Properties of squeezed number states and squeezed thermal states

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Much attention has been given in the past to two classes of squeezed states: the squeezed vacuum and coherent states. Here we study the effects of squeezing on number states and on thermal field states. The statistical properties of these various squeezed states are discussed using the second-order correlation functions. The quasiprobabilities of the Wigner, Q, and positive P representations are calculated and compared for the squeezed states. The Glauber P representation for the squeezed thermal state explicitly shows the limit of its applicability. The photon number distributions of the squeezed number and squeezed thermal states are extensively discussed and new interference effects in phase space are shown to lead to highly structured number distributions.

I. INTRODUCTION

There has been considerable interest in attempts to produce purely quantum-mechanical states of light such as the squeezed states, the sub-Poissonian states, and the photon number states.[1] The amplitude-squeezed states of the sub-Poissonian field have been observed by using a negative-amplitude feedback laser incorporating a photon nondemolition measurement of photon number.[2] The amplitude-squeezed states are described as having a "banana" shape in the quadrature operator phase space. These so-called banana states can ultimately tend to an annulus in phase space characterizing a number state.[3] Hong and Mandel[4] have successfully generated a localized one-photon state by use of an optical shutter, in which "twin" photons (the idler and signal photons) are produced by parametric down conversion, and the signal photon opens a photoelectric detection gate to the idler photon and produces a localized one-photon state.

The precisely defined photon number state can be used as an input field in a squeezing system, such as a parametric amplifier. For this reason, we calculate here the statistical properties of the squeezed number states. Historically, the squeezed number state was introduced by Yuen[5] but the properties of the squeezed number state have not to date been studied in any detail. The number state is determined by its photon number while the phase is completely random. By squeezing the photon number state the deterministic photon number is blurred and the quadrature uncertainty is squeezed in one direction. The properties of the state become phase dependent.

The thermal state can be expressed in terms of the Bose-Einstein weighted sum of the photon number states.[6] We can also consider a thermal input field to a squeezing device.[7-9] Yurke and co-workers[10] have developed a microwave Josephson junction parametric oscillator which has already generated substantial noise reduction in the thermal noise in a microwave cavity. One aim of such work is to develop nonclassical fields for interaction studies with Rydberg atoms, where the thermal noise in input fields is always large.

The plan of this paper is as follows. We review some well-known properties of the squeezed coherent state to compare with those of the squeezed number and the squeezed thermal states. We then calculate the quasiprobability functions of the squeezed states and the Glauber second-order correlation functions for the various squeezed states. The photon number distribution is extensively discussed. The pairwise oscillations resulting from the two-photon nature of the squeeze operator are explained in terms of phase-space interference.[11]

II. SUMMARY OF PROPERTIES OF THE SQUEEZED COHERENT STATE

The squeezed coherent state is defined as[12]

$$|\beta,r\rangle \equiv \hat{D}(\beta)\hat{S}(r)|0\rangle \tag{2.1}$$

where the squeeze operator[13] $\hat{S}(r)$ is given by

$$\hat{S}(r)\equiv\exp(\tfrac{1}{2}r\hat{a}^2-\tfrac{1}{2}r\hat{a}^{\dagger 2}) \tag{2.2}$$

and the Glauber displacement operator[14] $\hat{D}(\beta)$ by

$$D(\beta)\equiv\exp(\beta\hat{a}^{\dagger}-\beta^*\hat{a}) \ . \tag{2.3}$$

Here the squeeze parameter r has been assumed real for convenience. The squeezing operators provide a Bogoliubov transformation of the annihilation and creation operators as

$$\hat{S}^{\dagger}(r)\hat{a}\hat{S}(r)=\hat{a}\cosh r-\hat{a}^{\dagger}\sinh r \ , \tag{2.4a}$$

$$\hat{S}^{\dagger}(r)\hat{a}^{\dagger}\hat{S}(r)=\hat{a}^{\dagger}\cosh r-\hat{a}\sinh r \ . \tag{2.4b}$$

The displacement operator produces the operator transformations

$$\hat{D}^{\dagger}(\beta)\hat{a}\hat{D}(\beta)=\hat{a}+\beta \ , \tag{2.5a}$$

$$\hat{D}^{\dagger}(\beta)\hat{a}^{\dagger}\hat{D}(\beta)=\hat{a}^{\dagger}+\beta^* \ . \tag{2.5b}$$

The quadrature operators are defined by

$$\hat{X}_1=\hat{a}+\hat{a}^{\dagger} \ , \tag{2.6a}$$

$$\hat{X}_2=-i(\hat{a}-\hat{a}^{\dagger}) \ . \tag{2.6b}$$

For the squeezed coherent state, the variances $\langle(\Delta X_i)^2\rangle \equiv \langle \hat{X}_i^2\rangle - \langle \hat{X}_i\rangle^2$ are, from Eqs. (2.1), (2.4), and (2.5),

$$\langle(\Delta X_1)^2\rangle = \exp(-2r) , \tag{2.7a}$$

$$\langle(\Delta X_2)^2\rangle = \exp(2r) . \tag{2.7b}$$

The mean photon number for the squeezed coherent state is

$$\langle \hat{n}\rangle \equiv \langle \hat{a}^\dagger \hat{a}\rangle = |\beta|^2 + \sinh^2 r \tag{2.8}$$

and the photon number variance

$$\langle(\Delta n)^2\rangle = |\beta|^2[\exp(-2r)\cos^2\phi + \exp(2r)\sin^2\phi] + \tfrac{1}{2}\sinh^2(2r) \tag{2.9}$$

where $\beta = |\beta| e^{i\phi}$. It is a straightforward calculation to find the Glauber second-order correlation function[14]

$$g^{(2)} \equiv 1 + \frac{\langle(\Delta n)^2\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2} \tag{2.10}$$

using Eqs. (2.8) and (2.9),[15]

$$g^{(2)} = 1 - \frac{1}{\langle \hat{n}\rangle} + \frac{1}{\langle \hat{n}\rangle^2}[\,|\beta|^2(e^{-2r}\cos^2\phi + e^{2r}\sin^2\phi) + 2\sinh^2 r\cosh^2 r] . \tag{2.11}$$

