Dimensional study of some singular potentials

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Dimensional analysis is proposed as a technique for studying the functional dependence of energy on coupling constants for a class of singular potentials. Some exact consequences are derived. In conjunction with WKB a useful approximation to the x^6 anharmonic oscillator energies is proposed.

Bound-state-generating combined potentials are of great interest in many branches of physics. Because of the difficulties associated with their analytic solutions, perturbative techniques are commonly employed instead. These techniques assume the existence of suitable small parameter expansions. This rather strong assumption is usually hard to verify.

In this work we point out that for a useful class of singular potentials, dimensional considerations, based on some rudimentary knowledge of the compound potentials, can provide invaluable insight into the functional dependence of energy on coupling constants. This somewhat unorthodox approach to nonrelativistic quantum mechanics is derived from the ideology of constructing suitable covariants in the theory of relativity and in the domain of elementary particle physics. We shall develop the procedure through examples of combined potentials that are mixtures of power-law potentials that individually support bound states, irrespective of their strength.

We first consider anharmonic oscillators in one dimension.[1] As a typical representative, we take

$$V \equiv V_0 + V_6 = \tfrac{1}{2}\mu\omega^2 x^2 + \lambda x^6 \ . \tag{1}$$

We have a strictly bound-state problem here with one quantum number n and two distinct bound-state limits. There are two characteristic energy scales involved. The energy E is a function of the type $E(n,\omega,\lambda)$. If small parameter expansion exists, then it must be true that

$$\lim_{\lambda\to 0} E \to E_0, \quad \lim_{\omega\to 0} E \to E_6 \ , \tag{2}$$

where E_0 and E_6 represent the energy levels of the pure V_0 and V_6 problems, respectively.

Crucially important for the present work is the fact that E_0 is a linear function of ω, while E_6 varies as $\lambda^{1/4}$. Hence near the oscillator limit the energy E should scale as ω, while near the V_6 limit it should scale as $\lambda^{1/4}$.

We now construct the most general dimensionally permissible structure of E consistent with Eq. (2) and its reality and dimensionality. An example of a simple set of permissible candidates is

$$\left[\frac{E_0^p + E_6^p}{E_0^q + E_6^q}\right]^{1/(p-q)} ,$$

where p and q are real. Because of the limiting constraints imposed by Eq. (2), logarithmic, trigonometric, and exponential functions need not be considered. In fact, it is easy to see that the required structure of E can be written, in general, as

$$E(n) \equiv f(S_\alpha, S_\beta, \ldots) \ , \tag{3}$$

where

$$S_\nu = a_\nu(n)E_0^\nu(n) + b_\nu(n)E_6^\nu(n) + c_\nu \ ,$$
$$\nu = \alpha,\beta, \ldots \ . \tag{4}$$

The functions a_ν and b_ν are dimensionless. They may depend on the quantum number n and the index ν and obey the constraints imposed by Eq. (2). The arbitrary functions c_ν must vanish in both limits $\omega\to 0$ and $\lambda\to 0$. The function S_ν has the dimension of E^ν. The definition of the function S_ν can, in fact, be refined further. This, however, will not be required, as we shall see. The function f has the dimension of energy and the unique choice is dictated by the Schrödinger equation. In the present case this choice is not known.

Equations (2)–(4), which represent the entire limiting and dimensional requirements, are the basis of the present study. Their straightforward consequence is this: If the energy E is to have a power-series expansion around the oscillator limit in powers of λ, then the power-series expansion around the V_6 potential limit cannot be an expansion in powers of ω^2. However, an expansion in powers of ω^4 can be constructed. On the other hand, if we insist upon a small ω^2 expansion, as a conventional approach would require, then E becomes a necessarily nonanalytic function of λ. The type of expansion envisaged around one limit has a nontrivial bearing on the would be expansion around the other limit. We emphasize that this conclusion is an *exact* consequence of dimensional consistency and the existence of the limits.

However, given our present state of knowledge, no rationale can be provided to reject a small ω^2 expansion around the V_6 limit. As long as ω is small enough, the oscillator does indeed represent a perturbation. On the other hand, the qualifications of the parameter λ as an expansion parameter seem questionable. For example, by going to the momentum representation, we see that it multiplies the highest-derivative term in the momentum-space Schrödinger equation. Under such conditions the

small parameter expansion in λ is doubtful.[2] With this in mind, it them seems natural to infer that E should be nonanalytic in λ, whereas a small parameter expansion in powers of ω^2 should be permissible.

