic value in the limit of strongly colored noise.

I. INTRODUCTION

This paper deals with a specific aspect of the bifurcation theory which is suggested by the consideration of the experimental situation realized in lasers. In a semiclassical description, the laser first threshold is a steady bifurcation point, where the zero intensity solution loses its stability and the finite intensity solution becomes stable. In the good cavity limit many lasers are adequately described by a single nonlinear equation for the field amplitude E,

$$dE/dt = EF(E^2, A)\ , \tag{1.1}$$

which has two steady-state solutions. The parameter A is the optical pump parameter and $F(0,A)\neq 0$ except at the steady bifurcation point for which $A=A_c$. In this paper we shall concentrate our attention on the properties of the first laser threshold defined by the implicit equation $F(0,A_c)=0$.

In many situations, it is common to sweep the control parameter across the bifurcation point to avoid the critical dynamical effects associated with the bifurcation point. This sweeping modifies the position of the bifurcation point.[1-12] Despite this modification, the bifurcation remains, of course, an instability point of the zero solution, and therefore it can be analyzed by linearizing Eq. (1.1) around the trivial solution

$$\begin{aligned} E(t) &= \epsilon x(t) + O(\epsilon^2)\ ,\\ dx/dt &= \mu x(t) + O(\epsilon)\ , \end{aligned} \tag{1.2}$$

with $\mu = F(0,A)$. For the static case ($d\mu/dt=0$), the trivial solution is stable when $\mu<0$ with the bifurcation point occurring at $\mu_c=0$. On the contrary, when $\mu=\mu(t)$ is a monotonic function of its argument, the solution of Eq. (1.2) is

$$x(t) \equiv x(0)\exp\gamma(t) = x(0)\exp\left[\int_0^t \mu(t')dt'\right]\ . \tag{1.3}$$

When $\gamma(t)<0$, the linearized pertubation $x(t)$ converges to zero and the trivial steady state is stable. When $\gamma(t)>0$, the linearized solution $x(t)$ diverges and the trivial steady state is unstable. Thus the condition $\gamma(t^*)=0$ defines the dynamical bifurcation point $\mu(t^*)$. For a forward linear sweep defined by $\mu(t)=\mu(0)+vt$, with $\mu(0)<0$ and $v>0$ we have

$$t^* = 2\bar{t} = -2\mu(0)/v \quad \text{and} \quad \mu(t^*) = -\mu(0)\ , \tag{1.4}$$

where $\bar{t}$ is the time at which the static bifurcation point is reached: $\mu(\bar{t})=0$. Hence in a one-dimensional problem like Eq. (1.2) there is always a delay due to a forward sweep.

The purpose of this paper is to extend the study of the delayed bifurcation to the case where there is noise. Two cases can be considered, depending on whether the noise is additive or multiplicative. In the case of additive noise we consider the equation

$$dx/dt = \mu(t)x + \eta(t)\ , \tag{1.5}$$

whereas, for multiplicative noise we consider the equation

$$dx/dt = [\mu(t)+\eta(t)]x\ . \tag{1.6}$$

In both cases the stochastic function $\eta(t)$ satisfies the equation

$$\tau d\eta/dt = -\eta + \xi(t)\ , \tag{1.7}$$

where τ is the correlation time of the noise, and $\xi(t)$ is a Gaussian white-noise source defined by

$$\langle \xi(t)\rangle = 0, \quad \langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')\ . \tag{1.8}$$

When $\tau>0$, Eqs. (1.7) and (1.8) define a colored-noise process. Finally, we shall restrict our analysis to linear forward sweeps of the form $\mu(t)=\mu(0)+vt$ with $\mu(0)<0$ and $v>0$.

One way to study the Langevin equations, Eqs. (1.5) and (1.6), is to consider the associated Fokker-Planck equation for the probability distribution function $P(x,v,t)$.[13-16] The other way, which we shall follow in this paper, is to study directly the Langevin equations.[17] The linearity of these equations implies that the two-time bilinear moments $\langle x(t')\eta(t)\rangle$, $\langle x(t')x(t)\rangle$, and $\langle \eta(t')\eta(t)\rangle$ form a closed set which can be solved exactly. This is sufficient to obtain a solution for $\langle x^2(t)\rangle$, which in the laser case represents the average intensity. This is, in fact, the quantity of interest for most laser users. By limiting our characterization of the laser first threshold to the stability properties of its average intensity, we do not lose any significant information.

When solving the pair of equations, Eqs. (1.5) and (1.7), or Eqs. (1.6) and (1.7), we must specify the nature of the initial conditions. We consider that in the preparation of the system, the control parameter is kept fixed and the system evolves until it reaches its final state. This state provides the initial condition at the time $t=0$ when the sweep begins. Two types of experiments can be envisaged. In the first type of experiment, the system can be considered to be fundamentally deterministic and the noise is introduced from outside the system. Hence there is no noise present during the preparation of the system, and one may write $\langle x(0)\rangle = x(0)$ and $\langle x^2(0)\rangle = x^2(0)$. The statistical average is then over an ensemble of trajectories all starting from the same point. We shall refer to this situation as the external noise problem, or the deterministic initial condition. In the second type of experiment, the system is intrinsically noisy so that noise is already present during the preparation stage. Thus the initial conditions $x(0)$ and $x^2(0)$ are stochastic variables. The statistical average is then over an ensemble of trajectories with an "initial" spread. We shall refer to this situation as the internal noise problem, or the noisy initial condition.