The photon number distribution $P(l) \equiv \langle l|\hat{\rho}|l\rangle$ gives the probability of there being l photons in the field. For the squeezed coherent state, where the density operator is $\hat{\rho} \equiv |\beta, r\rangle\langle\beta, r|$, the photon distribution $P_{sc}(l)$ is, using Eqs. (2.4) and (2.5)[5]

$$P_{\rm SC}(l) = \frac{(\frac{1}{2}\tanh r)^l}{l!\cosh r}\exp[-|\beta|^2 - \tfrac{1}{2}(\beta^{*2}+\beta^2)\tanh r] \times \left| H_l\left[\frac{\beta+\beta^*\tanh r}{\sqrt{2\tanh r}}\right]\right|^2 \tag{2.12}$$

where H_l is the Hermite polynomial,[16] defined by

$$H_l(x) = \sum_m \frac{l!(-1)^m}{m!(l-2m)!}(2x)^{l-2m} . \tag{2.13}$$

Throughout the paper the factorials are defined only for non-negative integers, thus the upper and the lower limits of the summations, e.g., in Eq. (2.13), are determined by this limitation

$$0 \le m \le \frac{l}{2} .$$

III. SQUEEZED NUMBER STATES AND SQUEEZED THERMAL STATES

We define the squeezed number state density matrix by[11]

$$\hat{\rho}_n \equiv \hat{S}(r)|n\rangle\langle n|\hat{S}^\dagger(r) . \tag{3.1}$$

The squeezed thermal state is the Bose-Einstein weighted sum of the squeezed number states, with density matrix[6]

$$\hat{\rho}_{\rm th} \equiv (1+\bar{n})^{-1}\sum_{n=0}^{\infty}\left[\frac{\bar{n}}{1+\bar{n}}\right]^n \hat{\rho}_n \tag{3.2}$$

where $\bar{n}$ is the average photon number of the thermal input field. Alternatively, the density matrix $\hat{\rho}_{\rm th}$ can be expressed in a coherent basis as

$$\hat{\rho}_{\rm th} = \frac{1}{\pi\bar{n}}\int d^2\beta\exp(-\bar{n}^{-1}|\beta|^2)\hat{S}(r)|\beta\rangle\langle\beta|\hat{S}^\dagger(r) . \tag{3.3}$$

With the use of Eqs. (3.1), (2.4), and (2.5) for the squeezed number state, the variances of the quadrature operators are

$$\langle(\Delta X_1)^2\rangle = (2n+1)e^{-2r} , \tag{3.4a}$$

$$\langle(\Delta X_2)^2\rangle = (2n+1)e^{2r} . \tag{3.4b}$$

The variances for the squeezed thermal state are

$$\langle(\Delta X_1)^2\rangle = (2\bar{n}+1)e^{-2r} , \tag{3.5a}$$

$$\langle(\Delta X_2)^2\rangle = (2\bar{n}+1)e^{2r} . \tag{3.5b}$$

These results for the variances are in agreement with Fearn and Collett.[7]

The photon number variance $\langle(\Delta n)^2\rangle$ for the squeezed number state is

$$\langle(\Delta n)^2\rangle = \tfrac{1}{2}(n^2+n+1)\sinh^2 2r . \tag{3.6}$$

Note that for $n > 1$, the number uncertainty Δn grows linearly with n. This will determine many of the photon statistical properties we discuss later. When there is no squeezing, i.e., $r = 0$, the photon number variance is zero and the photon number is entirely deterministic for the photon number state. For $r \gg 1$, the photon number variance grows exponentially as the squeeze parameter increases. With the use of Eq. (3.6) in Eq. (2.10) we find for the second-order correlation function

$$g^{(2)} = 1 - \frac{\cosh(2r)}{\langle \hat{n}\rangle^2}n + \frac{\sinh^2 r}{\langle \hat{n}\rangle^2}[2n^2\cosh^2 r + 2n\cosh^2 r + \cosh(2r)] \tag{3.7}$$

where

$$\langle \hat{n}\rangle = n\cosh(2r) + \sinh^2 r . \tag{3.8}$$

When $r = 0$, we recover the second-order correlation function for the photon number state and when the squeezing is not significant, i.e., r small, the second-order correlation function can be less than unity, which indicates the light field has sub-Poissonian statistics. When $r \gg 1$, the second term of $g^{(2)}$ in Eq. (3.7) is negligible and

$$g^{(2)} \approx 1 + \frac{2(n^2+n+1)}{(2n+1)^2} . \tag{3.9}$$

For a large-photon number $g^{(2)}$ approaches 1.5. Similarly the second-order correlation function for the squeezed thermal state is obtained as

$$g^{(2)} = 2 + \frac{(2\bar{n}+1)^2}{\langle \hat{n}\rangle^2}\sinh^2 r\cosh^2 r \tag{3.10}$$

