Phase shift in the collision of two solitons propagating in a nonlinear transmission line

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An electric transmission line periodically 100-section loaded with variable capacitance diodes, i.e., a nonlinear lumped *LC* network, has been constructed to study nonlinear wave (soliton) propagation in the dispersive medium. Experimental results on the head-on collision of solitons with other solitons have been demonstrated. Besides the observation that solitons preserve their line shape and velocity during the collision, it has been observed for the first time that the solitons show a phase shift after the collision. The observed phase shift agrees with the theoretical expectation, which has been derived from an analysis of the nonlinear equation for a one-dimensional Toda lattice equivalent to the transmission line.

I. INTRODUCTION

Growing attention has been given to the propagation of nonlinear, solitary waves (solitons) in various branches of physics, e.g., in fluid dynamics, plasma physics, solid-state physics, and quantum electronics.[1] A soliton is known to exhibit the following unique properties, which are never observed for a linear wave.[2,3]

(i) A soliton travels in a nonlinear and dispersive medium at its own velocity depending on its amplitude, and a soliton of high amplitude travels faster than one of low amplitude.

(ii) Solitons pass through one another without losing their identities and integrities.

The second property regarding the interaction between solitons is a characteristic of the soliton; therefore this is frequently used to investigate whether the wave packet observed in a nonlinear medium is a solitary wave or not. According to this property, when two solitons collide with each other, they do not scatter but emerge from the collision having the same shapes and velocities with which they entered. This indicates that a soliton is not influenced by collision with another soliton. These properties have been experimentally observed for a soliton propagating in various nonlinear and dispersive media.

Toda and Wadati studied theoretically the collision of two solitons which obey the nonlinear Toda lattice equation, the Boussinesq equation, and the Korteweg–de Vries equation.[4,5] They showed that a solitary wave preserves its shape and velocity upon collision with another solitary wave, but that the phases of the two waves are changed. This is true not only for the head-on collision where two solitons run in the opposite directions, but also for the overtaking collision where two solitons run in the same direction and the faster soliton overtakes the slower one and outruns it. The appearance of the phase shift after collision has also been assured by computer experiments.[3,6,7]

So far, the preservation of the shape and velocity has been experimentally confirmed using a nonlinear transmission line.[3,7–10] No report, however, has been given on the observation of a phase shift, although the presence of a phase shift gives the only evidence that the soliton had experienced a collision with another soliton, since the soliton retains no traces of the collision in the shape and velocity. The present work was undertaken to confirm experimentally, using a nonlinear transmission line, whether the phase shift is present or not.

II. THE THEORY OF COLLISION OF TWO SOLITONS

A one-soliton solution, at the nth lattice point, of the nonlinear Toda lattice equation is described by[4]

$$f_n = 1 + A\exp(kn - \beta t), \quad A > 0,\ k > 0 .$$

When two solitons with k_1 and $k_2(<k_2)$ have a head-on collision with each other, the relative phase shift δ_1 for the k_1 soliton is given by

$$\delta_1 = \delta_1^+ - \delta_1^- = \log_{10}\{[\cosh(k_1-k_2)/4]/[\cosh(k_1+k_2)/4]\}^2 ,$$

where δ_1^- and δ_1^+ are the phase of the soliton wave before and after the collision, respectively. On the other hand, the relative phase shift δ_2 for the k_2 soliton results in

$$\delta_2 = -\delta_1 ,$$

indicating that the collision gives rise to the same amount of phase shift as the k_2 soliton.[4]

Since the coordinate of the k_i soliton is given by

$$n = \beta_i t/k_i - \delta_i^{(\pm)}/k_i ,$$

the change in the coordinate of the k_1 soliton, caused by the collision, is given by[11]

$$\Delta_1 = \delta_1/k_1 ,$$

whereas the change of the k_2 soliton is given by

$$\Delta_2 = \delta_2 / k_2 = -\delta_1 / k_2 \; .$$

Therefore the change in the coordinate is smaller for the soliton with the larger k. These results are true for the soliton propagating in our nonlinear transmission line, since the line is described by a differential equation which is equivalent to the Toda lattice equation.