Although there is no ambiguity in defining the deterministic bifurcation point, the presence of noise requires some caution. The deterministic criterion for the bifurcation point can also be expressed by the condition that $x(t^*)=x(0)$ for $t^*>0$. Indeed, as the sweep begins in the stable domain, the deviation $x(t)$ first decreases until the static bifurcation is reached at $t=\bar{t}$. This is the most stable point since it minimizes $\gamma(t)$. For times greater than $\bar{t}$, the deviation $x(t)$ increases until the initial condition $x(0)$ is recovered. This defines the bifurcation point. In the presence of noise, such a criterion may be difficult, or even impossible, to implement experimentally when $x(0)$ is a stochastic variable (internal noise problem). In this case it may be better to use a different criterion to define the bifurcation time t^*. We choose t^* as the time at which the average intensity reaches a predefined threshold level: $\langle x^2(t^*)\rangle = x_{\rm th}^2$. In the deterministic case, this condition yields

$$t^*=\bar{t}+\left[\bar{t}^2+\frac{1}{v}\ln\left[\frac{x_{\rm th}^2}{x^2(0)}\right]\right]^{1/2}, \tag{1.9}$$

instead of $t^*=2\bar{t}$. This result shows that the weight of the arbitrary threshold intensity $x_{\rm th}^2$ is influenced by the sweep rate. Because $\bar{t}=O(1/v)$, for sufficiently small sweep rates the influence of the threshold intensity vanishes. It is restored as $v \geq O(1)$. Finally, it seems natural to limit our study to multiplicative noise to the case of external noise as this can describe noise in the control (i.e., pump) parameter. For additive noise we study both the cases of internal and external noise.

This paper is divided in three sections. In Sec. II we analyze the effect of additive noise. Section II is divided in four subsections in which we consider the cases of external noise (deterministic initial condition) and internal noise (noisy initial condition), and in each case we separate the white-noise limit from the more general colored-noise case. In Sec. III we analyze the effect of multiplicative noise with no noise present during the preparation of the system. Two subsections are devoted to the white noise and the colored noise, respectively.

II. ADDITIVE NOISE

In the case of additive noise we analyze the following set of equations:

$$dx/dt=\mu(t)x+\eta(t), \tag{2.1}$$

$$\tau d\eta/dt=-\eta+\xi(t), \tag{2.2}$$

with

$$\langle \xi(t)\rangle = 0,\quad \langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'). \tag{2.3}$$

The formal solution of Eq. (2.1) leads to the average intensity

$$\begin{aligned}\langle x^2(t)\rangle &= \langle x^2(0)\rangle e^{2G(0,t)} \\ &\quad + 2e^{2G(0,t)}\int_0^t \langle x(0)\eta(s)\rangle e^{-G(0,s)}ds \\ &\quad + 2\int_0^t e^{2G(s,t)}\int_0^s e^{G(u,s)}\langle \eta(s)\eta(u)\rangle ds\, du,\end{aligned} \tag{2.4}$$

where we have defined $G(s,t)=\int_s^t \mu(u)du$. The first term in Eq. (2.4) describes the deterministic evolution of the initial intensity. The second term contains a correlation between the initial condition and the noise at later times. This term will give a finite contribution if the noise correlation time τ does not vanish. For Gaussian white noise, for instance, this second term identically cancels. Finally, the third term gives the usual influence of the noise via its integrated autocorrelation. Since the autocorrelation function of the colored noise is given by

$$\langle \eta(s)\eta(u)\rangle = (D/\tau)e^{-|s-u|/\tau},$$

the last integral in (2.4) can be evaluated independently of the properties of the initial condition. The result is

$$2\int_0^t e^{2G(s,t)}\int_0^s e^{G(u,s)}\langle\eta(s)\eta(u)\rangle ds\,du=(2\pi/v)^{1/2}\frac{D}{\tau}\exp\left[\frac{1}{v\tau^2}+\left[\frac{vt+\mu(0)}{\sqrt{v}}\right]^2\right]$$
$$\times\int_0^t dy\, e^{-[vy+\mu(0)+1/\tau]^2/(2v)}\left[\operatorname{erf}\left[\frac{vy+\mu(0)-1/\tau}{\sqrt{(2v)}}\right]-\operatorname{erf}\left[\frac{\mu(0)-1/\tau}{\sqrt{(2v)}}\right]\right]. \quad (2.5)$$

Further analysis of the mean intensity requires a knowledge of the initial condition.

A. External noise

In the case of external noise, we assume that the noise is added after the preparation of the system. Hence $x(0)$ is a deterministic variable and the following properties hold:

$$\langle x^2(0)\rangle=x^2(0),\quad \langle x(0)\eta(t)\rangle=0 \quad \text{for } t>0 .$$

At this point, we consider two possibilities directly related to the integration of Eq. (2.5): the case of additive white noise, which corresponds to the limit $\tau\to 0$, and the additive colored noise.

1. *Additive white noise*

In the limit $\tau\to 0$ we have $\eta(t)\to\xi(t)$, and using Eqs. (1.8), we obtain for (2.5) the result

$$2\int_0^t e^{2G(s,t)}\int_0^s e^{G(u,s)}\langle\eta(s)\eta(u)\rangle ds\,du =(\pi/v)^{1/2}D\exp\left[\left[\frac{vt+\mu(0)}{\sqrt{v}}\right]^2\right]\times\left[\operatorname{erf}\left[\frac{vt+\mu(0)}{\sqrt{v}}\right]-\operatorname{erf}\left[\frac{\mu(0)}{\sqrt{v}}\right]\right].$$

The integrals in Eq. (2.4) can now be performed completely, leading to

$$\langle x^2(t)\rangle=x^2(0)e^{(y^2-a^2)}+D(\pi/v)^{1/2}[\operatorname{erf}(y)-\operatorname{erf}(a)], \quad (2.6a)$$

where we have defined

$$a=\mu(0)/\sqrt{v},\quad y=\frac{\mu(0)+vt}{\sqrt{v}}=\frac{A(t)-1}{\sqrt{v}}. \quad (2.6b)$$

As explained in the introduction, the dynamical instability condition is chosen as $\langle x^2(t^*)\rangle=x_{\rm th}^2$ which is an implicit equation for the critical parameter $y(t^*)\equiv z$. Then Eq. (2.6a) gives

$$\alpha e^{-z^2}-b\operatorname{erf}(z)=e^{-a^2}-b\operatorname{erf}(a), \quad (2.7)$$

where $\alpha=x_{\rm th}^2/x^2(0)\geq 1$ and $b=[D/x^2(0)](\pi/v)^{1/2}$. We can easily verify that increasing α has no other significant role than to increase z, which is quite reasonable; the time for reaching higher values of $x_{\rm th}$ is larger.