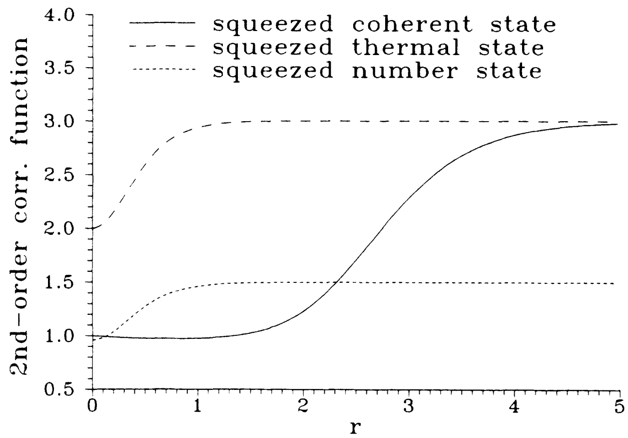

FIG. 1. Second-order correlation function for the squeezed coherent, the squeezed thermal, and the squeezed number state when the average photon number is 25, i.e., $n=25$, $\bar{n}=25$, and $\alpha=5$. r is the squeeze parameter.

where

$$\langle \hat{n} \rangle = \bar{n} \cosh 2r + \sinh^2 r \ , \tag{3.11}$$

$$\langle (\Delta n)^2 \rangle = \bar{n}^2 \cosh(4r) + \bar{n} \cosh(4r) + \tfrac{1}{2}\sinh^2(2r) \ . \tag{3.12}$$

For $r \neq 0$, the second term of $g^{(2)}$ in Eq. (3.10) is larger than zero so that the second-order correlation function for the squeezed thermal state is always greater than that for the thermal state. Squeezing the thermal field is responsible for larger fluctuations in the field intensity. For a limiting case, as $r \gg 1$, $g^{(2)} \to 3$. The factor 3 characteristic of super-Poissonicity was earlier discussed by Ekert and Rzążewski[17] and Janszky and Yushin[18] in the study of the statistical properties of the squeezed vacuum. Figure 1 compares the second-order correlation functions of the squeezed coherent, the squeezed number, and the squeezed thermal states. The squeezed coherent state shows sub-Poissonicity when r is small but becomes super-Poissonian as the squeezing increases. As the photon number state is squeezed, the photon number distribution is blurred and the photon statistics very rapidly becomes super-Poissonian, as can be seen in the dependence of $g^{(2)}$ on the squeeze parameter r in Fig. 1.

IV. QUASIPROBABILITY FUNCTIONS

A quasiprobability formulation of quantum mechanics has first been given by Wigner,[19] with a characteristic function, associated with the symmetrical order of the annihilation and creation operators, defined by[20]

$$C_W(\eta) \equiv \mathrm{Tr}[\hat{\rho} \exp(\eta \hat{a}^\dagger - \eta^* \hat{a})] = \mathrm{Tr}[\hat{\rho} \hat{D}(\eta)] \ . \tag{4.1}$$

The Wigner function is defined as the Fourier transform of the characteristic function $C_W(\eta)$

$$W(\alpha) \equiv \frac{1}{\pi^2} \int d^2\eta \exp(\alpha \eta^* - \alpha^* \eta) C_W(\eta) \ . \tag{4.2}$$

In order not to confuse the parameter α of the Wigner function with that of the displacement operator we have employed the notations β for the displacement parameter and α for the Wigner function argument. Using Eqs. (2.4) and (2.5) we obtain the characteristic function of the squeezed coherent state $|\beta, r\rangle \equiv \hat{D}(\beta)\hat{S}(r)|0\rangle$:[15]

$$C_W(\eta) = \exp(2i\eta_x \beta_x + 2i\eta_y \beta_y - \tfrac{1}{2}\eta_x^2 e^{-2r} - \tfrac{1}{2}\eta_y^2 e^{2r}) \tag{4.3}$$

where $\eta = \eta_x + i\eta_y$ and $\beta = \beta_x + i\beta_y$. Substituting Eq. (4.3) into the definition of the Wigner function, Eq. (4.2),

$$W_{\mathrm{SC}}(\alpha) = \frac{2}{\pi} \exp[-2e^{2r}(\alpha_x - \beta_x)^2 - 2e^{-2r}(\alpha_y - \beta_y)^2] \tag{4.4}$$

where α_x and α_y are the real and the imaginary parts of α, respectively. This is a Gaussian function with the maximum W_{SC} value at

$$(\alpha_x, \alpha_y) = (\beta_x, \beta_y) \ . \tag{4.5}$$

With use of the Taylor expansion of the Glauber displacement operator we find the transformation of the displacement operator by squeezing generates a similar operator $\hat{D}$, but with a different coefficient:

$$\hat{S}^\dagger(r)\hat{D}(\eta)\hat{S}(r) = \hat{D}(\xi) \tag{4.6}$$

with

$$\xi(\eta) = \eta \cosh r + \eta^* \sinh r \ . \tag{4.7}$$

If any two noncommuting operators $\hat{O}_1$ and $\hat{O}_2$ satisfy the conditions

$$[\hat{O}_1, [\hat{O}_1, \hat{O}_2]] = [\hat{O}_2, [\hat{O}_1, O_2]] = 0 \ , \tag{4.8}$$

then with the help of the Baker-Hausdorf theorem[21]

$$\exp(\hat{O}_1 + \hat{O}_2) = \exp(\hat{O}_1)\exp(\hat{O}_2)\exp(-\tfrac{1}{2}[\hat{O}_1, \hat{O}_2]) \tag{4.9}$$

and using Eqs. (3.1), (4.6), and (4.9) in Eq. (4.1) we find for the squeezed number state the characteristic function

$$C_W(\eta) = \exp(-\tfrac{1}{2}|\xi|^2)\langle n | e^{\xi \hat{a}^\dagger} e^{-\xi^* \hat{a}} | n \rangle \ . \tag{4.10}$$