III. EXPERIMENTAL RESULTS AND DISCUSSION

A lumped transmission line shown in Fig. 1, which is similar to that used by Kuusela *et al.*,[7] was constructed. The nonlinear LC circuit consists of 100 sections each of which has a nonlinear voltage-dependent capacitor, Toshiba varicap diode 1SV149, and a linear inductor of 22.0 μH with a resistance of 0.67Ω. The nonlinear capacitor satisfies the following relation in a limited voltage region:

$$C_d(V') = Q(V_0)/[F(V_0) + V' - V_0] \; , \qquad (1)$$

where V' is a reverse voltage and V_0 is a dc bias voltage. This relation gives the nonlinear wave equation which is equivalent to the Toda lattice equation.[8]

Our experiment was done at zero bias voltage (i.e., $V_0 = 0$) or at $V_0 = 2$ V. No appreciable difference was observed for the head-on collision of solitons in the two cases of different dc bias voltage. At zero bias voltage, the quantities that appeared in Eq. (1) are given by $C_d(V_0=0) = 833$ pF, $Q(V_0=0) = 1.0$ μC, and $F(V_0=0) = 1.2$ V in a region of $0 < V' < 3$ V. As a voltage source of solitons, rectangular pulses were fed from a pulse generator at one end of the transmission line. The wave form of voltage across a capacitor was observed by a digital oscilloscope and recorded by an X-Y recorder.

Figure 2 shows oscillogram voltage traces of two solitons, which approach each other from opposite directions, at various capacitor positions of 100 sections. In this case an input pulse is sent into the transmission line from each end. The two input pulses have the same amplitude. It is observed that the amplitude decreases as the soliton propagates in the line because of a dissipation effect in the inductors, each of which has a resistance of 0.67 Ω. In Fig. 2, the soliton shown at the left-hand side indicates that it appears earlier than the one at the right-hand side, i.e., the former means a soliton before the collision, whereas the latter means one after the collision. The two solitons collide with each other at the 50th section, which is just the middle of the line, producing a single waveform. The joint amplitude is greater than the sum of the amplitudes of the two solitons, in agreement with the theoretical result. In Fig. 3, we plot the space-time trajectories of these two solitons. From the figure it

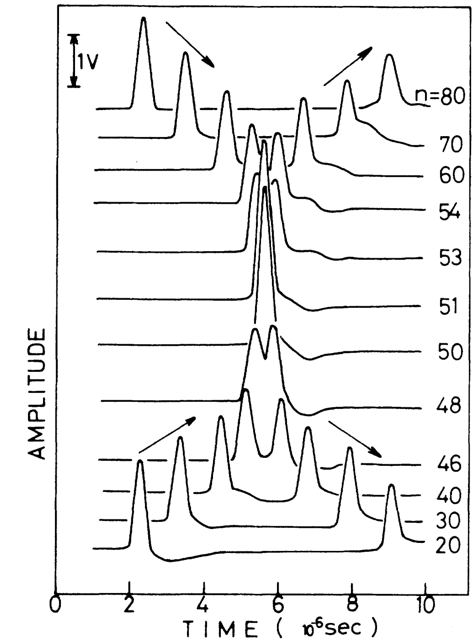

FIG. 2. Head-on collision of two solitons, measured at various capacitor positions under zero dc bias voltage ($V_0 = 0$), each of which was produced from an input rectangular pulse with amplitude of 3.0 V and frequency of 2.43 MHz (corresponding to the pulse width of 0.206 μsec). The origin (time $t = 0$) in the abscissa is taken to be the rising time of the input pulse. The same is true for Figs. 3–5.

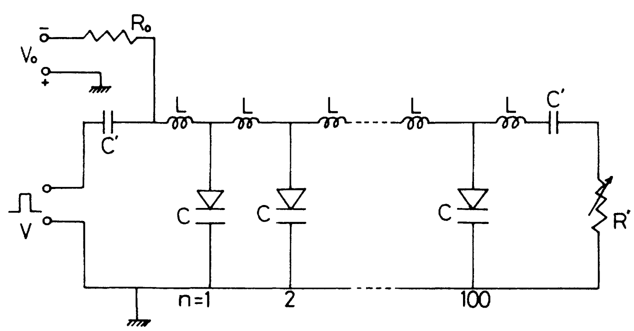

FIG. 1. Circuit diagram of nonlinear transmission line used in the experiment. V, input pulse; V_0, dc bias voltage, C, varicap; L, inductor of 22.0 μH; C', condenser of 0.01 μF; R_0, resistor of 1$k\Omega$; R', variable resistor up to 2k Ω; n, position of section.

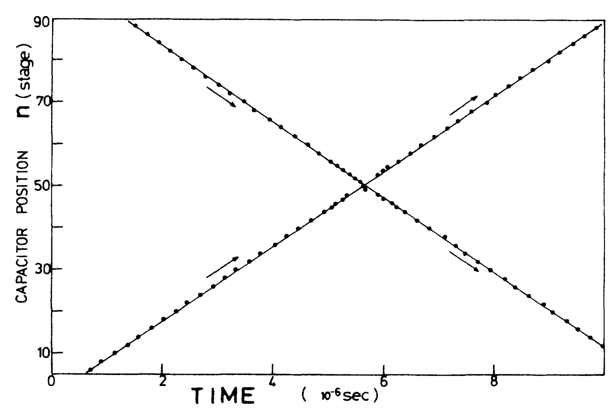

FIG. 3. Space-time trajectories of the peak positions of two soliton wave forms shown in Fig. 2.

