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Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase

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It is shown that the probability distribution for the rotated quadrature phase $[a^\dagger \exp(i\theta) + a\exp(-i\theta)]/2$ can be expressed in terms of quasiprobability distributions such as P, Q, and Wigner functions and that also the reverse is true, i.e., if the probability distribution for the rotated quadrature phase is known for every θ in the interval $0 \le \theta < \pi$, then the quasiprobability distributions can be obtained.

In quantum optics quasiprobability distributions such as the Glauber-Sudarshan P function, the Q function, and the Wigner function play an important role. With the help of these functions, expectation values of any products of creation operators $a^\dagger$ and annihilation operators a can be calculated. It is, however, impossible to measure these quasiprobability distributions directly. In this Rapid Communication we propose a method for obtaining these quasiprobability distributions by measuring appropriate probability distributions of certain quadrature phases.

Instead of the three different quasiprobability distributions mentioned above we use the s-parametrized quasiprobability distributions introduced by Cahill and Glauber.[1] These distributions may be defined as Fourier transforms of the characteristic functions (our definitions deviate from those of Ref. 1 by a factor of $1/\pi$)

$$\tilde{W}(\xi,s) = \mathrm{Tr}[\exp(\xi a^\dagger - \xi^* a + s|\xi|^2/2)\rho], \tag{1}$$

i.e.,

$$W(\alpha,s) = \frac{1}{\pi^2}\int \tilde{W}(\xi,s)\exp(\alpha\xi^* - \alpha^*\xi)d^2\xi, \tag{2}$$

where ρ is the density operator. As explained in detail in Ref. 1, $W(\alpha,1)$ is the Glauber-Sudarshan P function, $W(\alpha,0)$ is the Wigner function, and $W(\alpha,-1) = \langle\alpha|\rho|\alpha\rangle/\pi$ is the Q function. As also shown in this reference, any single-time expectation value of the s-ordered products $\langle[(a^\dagger)^n a^m]_s\rangle$ can be obtained by proper integration with weight $W(\alpha,s)$ in the complex α plane.

According to Refs. 2–5 a homodyne detector measures the following linear combination of the creation and annihilation operators;

$$\hat{x}(\theta) = \hat{x}(\theta)^\dagger = (a^\dagger e^{i\theta} + ae^{-i\theta})/2 = a_r\cos\theta - a_i\sin\theta, \tag{3}$$

where the a_r and a_i are the two quadrature phases

$$a_r = (a + a^\dagger)/2 = \hat{x}(0), \quad a_i = i(a^\dagger - a)/2 = \hat{x}(\pi/2) \tag{4}$$

introduced in connection with squeezing.[6,7] Because of (3), we call the operator $\hat{x}$ the rotated quadrature phase. The complete information for calculating any single-time expectation value of the rotated quadrature phase $\hat{x}(\theta)$ is given by its probability distribution $w(x,\theta)$. This distribution may be defined as the Fourier transform of the