We obtain the Wigner function for the squeezed number state from the Fourier transform of the characteristic function (4.10)

$$W_{\mathrm{SN}}(\alpha) = \frac{2}{\pi} \exp[\tfrac{1}{2}(\alpha - \alpha^*)^2 e^{-2r} - \tfrac{1}{2}(\alpha + \alpha^*)^2 e^{2r}] \times (-1)^n \mathcal{L}_n[(\alpha + \alpha^*)^2 e^{2r} - (\alpha - \alpha^*)^2 e^{-2r}] \tag{4.11}$$

where $\mathcal{L}_n$ is the Laguerre polynomial

$$\mathcal{L}_n(x) \equiv \sum_{m=0}^{n} (-1)^m \binom{n}{m} x^m / m! \ . \tag{4.12}$$

Similarly, using the definitions of the squeezed thermal state (3.3) and the characteristic function (4.1) and the relation (4.6), the Wigner function for the squeezed thermal state is obtained as

$$W_{\mathrm{ST}}(\alpha)=\frac{2/\pi}{1+2\bar{n}}\exp[-2(\alpha_y^2e^{-2r}+\alpha_x^2e^{2r})/(1+2\bar{n})]\ . \tag{4.13}$$

This is comparable with the Wigner function (4.4) for the squeezed coherent state. The squeezed thermal state Wigner function is also a Gaussian function with the maximum at $\alpha=(0,0)$ while the maximum value of the Gaussian squeezed coherent Wigner function is displaced by β. Both the squeezed thermal and the squeezed coherent Wigner functions are seen to be stretched by the action of the squeezing. As the average photon number $\bar{n}$ of the initial thermal state gets larger the Wigner function $W_{\mathrm{ST}}(\alpha)$ of the squeezed thermal state is more widely stretched so that $W_{\mathrm{ST}}(\alpha)$ is more slowly varying than $W_{\mathrm{SC}}(\alpha)$ with phase parameters α_x^2, α_y^2. The Wigner functions for the squeezed number, the squeezed thermal, and the squeezed coherent states are plotted in Fig. 2.

The Q representation is another quasiprobability formulation and is defined as the Fourier transformation of the antinormal-ordered characteristic function.[20] Alternatively, the Q representation can be defined as

$$\pi Q(\alpha)\equiv\langle\alpha|\hat{\rho}|\alpha\rangle\ . \tag{4.14}$$

From this definition we see that the Q representation is always non-negative. Using the definition of the density matrix of the squeezed coherent state $|\beta,r>$ we obtain the Q representation $Q_{\mathrm{SC}}(\alpha)$ for the squeezed coherent state

$$Q_{\mathrm{SC}}(\alpha)=\frac{1}{\pi}|\langle\alpha-\beta|\hat{S}(r)|0\rangle|^2\ . \tag{4.15}$$

Considering the definition of the squeeze operator in Eq. (2.2) it is not obvious how to solve for $\hat{S}(r)|0\rangle$. We now factorize the squeeze operator into a product of exponentials following Schumaker and Caves:[22]

$$\hat{S}(r)=\frac{1}{\sqrt{\cosh r}}\exp[-\tfrac{1}{2}(\tanh r)\hat{a}^{\dagger 2}](\cosh r)^{-\hat{a}^\dagger\hat{a}}\times\exp[\tfrac{1}{2}(\tanh r)\hat{a}^2]\ . \tag{4.16}$$

Substituting Eq. (4.16) into Eq. (4.15) we find immediately that

$$Q_{\mathrm{SC}}(\alpha)=\frac{1}{\pi\cosh r}\exp[-2(\alpha_y-\beta_y)^2/(1+e^{2r})-2(\alpha_x-\beta_x)^2/(1+e^{-2r})]\ . \tag{4.17}$$

Similarly the factorization of the squeeze operator (4.16) enables us to find the Q representation for the squeezed number state

$$Q_{\mathrm{SN}}(\alpha)=\frac{\exp(-|\alpha|^2)}{\pi\cosh r}n!\exp[-\tfrac{1}{2}\tanh r(\alpha^2+\alpha^{*2})]\times\left|\sum_{k=0}\left[\frac{\tanh r}{2}\right]^k\frac{(\alpha^*)^{n-2k}}{(n-2k)!k!}\times\left[\frac{1}{\cosh r}\right]^{n-2k}\right|^2\ . \tag{4.18}$$