Without loss of generality, we can consider the case for which $\alpha=1$, which finally gives the implicit equation[17] for the dynamical bifurcation parameter t^*

$$\begin{aligned} &e^{-z^2}-b\operatorname{erf}(z)=e^{-a^2}-b\operatorname{erf}(a),\\ &a=\mu(0)/\sqrt{v},\\ &b=\frac{D}{x^2(0)}(\pi/v)^{1/2},\\ &z\equiv y(t^*)=\frac{\mu(0)+vt^*}{\sqrt{v}}=\frac{A(t^*)-1}{\sqrt{v}}. \end{aligned} \quad (2.8)$$

We have solved this equation numerically, and Fig. 1 is a plot of the variable z as a function of the parameter a for three values of b. One can compare directly the value of z with the corresponding deterministic critical control parameter, and note that the delay introduced by the sweeping is reduced by the noise. From this figure, we can explain the role of the parameters a and b.

(i) The quantity a measures how long the noise can act: when $|a|$ increases, the effect of the noise increases but saturates to a plateau value $z_{\max}$. A critical value $a_{\lim}$ characterizes a domain of blurred transitions for which z is negative. This means a dynamical bifurcation point below the static bifurcation point.

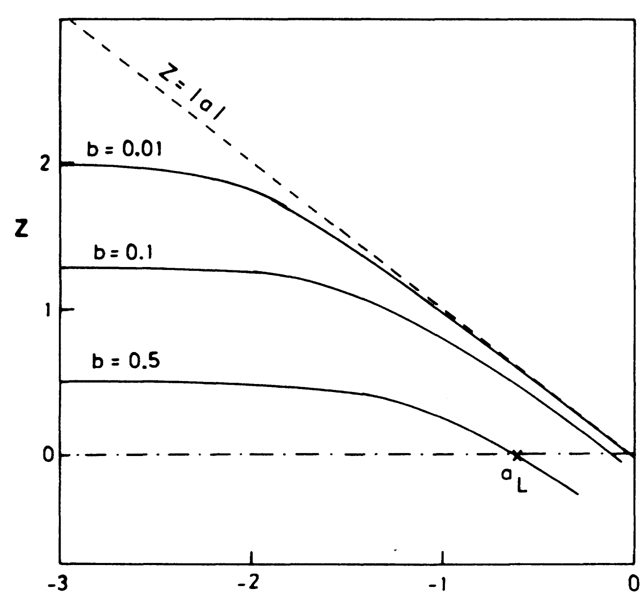

FIG. 1. Critical bifurcation parameter $z=\mu(t^*)/\sqrt{v}$ vs $a=\mu(0)/\sqrt{v}$ for different values of $b=[D/x^2(0)](\pi/v)^{1/2}$. The diagonal corresponds to $b=0$ and represents the deterministic case. The initial condition is deterministic and white noise is added during the sweep.

(ii) The value $a_{\lim}$ is a function of the parameter b, which measures the competition between the intensity of the noise and the initial position $x^2(0)$; when this ratio increases, $|a_{\lim}|$ increases. When $b>1$ and $\alpha=1$, we obtain $z<0$. For $x_{\rm th}>x(0)$, the condition $z>0$ implies $b<\alpha$.

When $b<<1$, one easily verifies from (2.8) that $z_{\max}\sim[\ln(1/2b)]^{1/2}$ and $a_{\lim}\sim-2b/\sqrt{\pi}$. However, for b very near to unity, though strictly less than 1, $z_{\max}\sim 2(1-b)/\sqrt{\pi}$ and $a^2_{\lim}\sim \ln 1/(1-b)$. The parameter $z_{\max}$ depends on $x_{\rm th}$ but is independent of the initial condition $x(0)$, as we will see in Sec. II B.

2. *Additive colored noise*

In this case, the integration of the right-hand side of Eq. (2.5) cannot be performed exactly, but using the same notation as for the white-noise case we derive the following implicit equation for the critical bifurcation parameter z:

$$\alpha e^{-z^2}-e^{-a^2}-\sqrt{2}bce^{c^2}\times\int_a^z ds\, e^{-(s+c)^2/2}\left[\operatorname{erf}\left[\frac{s-c}{\sqrt{2}}\right]-\operatorname{erf}\left[\frac{a-c}{\sqrt{2}}\right]\right]=0 \;. \tag{2.9}$$

We shall again take $\alpha=1$, and we have introduced a new parameter which contains the time correlation of the noise, namely, $c=1/(\tau\sqrt{v})$. Two limits will be considered.

(i) $c\to\infty$ or $\tau<<v^{-1/2}$: This limiting case describes either a situation of slightly colored noise ($\tau\to 0$ for fixed v) or a small value of the sweeping rate ($v\to 0$ for fixed τ). Equation (2.9) can then be expanded in powers of c^{-1} as

$$e^{-z^2}\left[1+\frac{b}{c\sqrt{\pi}}\right]-b\operatorname{erf}(z)=e^{-a^2}\left[1-\frac{b}{c\sqrt{\pi}}\right]-b\operatorname{erf}(a)+O(c^{-2}) \;. \tag{2.10}$$

This last equation is a first-order perturbation of Eq. (2.8) in powers of a small parameter $\epsilon=b/c\sqrt{\pi}$. At this order, the solution can be expanded as

$$z\sim z_{\rm wn}+\epsilon\frac{1+\exp(z^2_{\rm wn}-a^2)}{2(z_{\rm wn}+b/\sqrt{\pi})}+O(\epsilon^2) \;. \tag{2.10$'$}$$

This result indicates that the delay is increased with respect to the white-noise situation, causing the plateau to occur at a later time than in the white-noise case. The critical value $|a_{\lim}|$ is also higher; the colored noise can therefore transform a blurred transition into a delayed bifurcation.