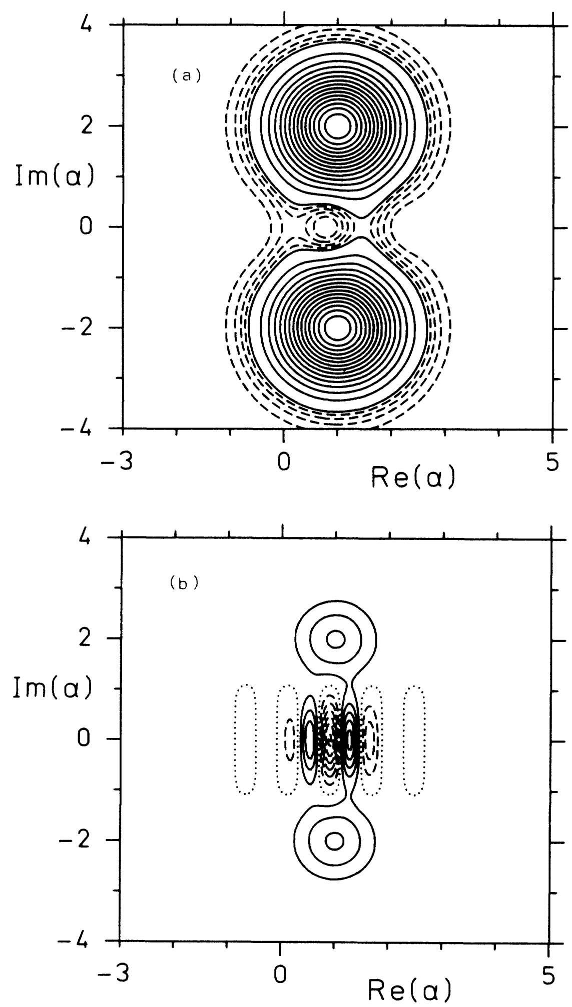

FIG. 1. Contour lines of the quasiprobability distribution (15) in the complex α plane for the superposition (14) of the two coherent states for $a = 1$ and $b = 2$. In (a) the contour lines are shown for $s = -1$ (Q function) and have the heights 0.01,0.02,... (solid lines), 0.002,0.004,0.006,0.008 (dashed lines); in (b) they are shown for $s = 0$ (Wigner function) and have the heights 0 (dotted lines), 0.1,0.2,... (solid lines), $-0.1, -0.2, \ldots$ (dashed lines).

characteristic function

$$\tilde{w}(\eta,\theta)=\mathrm{Tr}\{\exp[i\eta\hat{x}(\theta)]\rho\}\ , \tag{5}$$

i.e.,

$$w(x,\theta)=\frac{1}{2\pi}\int\tilde{w}(\eta,\theta)e^{-i\eta x}d\eta\ . \tag{6}$$

Because $\hat{x}(\theta)$ is a Hermitian operator, this probability distribution is well behaved and positive everywhere.

It is now an easy matter to establish a one-to-one correspondence between the quasiprobability distribution $W(\alpha,s)$ and the probability distribution $w(x,\theta)$. This is most easily seen by looking at the characteristic functions (1) and (5). Using the definition (3), we immediately obtain

$$\tilde{w}(\eta,\theta)=\tilde{W}(i\eta e^{i\theta}/2,s)\exp(-s\eta^2/8)\ . \tag{7}$$

In real notation

$$\xi=\xi_r+i\xi_i;\quad \tilde{W}(\xi,s)=\tilde{W}(\xi_r,\xi_i,s)\ , \tag{8}$$

we have

$$\tilde{w}(\eta,\theta)=\tilde{W}(-\tfrac{1}{2}\eta\sin\theta,\tfrac{1}{2}\eta\cos\theta,s)\exp(-s\eta^2/8)\ . \tag{9}$$

If $\tilde{w}(\eta,\theta)$ is known for all η values in the range $-\infty<\eta<\infty$ and for all θ values in the range $0\le\theta<\pi$, the characteristic function $\tilde{W}(\xi_r,\xi_i,s)$ is known in the whole complex ξ plane, i.e., for ξ_r and $\xi_i\in(-\infty,\infty)$. Thus, there is a one-to-one correspondence between the characteristic functions (1) and (5), and therefore we also have a one-to-one correspondence between the quasiprobability distributions (2) and the probability distribution (6). Using the Fourier transform of (9), inserting the inverse Fourier transform of (2), and changing integration variables, we obtain

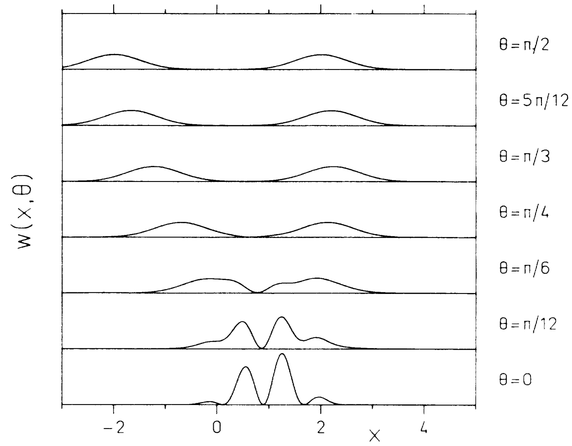

FIG. 2. The distribution (16) for the indicated θ values and for $a=1$, $b=2$.