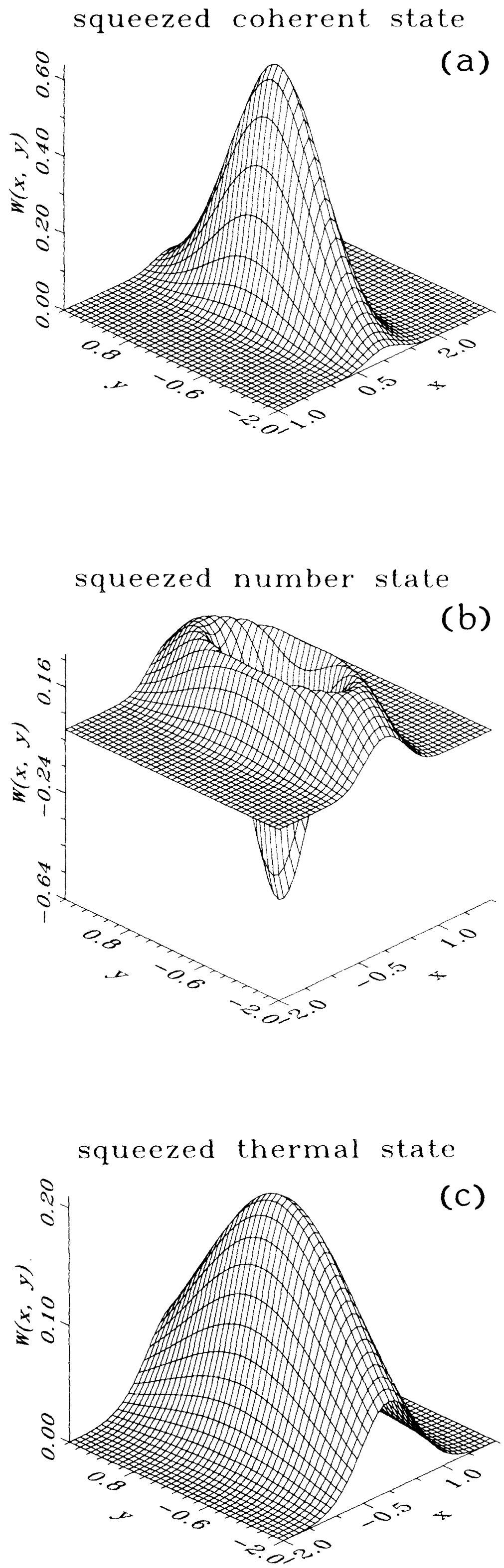

FIG. 2. Wigner function $W(x,y)$ for (a) squeezed coherent state when $\alpha=1$, (b) squeezed number state when $n=1$, and (c) squeezed thermal state when $\bar{n}=1$. The squeeze parameter $r=0.5$. Here $x=\mathrm{Re}(\alpha)$ and $y=\mathrm{Im}(\alpha)$.

As a simple limit we find the Q representation for the squeezed one-photon number state $\hat{S}(r)|1\rangle$ as

$$Q_{SN}(\alpha)=\frac{1}{\pi}\exp(-|\alpha|^2)\times\exp[-\tfrac{1}{2}\tanh r(\alpha^2+\alpha^{*2})]\frac{|\alpha|^2}{(\cosh r)^3} . \quad (4.19)$$

As Fig. 3(b) shows the cylindrical shape of the Q representation for the photon number state is squeezed as the squeeze parameter r increases. The squeezed thermal state is described by the integration of the squeezed coherent states with their appropriate weights given by Eq. (3.3). Thus we find after straightforward algebra using Eqs. (3.3) and (4.17) the Q representation $Q_{ST}(\alpha)$ for the squeezed thermal state

$$Q_{ST}(\alpha)=\frac{(\pi\bar{n}\cosh r)^{-1}}{[(1+1/\bar{n})^2-\tanh^2 r]^{1/2}}\exp\left[-\tfrac{1}{2}\tanh r(\alpha^2+\alpha^{*2})-|\alpha|^2+\frac{1/\cosh^2 r}{(1+1/\bar{n})^2-\tanh^2 r}[(1+1/\bar{n})|\alpha|^2-\tfrac{1}{2}(\alpha^2+\alpha^{*2})\tanh r]\right] . \quad (4.20)$$

The Q representations for the squeezed coherent and the squeezed thermal states are Gaussian as are their Wigner functions. The Q representations are plotted in Fig. 3. The squeezed number state has two peaks with a trough at (0,0) while the squeezed thermal and the squeezed coherent states are Gaussian. The peak of the squeezed coherent state has been displaced by β as expected. The Q representation of the squeezed thermal state is largely stretched along the y axis (α_y axis) due to the small coefficient of α_y^2.

Glauber[20] and Sudarshan,[23] independently, have introduced the diagonal P representation for the probability density. The P representation is defined as the Fourier transform of the normal-ordered characteristic function. Using the definition (3.3) of the squeezed thermal state we obtain the characteristic function $C_p(\eta)$ for the squeezed thermal state as

$$C_p(\eta)=\exp[-(e^r\sinh r+\bar{n}e^{2r})\eta_x^2-(\bar{n}e^{-2r}-e^{-r}\sinh r)\eta_y^2] . \quad (4.21)$$

Considering the squeezing in one direction, namely $r>0$, the coefficient of η_x^2 is always negative but that of η_y^2 can become positive, which makes the characteristic function diverge. Thus only when

$$\bar{n}e^{-2r}-e^{-r}\sinh r>0 \quad (4.22)$$

can we Fourier transform $C_p(\eta)$ and find the P representation $P_{ST}(\alpha)$ for the squeezed thermal state as follows:

$$P_{ST}(\alpha)=\frac{1/\pi}{[(\sinh r+\bar{n}e^r)(\bar{n}e^{-r}-\sinh r)]^{1/2}}\times\exp[-\alpha_y^2e^{-r}/(\sinh r+\bar{n}e^r)-\alpha_x^2e^r/(-\sinh r+\bar{n}e^{-r})] . \quad (4.23)$$

The diagonal P representation is well defined for a classical state, but either it is negative or does not exist for states exhibiting nonclassical behavior.[24] The condition (4.22) can be written as

$$(2\bar{n}+1)e^{-2r}>1 . \quad (4.24)$$

The left-hand side of Eq. (4.24) is the variance of the quadrature operator for the squeezed thermal state [see Eqs. (3.5)] and the factor 1 of the right-hand side is the quadrature variance of the vacuum. If the quadrature variances are larger than the minimum uncertainty limit, it is possible to describe the squeezed thermal state in terms of a well-behaved P representation which is positive everywhere. The squeezed thermal state can show either a classical or a quantum behavior depending on the sizes of $\bar{n}$ and r.