(ii) $c\to 0$: This limit describes either a strongly colored noise (v fixed and $\tau\to\infty$) or a fast sweep (τ fixed and $v\to\infty$). Equation (2.9) can then be written as

$$e^{-z^2}-e^{-a^2}=\eta\left[\operatorname{erf}\left[\frac{z}{\sqrt{2}}\right]-\operatorname{erf}\left[\frac{a}{\sqrt{2}}\right]\right]^2+O(\eta^2) \;, \tag{2.11}$$

and the small parameter is $\eta=2cb/\sqrt{\pi}$. The solution z to first order in η is

$$z=|a|-2e^{a^2}\eta\frac{\operatorname{erf}^2(a/\sqrt{2})}{|a|}+O(\eta^2) \;. \tag{2.11$'$}$$

Thus, in this case, the delay z is always decreased with respect to the deterministic delay $|a|$.

Between these two limiting cases, the derivative of z with respect to the parameter c is always negative. Hence z has a monotonic behavior between its lower limit of the white-noise case and its upper limit corresponding to the deterministic limit. This derivative also gives the rate of change of z between these two limits: when $c\to 0$, $dz/dc\sim O(c^{-1})$, and therefore z depends strongly on c for small values of c (high τ). On the other hand, when $c\to\infty$, $dz/dc\sim O(c^{-2})$, and the variation of z with respect to c is very small. We can conclude that the effect of the coloration starts out weakly for small τ and becomes very strong for higher values of τ.

More detailed results for the critical bifurcation parameter z versus $\ln(c)$ are represented on Fig. 2. These results were obtained by a numerical integration of Eq. (2.9) for a selected set of values for the parameters a and b. To give an idea of the orders of magnitude, we note that the white-noise limit occurs for $c\sim 3$ when $b=0.01$, and $c\sim 6$ when $b=0.5$, whereas the deterministic limit is reached for $c\sim 10^{-5}$ when $a=-3$, and $c\sim 10^{-12}$ when $a=-5$.

We also note that b has no influence in the deterministic limit but determine the values of z in the white-noise limit. On the other hand, the value of a affects only the

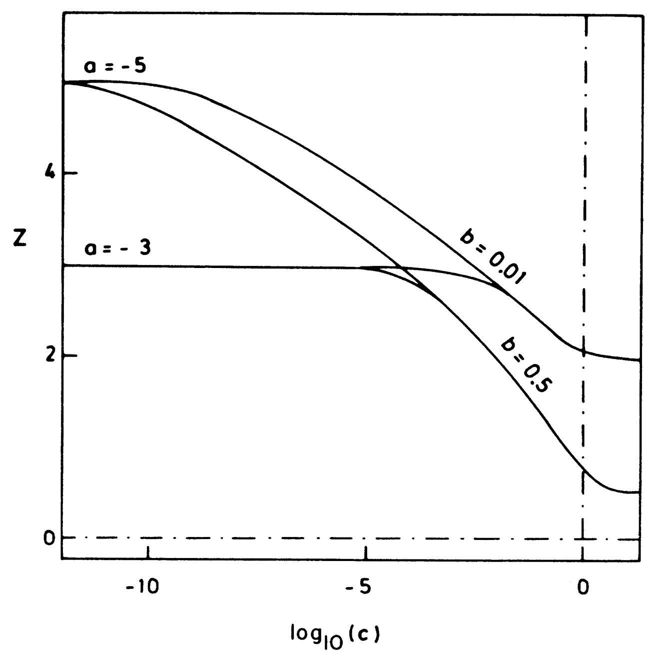

FIG. 2. Critical bifurcation parameter $z=\mu(t^*)/\sqrt{v}$ vs $\ln[1/(\tau\sqrt{v})]$ for different values of b and a. The initial condition is deterministic and colored noise (with correlation time τ) is added during the sweep.

deterministic results. By increasing the coloration of the noise, one can thus determine the role of a. This effect was predicted in Ref. 13. The influence of the threshold value is progressively lost as the coloration of the noise increases.

B. Internal noise

In the case of internal noise, $x(0)$ is a stochastic variable and depends on the intensity of the noise. Then, Eq. (2.4) contains an extra term related to the effect of the initial state on the time evolution of the system.

1. *White-noise case*

To determine the average of the "initial" condition $\langle x^2(0)\rangle$, we have to solve the pair of equations, Eqs. (2.1) and (2.2), with $v=0$, since during the preparation of the system the noise is acting but the control parameter is kept fixed: $\mu=\mu(0)$. The use of the results of Sec. II A easily leads to the long-time-limit value $\langle x^2(t)\rangle \to D/|\mu(0)|$. This expression becomes the initial condition for the problem with swept parameter: $\langle x^2(0)\rangle = D/|\mu(0)|$ and $\langle x(0)\xi(t)\rangle=0$. Using the notation y and a introduced in (2.6b), Eq. (2.4) leads in the present case to

$$\frac{\langle x^2(t)\rangle}{D}|\mu(0)| = e^{y^2-a^2}+\sqrt{\pi}|a|e^{y^2}[\mathrm{erf}(y)-\mathrm{erf}(a)] .$$

With the dynamical instability condition defined by $\langle x^2(t^*)\rangle = x_{\text{th}}^2$ corresponding to a delay $z=y(t^*)$, the ratio $\alpha=x_{\text{th}}^2/\langle x^2(0)\rangle$ can be written as $|a|\sqrt{\pi}/\beta$, where β is now defined by $(D/x_{\text{th}}^2)(\pi/v)^{1/2}$. The parameter α is restricted to values larger than unity. The implicit equation giving the critical control parameter z at the threshold x_{th}^2 is

$$\frac{e^{-z^2}}{\beta}-\mathrm{erf}(z)=\frac{e^{-a^2}}{|a|\sqrt{\pi}}-\mathrm{erf}(a) . \tag{2.12}$$