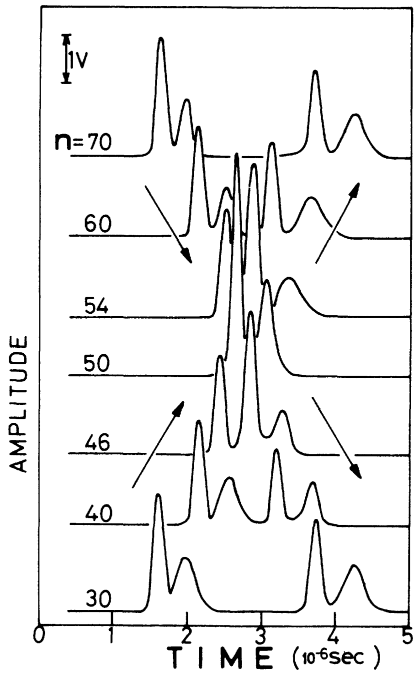

FIG. 4. Head-on collision of two soliton groups, measured at various capacitor position under zero dc bias voltage ($V_0 = 0$), each of which was produced from decomposition of an input rectangular pulse with an amplitude of 3.0 V and a frequency of 1.14 MHz (corresponding to the pulse width of 0.438 μsec) into two solitons.

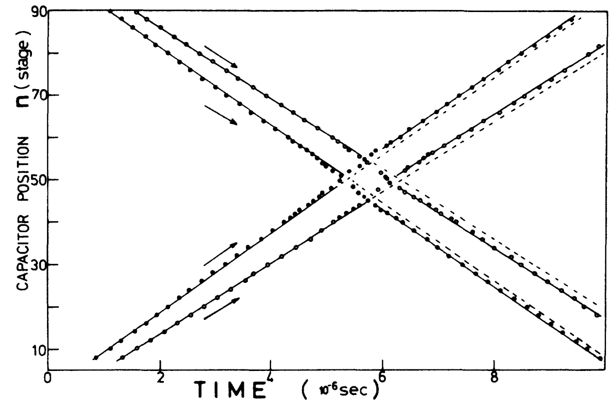

FIG. 5. Space-time trajectories of the peak positions of four soliton wave forms shown in Fig. 4. Closed and open circles indicate the faster and slower solitons, respectively.

is observed that the two solitons approach and collide. Each soliton then moves forward in its moving direction after the collision. No appreciable phase shift is observed for the two solitons after the collision.

To try to observe the phase shift due to collision, we decompose an input pulse into two solitons using a wider input-pulse width than that used in Fig. 2. It is well known that the input pulse is decomposed into a finite train of solitary waves; the number of the produced wave depends on both the pulse width and the pulse amplitude.[3,6,12,13] The wider the pulse becomes, the more numerous are the decomposed solitons. In our case, of the produced two solitons, the amplitude of the first soliton is about twice as large as that of the second soliton, and the former is created earlier than the latter. From the first property is regarding the soliton mentioned in Sec. I, we call the first soliton the faster soliton hereafter, whereas the other one we call the slower soliton. Figure 4 shows the head-on collision of two wave groups containing the faster and slower solitons in each group, where both the faster and slower solitons of one group are seen to collide with those of the other group, successively.

In Fig. 5 we plot the space-time trajectories of the four solitons shown in Fig. 4. It is observed that the velocity of a soliton is increased when it approaches another soliton, and immediately after, it collides with the latter one. Additionally, the soliton advances in phase by the double collisions with other solitons moving from opposite directions. This is true for both the faster and slower solitons. The amount of change in the coordinate for the slower soliton is found to be about two times as large as that for the faster soliton. The result is that the weighted sum of the phase shifts of individual solitons is zero, because the amplitude of the faster soliton is about two times as large as that of the slower one. This is consistent with the theoretical result, which is described in Sec. II. It is concluded that the phase shift of moving solitary waves by collision between solitons was observed for the collision between solitons with different amplitudes.

ACKNOWLEDGMENTS

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[1]See, for example, S. E. Trullinger, V. E. Zakharov, and V. L. Pokrovsky, *Solitons* (North-Holland, Amsterdam, 1986); G. L. Lamb, Jr., *Elements of Soliton Theory* (Wiley, New York, 1980).

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