As a possible way to avoid the limit of the applicability of the P representation, Drummond and Gardiner[25] have suggested the so-called positive P representation. The positive P representation is defined over a double-phase space. The positive P representation is always positive as is the Q representation. The density operator can be expressed in terms of the positive P representation and the projection operators $\Lambda(\alpha,\gamma)$ as[26]

$$\hat{\rho}=\int d^2\alpha\, d^2\gamma\, P(\alpha,\gamma)\Lambda(\alpha,\gamma) \quad (4.25)$$

where $P(\alpha,\gamma)$ is the positive P representation and the projection operator $\Lambda(\alpha,\gamma)$ is defined as

$$\Lambda(\alpha,\gamma)=\frac{|\alpha\rangle\langle\gamma^*|}{\langle\gamma^*|\alpha\rangle} . \quad (4.26)$$

After a little algebra we find that a positive P representation exists for any quantum density operator $\hat{\rho}$ in the form

$$P(\alpha,\gamma)=\frac{1}{4\pi^2}\exp\left[-\frac{|\alpha-\gamma^*|^2}{4}\right]\left\langle\frac{\alpha+\gamma^*}{2}\right|\hat{\rho}\left|\frac{\alpha+\gamma^*}{2}\right\rangle . \quad (4.27)$$

The positive P representation (4.27) can be rearranged with the help of the Q representation (4.14):

$$P(\alpha,\gamma)=\frac{1}{4\pi}\exp\left[-\frac{|\alpha-\gamma^*|^2}{4}\right]Q\left[\frac{\alpha+\gamma^*}{2}\right] . \quad (4.28)$$

For the plane $\alpha=\gamma^*$ in the double-phase space of α and γ, the positive-P representation is proportional to Q representation,

$$P(\alpha)=\frac{1}{4\pi}Q(\alpha) . \quad (4.29)$$

V. PHOTON NUMBER DISTRIBUTION

The photon number distribution, defined in Sec. II, for the squeezed number state is obtained using the factorization of the squeeze operator (4.16):

$$P_{\rm SN}(l)=|\langle l|\hat{S}(r)|n\rangle|^2=\begin{cases}\dfrac{n!l!}{(\cosh r)^{2l+1}}(\tfrac{1}{2}\tanh r)^{n-l}S(r,l,n) & \text{when } |l-n| \text{ is even}\\ 0 \quad \text{when } |l-n| \text{ is odd}\end{cases} \quad (5.1)$$

where

$$S(r,l,n)=\left|\sum_m \frac{(-1)^m(2^{-1}\sinh r)^{2m}}{m!(l-2m)![m+(n-l)/2]!}\right|^2 . \quad (5.2)$$

The factorials in Eq. (5.1) are valid for non-negative integers, so that $\frac{1}{2}(l-n)\le m\le\frac{1}{2}l$. The photon number distribution can, alternatively, be written as

$$P_{\rm SN}(l)=\frac{n!l!}{(\cosh r)^{2l+1}}(\tfrac{1}{2}\tanh r)^{n-l}\times S(r,l,n)\cos^2\frac{(n-l)\pi}{2} . \quad (5.3)$$

The cosine term is responsible for the vanishing value of $P_{\rm SN}(l)$ when $|n-l|$ is odd. When $r=0$, we recover the photon number state result $P_{\rm SN}(l)=\delta_{ln}$. Another simple case of $P_{\rm SN}(l)$ is found for $n=1$; the photon number distribution for the squeezed one-photon number state is

$$P_{\rm SN}(l)=\frac{l!(-\frac{1}{2}\tanh r)^{l-1}}{\cosh^3 r\left[\left[\frac{l}{2}-\frac{1}{2}\right]!\right]^2}\cos^2\frac{(1-l)\pi}{2} . \quad (5.4)$$

For the squeezed thermal state we find, using Eq. (3.2), the photon distribution $P_{\rm ST}(l)$ to be given by

$$P_{\rm ST}(l)=(1+\bar{n})^{-1}\sum_n P_{\rm SN}(l,n)\left[\frac{\bar{n}}{1+\bar{n}}\right]^n . \quad (5.5)$$

The photon number distribution $P_{\rm SV}(l)$ for the squeezed vacuum is obtained by setting $n=0$ in $P_{\rm SN}(l)$:

$$P_{\rm SV}(l)=\frac{(-\frac{1}{2}\tanh r)^l l!}{(\frac{1}{2}l)!^2\cosh r}\cos^2\left[\frac{l}{2}\pi\right] . \quad (5.6)$$

When l is odd $P_{\rm SV}(l)$ is zero, otherwise $P_{\rm SV}(l)$ may be nonzero. These so-called pairwise oscillations are the result of the quadratic, or two-photon nature of the squeeze operator $\hat{S}(r)$.[5] We have plotted the photon distributions $P_{\rm SN}(l)$, $P_{\rm ST}(l)$ as functions of photon number l in Fig. 4. Both the distributions show pairwise oscillations. A more noticeable feature is the large-scale macroscopic oscillations of the squeezed number state photon distribution, analogous to those found by Schleich and Wheeler[27] for the squeezed coherent state.