Some information can be deduced from the derivative of z with respect to the parameters a and β. The parameter β plays nearly the same role as for the deterministic initial condition, but the influence of the parameter a is modified, being more important when the value of $|a|$ is less than unity. These analytical results are verified on Fig. 3, where Eq. (2.12) has been numerically integrated and z is represented versus the parameter a for two values of β ($\sim$0.9 and 0.35). On the same graph we plot the critical control parameter for the deterministic case derived from (1.9), which we rewrite as

$$z_d=\left[a^2+\ln\left[\frac{\pi|a|}{b}\right]\right]^{1/2}$$

The noisy curves are always below the deterministic curves; the noise reduces the delay introduced by the sweeping. The two curves are separated whatever the value of a for which z exists. The difference between the two situations increases when β increases, and grows with the parameter a because of the plateau which occurs already for $|a|>1.5$ in the noisy case and because of the

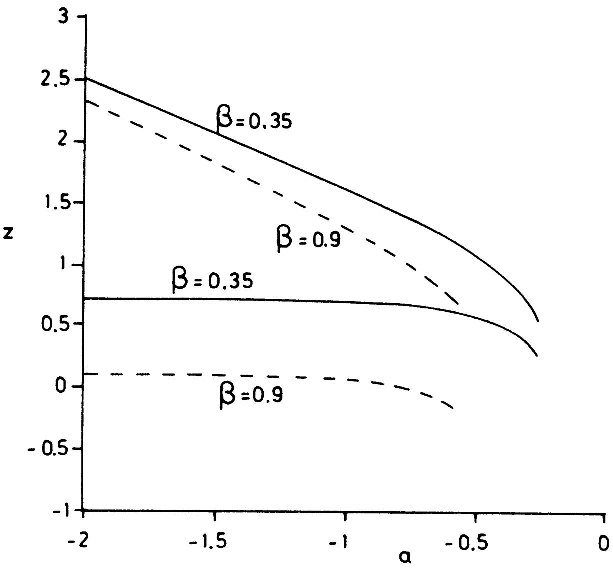

FIG. 3. Same as Fig. 1 but for a noisy initial condition. The two upper curves give the fully deterministic solution, whereas the two lower curves give the influence of additive white noise present during the preparation of the system and during the sweep.

absence of a plateau in the deterministic case. Comparing Figs. 1 and 3 indicates that the plateau occurs for smaller values of $|a|$ when the initial condition is noisy, and the influence of a is maximum when $|a|<1$. The asymptotic values of $z_{\max}$ for $\beta\leq 1$ and for $\beta<<1$ are still given by $(2/\sqrt{\pi})(1-\beta)$ and $[\ln(1/2\beta)]^{1/2}$, respectively: $z_{\max}$ is clearly independent of the initial condition. We see that the noisy initial condition gives qualitatively the same results as the deterministic one. The only difference is a strengthening of the noise effects which corresponds to a modification of the influence of a and thus of $\mu(0)$.

2. *Colored-noise case*

In this situation, the three terms in Eq. (2.4) give a nonvanishing contribution. Equation (2.5) gives the last term. For the other two terms we proceed as follows. For $t<0$, the system is described by

$$\frac{dx(t)}{dt}=\mu(0)x(t)+\eta(t) ,$$

and the noise is given by Eq. (1.7). The correlation between the variable $x(t)$, and the noise $\eta(t)$, and the mean value of $x^2(t)$ obey the equations

$$\frac{d}{dt}\langle x(t)\eta(t)\rangle=\left[\mu(0)-\frac{1}{\tau}\right]\langle x(t)\eta(t)\rangle+\frac{D}{\tau} ,$$

$$\frac{d}{dt}\langle x^2(t)\rangle=2\mu(0)\langle x^2(t)\rangle+2\langle x(t)\eta(t)\rangle .$$

For the preparation of the system, the following long-time limit or stationary value has to be taken:

$$\langle x(t)\eta(t)\rangle_{\rm st}=D/1-\mu(0)\tau$$

and

$$\langle x^2\rangle_{\rm st}=D/\mu(0)[\mu(0)\tau-1] .$$

This long-time limit for the mean intensity $\langle x^2\rangle_{\rm st}$, determined with a constant $\mu(0)$, becomes the initial value $\langle x^2(0)\rangle$ for the swept-parameter problem. In the same way, for $t'<t$,

$$\frac{d}{dt}\langle x(t')\eta(t)\rangle=-\frac{1}{\tau}\langle x(t')\eta(t)\rangle ,$$

so that

$$\langle x(t')\eta(t)\rangle=e^{-(t-t')/\tau}\langle x(t')\eta(t')\rangle .$$

In the long-time limit $t>t'$ and $t'\to+\infty$ we obtain

$$\langle x(t')\eta(t)\rangle=\frac{D}{1-\mu(0)\tau}e^{-(t-t')/\tau} .$$

Since this long-time-limit correlation function depends only on the difference $t-t'$ where t' is related to the preparation time, we can use in the swept experiment the result

$$\langle x(0)\eta(t)\rangle=\frac{D}{1-\mu(0)\tau}e^{-t/\tau} ,$$

expressing the fact that the end of the preparation period corresponds to the beginning of the sweep. Using the notations defined previously for a, β, c, and z, the equation governing the critical parameter z of the system can be written as

$$\frac{1}{\beta}e^{-z^2}-\frac{c/\sqrt{\pi}}{a^2-ac}e^{-a^2}-\frac{\sqrt{2}c}{c-a}e^{-a^2}e^{-(a+c)^2/2}\left[\operatorname{erf}\left[\frac{z+c}{\sqrt{2}}\right]-\operatorname{erf}\left[\frac{a+c}{\sqrt{2}}\right]\right]$$
$$-\sqrt{2}ce^{c^2}\int_a^z ds\, e^{-(s+c)^2/2}\left[\operatorname{erf}\left[\frac{s-c}{\sqrt{2}}\right]-\operatorname{erf}\left[\frac{a-c}{\sqrt{2}}\right]\right]=0 . \quad (2.13)$$

The influence of the noise coloration appears not only in the initial condition $\langle x^2(0)\rangle$ and in the autocorrelation function of the noise $\langle\eta(t)\eta(t')\rangle$, but also in an extra term which corresponds to the influence of the noise present for $t<0$. The integral can be performed only by a perturbative expansion, and two limiting cases are considered.