Unfortunately, the potentials V and V_6 cannot be solved analytically.[3] Hence, to explore this situation further, we devise a dimensionally inspired empirical approach. We compute the energy levels of the pure V_6 problem numerically. Using these, we compare the performance of several dimensional candidates such as E_0+E_6, $(E_0^2+E_6^2)^{1/2}$, ... against the numerically obtainable energies of the x^6 anharmonic oscillator, for a large number of levels and a wide range of couplings. In this way, we find that the combination

$$E=a(E_0^2+E_6^2)^{1/2}+b(E_0^4+E_6^4)^{1/4}\ , \tag{5}$$

with

$$a+b=1\ , \tag{6}$$

provides a remarkably accurate description of the anharmonic oscillator of Eq. (1). Notice that Eq. (6) is dictated by Eq. (2). As mentioned earlier, a and b can be varied from level to level. However, with our present goal in view, a relevant indication of the agreements obtainable is already provided by simply setting $a=b=\frac{1}{2}$ for *all* levels and couplings. With this choice, we present in Table I a sample comparison for the first six even levels for various couplings. The dimensionless parameter β used in Table I is defined through the relation $\beta^2\equiv 2\hbar^2\lambda/\mu^3\omega^4$. For β values outside the range indicated, the fits are even better. A similar result holds for the missing odd levels. For higher states the fits actually improve, as is to be expected. As noted above, the small departures can be rectified by allowing a very minor variation in a and b consistent with Eq. (6). Equation (5), in essence, represents a one-parameter fit for all levels and for all β.

Let us recall that the WKB result for the V_6 problem is good to well within 1% for all levels $n>2$. Hence Eq. (5), with E_6 replaced by the known WKB result, provides a good description of the problem at hand for all $n>2$ and for all β.

Now Eq. (5) explicitly indicates that E has an essential singularity at $\lambda=0$ but a small ω^2 expansion is permissible. Given its proximity to reality, it seems reasonable to infer that the true energy is indeed nonanalytic in λ.

The study of other anharmonic oscillators can be carried out along the same lines. If we *assume* that small ω^2 expansions exist, then an essential singularity of E in λ at $\lambda=0$ follows in each case as a strict consequence. These results are in accord with several previous studies of these potentials that directly studied the analytic properties of the energy in the complex coupling constant plane.[4]

Next, we briefly consider a second class of potentials which consist of a combination of the Coulombic and power-law potentials. Specifically, consider a two-dimensional problem with

$$V=V_c+V_0=-\frac{e^2}{\rho}+\frac{1}{2}\mu\omega^2\rho^2\ . \tag{7}$$

This is again a purely bound-state problem with two quantum numbers (k,m) and two well-defined bound-state limits with energies E_c and E_0, respectively, where

$$E_c=-\mathcal{R}/(k+|m|+\tfrac{1}{2})^2,\quad E_0=\hbar\omega(2k+|m|+1)\ , \tag{8}$$

where $\mathcal{R}$ is the Rydberg constant.

In general, $E\equiv E(k,m,\omega,e)$ such that $E\to E_0$ as $e\to 0$ and $E\to E_c$ as $\omega\to 0$. As before, we write

$$E=f(S_\alpha,S_\beta\ldots)\ , \tag{9}$$

where

$$S_\nu=a_\nu(k,m)|E_c|^\nu+b_\nu(k,m)E_0^\nu+c_\nu\ ,\qquad \nu=\alpha,\beta,\ldots\ . \tag{10}$$

Remembering that E_c is negative, and following the procedure of the previous example, we arrive at a similar conclusion. No combination of any finite number of dimensional terms can permit a power-series expansion in ω^2 around the Coulomb limit. A power-series expansion in e^2 around the oscillator limit is possible, but then E becomes nonanalytic in ω. However, suitable small parameter expansions around both limits are constructible. We stress that the exact form of f is not required for this deduction but the limiting constraints play a decisive role.

To investigate the nature of possible expansions one can proceed as earlier. This time the numerical computation of the levels of the full potential alone is required, the limiting solutions being known. One then examines the performance of various dimensional possibilities, such as E_0+E_c, $(E_0^2+E_c^2)/(E_0+E_c)$, ... Finally, one has to look for a suitable combination of these terms that would reproduce the numerical data.

Fortunately, however, this elaborate procedure can be circumvented, since the problem at hand has an analytic solution.[5] This deceptively simple result corresponds to an elementary dimensional term $E=E_0+E_c$ that at once confirms the claims made on the basis of the dimensional considerations above.

To sum our findings, we have shown that dimensional consideration, together with the limiting constraints pro-

TABLE I. Comparison of the energy predicted by Eq. (5) $\equiv E_p$ and the energy computed numerically $\equiv E_n$ for the first six even levels of the x^6 anharmonic oscillator [$\varepsilon_6\equiv(8\mu^3/\lambda\hbar^6)^{1/4}E_6$, $\beta\equiv(2\lambda\hbar^2/\mu^3\omega^4)^{1/2}$, $a=b=\frac{1}{2}$].

Level No.	ε_6 (numerical)	$\frac{E_p}{E_n}$ $\beta=0.1$	$\beta=1$	$\beta=10$
0	1.144	1.017	0.977	0.994
2	9.06	1.019	0.985	0.998
4	21.70	1.0095	0.990	0.999
6	37.59	0.9993	0.993	0.9995
8	56.17	0.9928	0.994	0.9996
10	77.10	0.9890	0.995	0.9996

vide a very powerful approach for the investigation of the functional dependence of energy on coupling constants for a combination of power-law potentials. Furthermore, if the problem happens to have a simple combination of dimensional terms as the actual solution, then it can be traced quite easily. Finally, the very procedure of constructing dimensional terms ensures that the resulting energy cannot diverge.

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