Schleich and Wheeler[27] have interpreted the large-scale oscillations of the photon number distribution for the squeezed coherent state using the Bohr-Sommerfeld phase-space picture. In phase space, when the coherent state is appropriately squeezed the photon number state annular overlaps twice with the squeezed coherent state ellipse. According to basic quantum mechanics, the photon number distribution, which is a probability function, is not merely the sum of the areas of overlap but the sum of the probability amplitudes, fixed by the areas of overlap, with appropriate phases. It is straightforward to show that the pairwise oscillation discussed by Yuen[5] for the squeezed vacuum photon number distribution has a similar interpretation. The phase-space contour for the number state is dominated for these purposes by an annulus representing the leading effect of the Laguerre polynomial representation of the appropriate Wigner function whose radius is determined by the photon number l, whereas the Wigner contour for the squeezed vacuum state is an ellipse centered on, and maximized at the origin. There can be two areas of overlap or none between the number state and the squeezed vacuum in the phase space. The sizes of the two areas are the same due to their property of symmetry. If the area of each overlap in the phase space of X_1 and X_2 is A_l, and the phase is φ_l, Schleich and Wheeler[27] show that

$$P_{\rm SV}(l)\approx|A_l^{1/2}\exp(i\varphi_l)+A_l^{1/2}\exp(-i\varphi_l)|^2 =4A_l|\cos\varphi_l|^2 , \quad (5.7)$$

where

$$\varphi_l=\int_0^{\zeta_l}dX_1(2l+1-X_1^2)^{1/2}-\frac{\pi}{4} \quad (5.8)$$

with $\zeta_l=(2l+1)^{1/2}$. [Although the squeezed vacuum is beyond the original coherent state large $\beta\gg l\gg 1$ condition, under which Schleich and Wheeler obtained the approximation (5.7), the approximation still works well for the trend of the photon number distribution of the squeezed vacuum.] Calculating Eq. (5.8) we find that the phase $\varphi_l=l\pi/2$ for the squeezed vacuum. This agrees

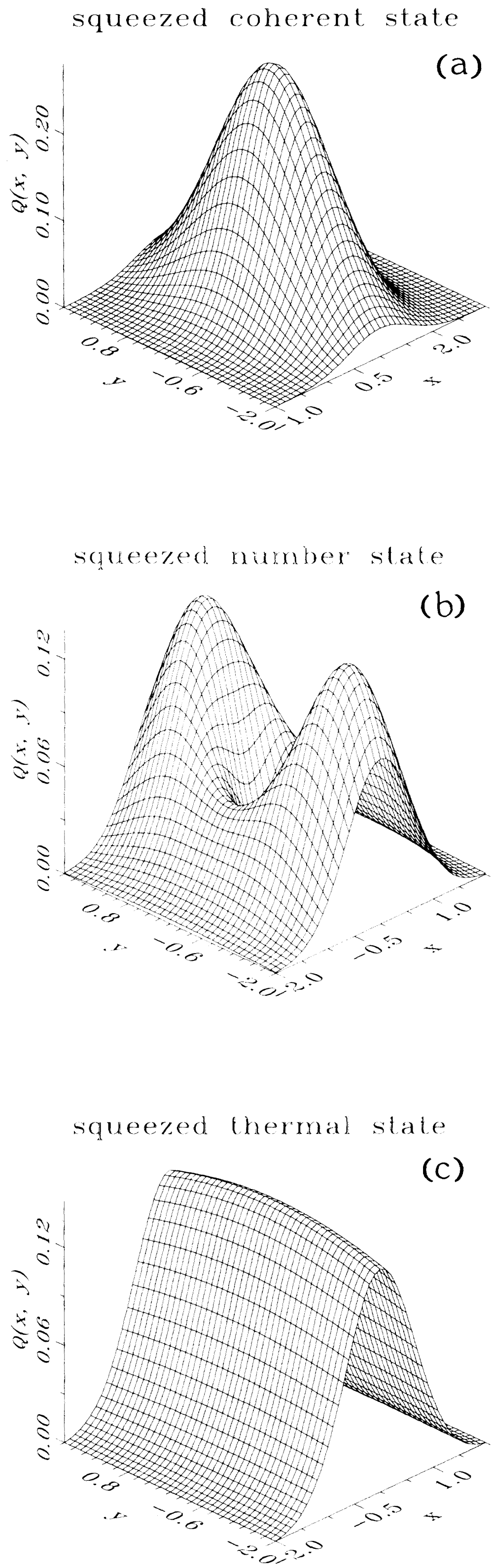

FIG. 3. Q representation $Q(x,y)$ for (a) squeezed coherent state when $\alpha=1$, (b) squeezed number state when $n=1$, and (c) squeezed thermal state when $\bar{n}=1$. The squeeze parameter $r=0.5$. Here $x=\mathrm{Re}(\alpha)$ and $y=\mathrm{Im}(\alpha)$.

with our earlier result (5.6) and shows the pairwise oscillations.

For the squeezed number state, the photon number distribution $P_{\mathrm{SN}}(l)$ in Eq. (5.3) is not easy to analyze so that we suggest some approximations leading to a tractable form of the equation. If we squeeze a large-photon number state, i.e., $n \gg 1$, and consider $P_{\mathrm{SN}}(l)$ for $l<n$,

$$\tfrac{1}{2}(n-l)+m \approx \frac{n}{2} \gg 0 \; . \tag{5.9}$$

For $l<n$, the summation range in Eq. (5.2),

$$\tfrac{1}{2}(l-n) \le m \le \frac{l}{2} \rightarrow 0 \le m \le \frac{l}{2} \; , \tag{5.10}$$

where it has been taken into account that m is a nonnegative integer. Considering the condition (5.9) we approximate the sum (5.2) as

$$S(r,l,n) \approx \frac{1}{\left[\dfrac{n}{2}!\right]^2} \left| \sum_m \frac{(-1)^m}{m!(l-2m)!}(2\xi)^{-2m} \right|^2 \tag{5.11}$$