(i) $c\to+\infty$: Eq. (2.13) can be expanded to first order in c^{-1}:

$$\left[1+\frac{\beta}{c\sqrt{\pi}}+\frac{z\beta}{c^2\sqrt{\pi}}\right]e^{-z^2}+\beta\left[1+\frac{1}{2c^2}\right]\operatorname{erf}(z)$$
$$=e^{-a^2}-\beta\left[1+\frac{1}{2c^2}\right]\operatorname{erf}(a)+O(c^{-3}) . \quad (2.14)$$

When comparing Eqs. (2.10) and (2.14), we notice that the main difference up to order c^{-1} is a modification of the coefficient of e^{-a^2}. In the case of the deterministic initial condition, the e^{-a^2} terms originated only from the initial intensity contribution, i.e., the first term on the right-hand side of Eq. (2.4). In the present case of a noisy initial condition, additional contributions arising from $\langle x(0)\eta(t)\rangle$ cancel exactly the c^{-1} as well as the c^{-2} corrections to the dominant coefficient of e^{-a^2}. The result is that the influence of the color will start out later than for the deterministic initial condition. A perturbative calculation of z gives the correction to the white-noise delay $z_{\rm wn}$:

$$z=z_{\rm wn}+\frac{\beta c^{-1}}{\sqrt{\pi}}\frac{1}{2\left[z_{\rm wn}+\frac{\beta}{\sqrt{\pi}}\right]}+O(c^{-2}) ,$$

where we can verify that the correction to $z_{\rm wn}$ is always positive and is now smaller than for the deterministic initial condition given by Eq. (2.10′).

(ii) $c\to 0$: Eq. (2.13) leads to dominant order $\beta^{-1}e^{-z^2}\simeq 0$. Then z has to take large values, and the critical control parameter is large. We can explain this result easily by noticing that this limit makes the initial condition $\langle x^2(0)\rangle=D/\mu(0)[\mu(0)\tau-1]$ very close to the zero steady state. The closer the initial condition is to the zero solution, the longer the delay will be. A more precise calculation gives to first order in c

$$e^{-z^2}=c\sqrt{\pi}\left\{\frac{e^{-a^2/2}}{a\sqrt{\pi}}-\frac{1}{\sqrt{2}}\left[\operatorname{erf}\left[\frac{z}{\sqrt{2}}\right]-\operatorname{erf}\left[\frac{a}{\sqrt{2}}\right]\right]\right\}^2$$
$$+O(c^2) . \quad (2.15)$$

Since z has to be large, we can approximate the function $\operatorname{erf}(z)$ by 1 in the last equation, so that the behavior of z is well described by the relation

$$z\simeq\left[\ln\frac{1}{\sqrt{\pi}c\beta\left\{\frac{e^{-a^2/2}}{a\sqrt{\pi}}-\frac{1}{\sqrt{2}}\left[1-\operatorname{erf}\left[\frac{a}{\sqrt{2}}\right]\right]\right\}^2}\right]^{1/2} .$$

This expression diverges when $c\to 0$. When the parameter a corresponds to the plateau ($|a|>3$), z can be written as

$$z\sim\left[a^2+\ln\frac{\pi|a|}{\beta}+\ln\frac{|a|}{\beta}\right]^{1/2} ,$$

where the first two terms under the square root are those of the deterministic case (no noise at all). We can con-

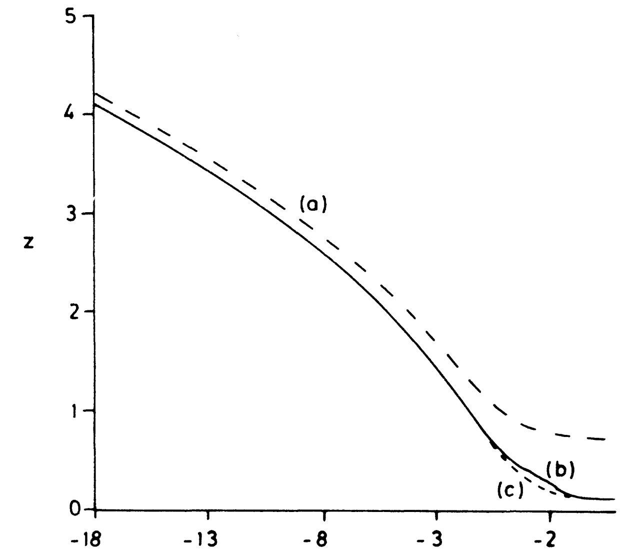

FIG. 4. Same as Fig. 2 but for a noisy initial condition. Curve (a), $\beta=0.35$, $a=-10$; curve (b), $\beta=0.9$, $a=-10$; curve (c), $\beta=0.9$, $a=-3$.

clude that when the parameter c goes to zero, the variable z increases without limit and can reach a higher value than in the fully deterministic case, which we define as the situation where no noise is ever present in the system.

The main differences between the two kinds of initial conditions can be summarized as follows: (i) the effect of the noise correlation time starts out for higher values of τ; (ii) in the case of a deterministic initial condition z reaches the fully deterministic limit for strong coloration. On the contrary, for the noisy initial condition, z increases without bound in the limit of strong coloration.