$$w(x,\theta)=\frac{1}{2\pi}\int\int\int W(u\cos\theta-v\sin\theta,u\sin\theta+v\cos\theta,s)\ \exp[-s\eta^2/8+i(u-x)\eta]du\,dv\,d\eta\ . \tag{10}$$

For $s\ge 0$ we can perform the integration over η and thus arrive at the simpler expression

$$w(x,\theta)=\left(\frac{2}{\pi s}\right)^{1/2}\int\int W(u\cos\theta-v\sin\theta,u\sin\theta+v\cos\theta,s)\exp[-2(u-x)^2/s]du\,dv\ . \tag{11}$$

For $s=0$, i.e., for the Wigner function, (10) reduces to

$$w(x,\theta)=\int W(x\cos\theta-v\sin\theta,x\sin\theta+v\cos\theta,0)dv\ . \tag{12}$$

By similar steps we obtain from (9) the inverse of (10)

$$W(\alpha_r,\alpha_i,s)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{\pi}w(x,\theta)\exp[s\eta^2/8+i\eta(x-\alpha_r\cos\theta-\alpha_i\sin\theta)]\,|\eta|\,dx\,d\eta\,d\theta\ . \tag{13}$$

Thus, the quasiprobability distributions $W(\alpha,s)$ are uniquely determined by the probability distribution $w(x,\theta)$ and vice versa.

As an example we consider a superposition of the two coherent states $|a\pm ib\rangle$, i.e.,

$$|\psi\rangle=\frac{|a+ib\rangle+|a-ib\rangle}{\{2[1+\cos(2ab)\exp(-2b^2)]\}^{1/2}}\ . \tag{14}$$

For this example the quasiprobability distributions take the form

$$W(\alpha_r,\alpha_i,s)=\frac{\exp[-2(\alpha_r-a)^2/(1-s)]}{\pi(1-s)[1+\cos(2ab)\exp(-2b^2)]}\left[\exp\left(-\frac{2}{1-s}(\alpha_i-b)^2\right)+\exp\left(-\frac{2}{1-s}(\alpha_i+b)^2\right)+2\exp\left(\frac{2}{1-s}(sb^2-\alpha_i^2)\right)\cos\left(\frac{4b}{1-s}\alpha_r-2\frac{1+s}{1-s}ab\right)\right], \tag{15}$$

whereas the distribution (6) reads

$$w(x,\theta)=\left(\frac{2}{\pi}\right)^{1/2}\frac{\exp[-2(x-a\cos\theta)^2-2b^2\sin^2\theta]}{1+\cos(2ab)\exp(-2b^2)}\{\cosh[4b\sin(\theta)x-2ab\sin2\theta]+\cos[4b\cos(\theta)x-2ab\cos2\theta]\}\ . \tag{16}$$

Figure 1 shows $W(\alpha,-1)$ (Q function) and $W(\alpha,0)$ (Wigner function) for the superposition (14). As already demonstrated in Ref. 8, the Wigner function oscillates in between the two maxima. The plots of $w(x,\theta)$ are shown in Fig. 2. For $\theta=0$ the distribution $w(x,\theta)$ oscillates rapidly, similar to the Wigner function along the real axis, though no negative values occur here. For $\theta=\pi/2$, two maxima are clearly visible similar to those of the Q function. It follows from the symmetry of the example that we have $w(x,\theta)=w(-x,\pi-\theta)$. Thus, $w(x,\theta)$ is only plotted for θ values in the range $0\leq\theta\leq\pi/2$.

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