Equation (2.13) has been integrated numerically, and Fig. 4 represents z versus $\ln_{10}(c)$ for three cases: (a) $a=-10$ and $\beta=0.35$; (b) $a=-10$, $\beta=0.9$; (c) $a=-3$, $\beta=0.9$. Comparison of curves (a) and (b) demonstrates the influence of the parameter β. The influence is largest for small values of the time correlation of the noise (white-noise limit). It then becomes uniform in the case of very-colored noise. However, this influence is always present and does not disappear completely with high coloration, as was the case for the deterministic initial condition.

The parameter a has no influence in the white-noise limit (because of the plateau) but also in the very-colored-noise limit, which is not the case with the deterministic initial condition. In fact, the influence of a is localized in a domain of c values corresponding to $\ln(c)$ between -2.5 to 2.5, and is of the order of 0.1 for the critical control parameter. In fact, for smaller values of the parameter β, this influence of a diminishes and disappears.

III. MULTIPLICATIVE NOISE

In this case, we consider the set of equations

$$\frac{dx}{dt}=[\mu(t)+\eta(t)]x(t) , \tag{3.1}$$

$$\tau\frac{d\eta}{dt}=-\eta(\tau)+\xi(t) . \tag{3.2}$$

The noise is Gaussian and $\xi(t)$ is the white noise characterized by its mean value and autocorrelation function given in (2.3). The mean value of x^2 is given by

$$\langle x^2(t)\rangle = x^2(0)e^{vt^2+2t[\mu(0)+2D]-4D\tau(1-e^{-t/\tau})} , \tag{3.3}$$

where we have made the assumption of external noise; there is no noise during the preparation of the system $(t<0)$, and $x(0)$ is independent of the noise present during the sweep. The dynamical instability condition can be expressed as $\langle x^2(t^*)\rangle = x_{\text{th}}^2$, and the critical control parameter is given by the equation

$$(vt^*)^2+2t^*[\mu(0)+2D]-4D\tau(1-e^{-t^*/\tau}) = \ln\frac{x_{\text{th}}^2}{\langle x^2(0)\rangle}\equiv\ln\alpha . \tag{3.4}$$

We shall consider separately the white-noise case $(\tau\to 0)$ and the colored-noise case.

A. Multiplicative white noise

Equation (3.4) can be solved exactly in the limit $\tau=0$ and gives

$$t^*=-\frac{\mu(0)+2D}{v}+\left[\left[\frac{\mu(0)+2D}{v}\right]^2+\frac{1}{v}\ln\alpha\right]^{1/2} . \tag{3.5}$$

The influence of the ratio α is related to the sweeping rate in the same way as in the fully deterministic situation. The effect of the multiplicative white noise can be viewed as a modification of the characteristic time $\bar{t}=-\mu(0)/v$ which is replaced by $\tilde{t}=-[\mu(0)+2D]/v$ which is smaller than $\bar{t}$ since $\mu(0)<0$. The delay introduced by the sweeping is then reduced, by an amount which is function of the ratio D/v. The influence of the noise is more important when $|\mu(0)|$ is greater than but near to $2D$ since the intensity of the noise is directly compared to $\mu(0)$. When $\alpha=1$, the critical time is given by

$$t^*=2\bar{t}-4D/v .$$

In terms of the parameter a and the variable z, this last equation can be written as

$$z=|a|-4D/\sqrt{v} . \tag{3.6}$$

An important difference with the additive noise case [ruled by Eq. (2.8)] is that now the delay z does not depend on $x(0)$.

B. Colored multiplicative noise

Equation (3.4) cannot be solved analytically but a graphical solution can be obtained by rewriting this equation as

$$\begin{aligned} y&=vt^{*2}+2t^*[\mu(0)+2D]-\ln\alpha ,\\ y&=4D\tau(1-e^{-t^*/\tau}) , \end{aligned} \tag{3.7}$$

and finding the points where the two curves coincide.

This graphical approach leads to the condition $|\mu(0)|>2D$ to have a positive value for t^*. As for the additive colored noise, we could use the parameters a, z, c and, in this case, the ratio $D/\sqrt{v}$. However, it is interesting to solve the problem directly in terms of t^* and τ, avoiding the rescaling due to the sweeping rate. We again consider the situation where $\alpha=1$. If we define γ as $\mu(0)/v$ and δ as D/v, we can expand Eq. (3.4) in two limits.

(i) When τ is sufficiently small, Eq. (3.4) can be written to first order as

$$t^{*^2}+2(\gamma+2\delta)t^*-4\delta\tau+0(\tau^2)=0 ,$$

and the solution reads

$$t^*=2\overline{t}-4\gamma-\tau\frac{2\delta}{\gamma+2\delta}+O(\tau^2) .$$

The correction to the white-noise critical time is strictly positive and the delay is increased.

(ii) When τ is sufficiently large, then Eq. (3.4) can be expanded as

$$(t^*)^2(1+2\delta\tau^{-1})+2\gamma t^*+O(\tau^{-2})=0 ,$$

and the solution is

$$t^*=-2\gamma+\tau^{-1}4\delta\gamma+O(\tau^{-2}) .$$

Since γ is negative, the correction to the deterministic critical time is now negative.

Thus we see that when the color increases, the critical time of the system goes from its white-noise value to the deterministic value. The maximum decrease of the critical time $\Delta=4D/v$, which is introduced by the white noise, is progressively reduced by the color. This is similar to the additive case. On the other hand, $dt^*/d\tau\rightarrow-2\delta/\gamma+2\delta>1$ when $\tau\rightarrow0$, and $dt^*/d\tau\rightarrow0$ when $\tau\rightarrow\infty$. The rate of the reestablishment of the delay is then higher for small τ than for large values of τ, which is just the opposite of what happens for the additive case. This is made apparent in Fig. 5.

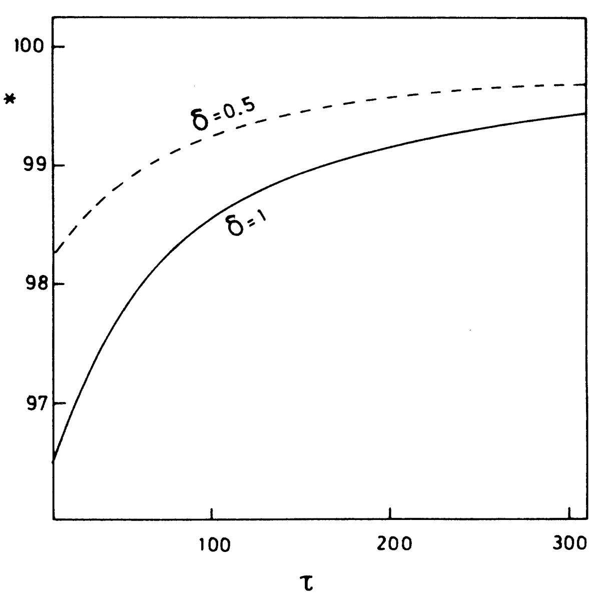

FIG. 5. Critical bifurcation time vs the correlation time of the multiplicative noise for two values of $\delta=D/v$.

IV. CONCLUSIONS

The calculations we have performed in terms of the second moment of the stochastic variable $x(t)$ have improved our understanding of the effects of the noise and its dependence on the different parameters of the problem. First, we have stressed the role played by the sweeping rate. In the fully deterministic situation, increasing the sweep rate decreases the critical time, whereas in the noisy situation we observe the opposite effect; the noise has less time to act. Furthermore, for the noisy additive case, the sweep rate scales all relevant variables, meaning that it imprints a new time scale on the system.

We have seen that the relation between the different parameters is not simple. For the same dynamical instability condition, the effect of multiplicative noise is independent of the initial condition, whereas the effects of additive noise depend on the initial condition. As far as the dependence of the delay on D (the noise intensity) is concerned, we notice that in the case of a multiplicative noise, the delay is a function of the difference $|\mu(0)|-D$, whereas, in the additive case, the delay is a function of the ratio $D/x_{\rm th}^2$. At constant D, the effect of additive noise first increases with $|\mu(0)|/\sqrt{v}$, and then saturates. On the contrary, the delay is a constant for a fixed D when the noise is multiplicative.

The color of the noise has a quite clear effect in that it reintroduces the delay destroyed by the white noise. In the multiplicative case, $dz/d\tau$ is large for small τ and goes to zero for larger values of the correlation time. The delay is therefore increased significantly for reasonable values of τ. For the case of additive noise, however, the sensitivity to the correlation time occurs at larger values of τ. This justifies the use of a logarithmic scale in Fig. 2 since very large τ are required to observe the influence of the coloration. It also explains why the authors of Ref. 15 did not find any dependence on the color since the value of τ used in their analog simulations was too small. The best range of parameters to measure a difference due to the color of the noise seems to be $v\sim10^{-2}$ and $\mu(0)\sim-1$ with higher values of τ (~50).

The noise coloration induces a modification of the influence of the parameters a and b. For the deterministic initial condition, the dependence on b disappears when τ increases. In the case of the noisy initial condition, the influence of β decreases but with a finite lower bound. On the contrary, the parameter a has, qualitatively, the opposite dependence on the noise coloration. However, in the noisy initial condition case, z depends weakly on a as the correlation time increases. This last effect has been predicted in Ref. 13 for the parameter $\mu(0)$ on qualitative grounds and is verified in the present approach. Another conclusion is that the influence of the noisy initial condition is weaker in the case of white noise than in the case of colored noise.

ACKNOWLEDGMENTS

Partial support from the Fonds National de la Recherche Scientifique (FNRS) (Belgium), and the Interuniversity Attraction Pole (IAP) program of the Belgian government is gratefully acknowledged.

[1]P. Mandel and T. Erneux, Phys. Rev. Lett. **53**, 1818 (1984).
[2]T. Erneux and P. Mandel, Phys. Rev. A **30**, 1893 (1984).
[3]T. Erneux and P. Mandel, SIAM J. Appl. Math. **46**, 1 (1986).
[4]P. Mandel, in *Frontiers in Quantum Optics,* edited by E. R. Pikes and S. Sarkar (Hilger, Bristol, 1986), p. 430.
[5]A. I. Neishtadt, Diff. Eq. **23**, 1385 (1987).
[6]P. Mandel and T. Erneux, J. Stat. Phys. **48**, 1059 (1987).
[7]M. Lefèbvre, D. Dangoisse, and P. Glorieux, Phys. Rev. A **29**, 758 (1984).
[8]T. Midavaine, D. Dangoisse, and P. Glorieux, Phys. Rev. Lett. **55**, 1989 (1985).
[9]P. Glorieux and D. Dangoisse, IEEE J. Quantum Electron. **QE-21**, 1486 (1985).
[10]B. Morris and F. Moss, Phys. Lett. **118A**, 117 (1986).
[11]W. Sharpf, M. Squicciarini, D. Bromley, C. Green, J. R. Tredicce, and L. M. Narducci, Opt. Commun. **63**, 344 (1987).
[12]F. T. Arecchi, W. Gadomski, R. Meucci, and J. A. Roversi, Opt. Commun. **70**, 155 (1989).
[13]G. Broggi, A. Colombo, L. A. Lugiato, and P. Mandel, Phys. Rev. A **33**, 3635 (1986).
[14]D. K. Kondepudi, F. Moss, and P. V. E. McClintock, Phys. Lett. **114A**, 68 (1986).
[15]R. Mannella, F. Moss, and P. V. E. McClintock, Phys. Rev. A **35**, 2560 (1987).
[16]M. C. Torrent and M. San Miguel, Phys. Rev. A **38**, 245 (1988).
[17]C. Van den Broeck and P. Mandel, Phys. Lett. **122A**, 36 (1987).