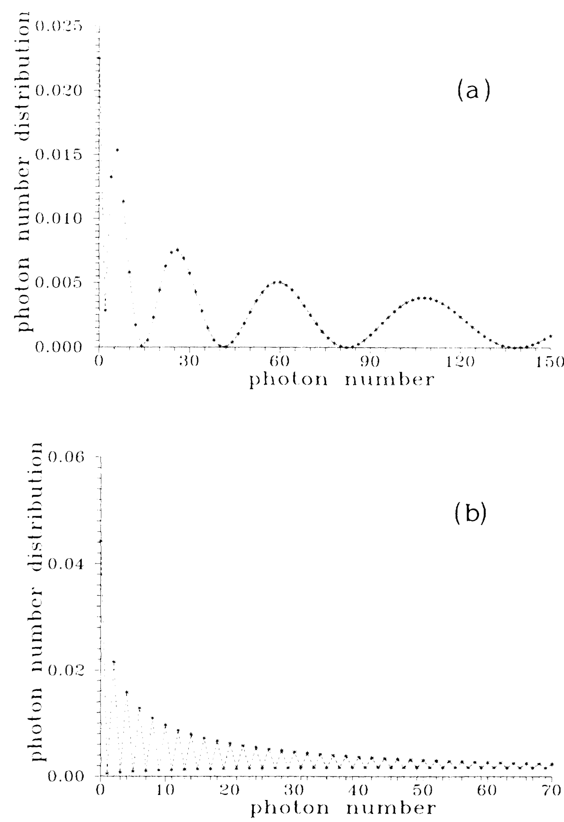

FIG. 4. (a) Photon number distribution $P_{\mathrm{SN}}(l)$ of the squeezed number state when the squeeze parameter $r=2$ and the initial photon number $n=20$. This photon number distribution is valid only for even integers, i.e., $l=0,2,4,\ldots$. (b) Photon number distribution $P_{\mathrm{ST}}(l)$ of the squeezed thermal state when $r=3$ and the initial average photon number $\bar{n}=2$.

where

$$\xi = \frac{1}{\sinh r} . \tag{5.12}$$

In terms of the Hermite polynomial (2.13), Eq. (5.11) is simply written as

$$S(r,l,n) = \frac{H_l^2(\xi)}{\left[\frac{n}{2}!\right]^2 l!^2 (2\xi)^{2l}} . \tag{5.13}$$

Substituting Eq. (5.13) into Eq. (5.3) we find an approximation of $P_{SN}(l)$ as

$$P_{SN}(l) = \frac{n!\left[\frac{n}{2}!\right]^{-2}}{l!\cosh r}\left[\tfrac{1}{2}\tanh r\right]^{n+l} \times \cosh^2\frac{(n-l)\pi}{2} H_l^2(\xi) . \tag{5.14}$$

Schleich and Wheeler[28] have found an improved asymptotic formula for the Hermite polynomial, for $l \gg 1$,

$$H_l(\xi) = (4\pi)^{1/4}(2^l l!)^{1/2}\left[\frac{-t_l}{\xi^2 - \zeta_l^2}\right]^{1/4} \times A_i(-t_l)\exp(\xi^2/2) \tag{5.15}$$

where $A_i(-t_l)$ is an Airy function and t_l an integral function of ξ, defined as

$$t_l \equiv \left[\tfrac{3}{2}\int_\xi^{\zeta_l} dX_1 p_l(X_1)\right]^{2/3} \tag{5.16}$$

and we have assumed $(\sinh r)^{-1} = \xi \le \zeta_l$. The parameter, which has been introduced in Eq. (5.15), is defined as

$$p_l(X_1) \equiv (\zeta_l^2 - X_1^2)^{1/2} . \tag{5.17}$$

If l is appropriately larger than ξ, the Airy function simplifies to[16]

$$A_i(-t_l) = \pi^{-1/2} t_l^{-1/4} \cos\left[\frac{2}{3}t_l^{3/2} - \frac{\pi}{4}\right] . \tag{5.18}$$

Using Eqs. (5.15) and (5.18), the photon number distribution becomes

$$P_{SN}(l) = \frac{2}{\sqrt{\pi}} \frac{2^{-n} n!}{\left[\frac{n}{2}!\right]^2} \frac{(\tanh r)^{n+l}}{\cosh r}\left[\frac{1}{\zeta_l^2 - \xi^2}\right]^{1/2} \times \exp(\xi^2)\cos^2\frac{(n-l)\pi}{2}\cos^2\phi_l \tag{5.19}$$

where

$$\phi_l \equiv \int_\xi^{\zeta_l} dX_1(\zeta^2 - X_1^2)^{1/2} - \frac{\pi}{4} . \tag{5.20}$$

This approximation has been derived when $n > l \gg 1$ and the squeeze parameter r is appropriately large. From Eq. (5.19) we see that there are two phases, $(n-l)\pi/2$ and ϕ_l, in the photon number distribution for the squeezed number state. The pairwise oscillations are determined by the former and the large-scale oscillations are fixed by the latter. The size of the phase ϕ_l depends on the squeeze parameter r. As $r \to \infty$, the integration range of ϕ_l is between 0 and ζ_l, which is the same for the photon number distribution of the squeezed vacuum in Eq. (5.8).

As in Fig. 5, there are zero, two, or four areas of overlap between the annular phase-space contour of the number state and the compressed annulus of the squeezed number state phase-space contour. The areas are the same but the phases are different. Extending the work of Schleich and Wheeler,[27] where the phase is determined by the enclosed area to the right of the intersections, we may say that the two overlaps on the right have the phase ϕ_l while the other two on the left have the different phase ϕ_l'. The photon number distribution is governed by the areas A_l and phases ϕ_l, ϕ_l' so that

$$P_{SN}(l) = 4A_l|\cos\phi_l + \cos\phi_l'|^2 . \tag{5.21}$$

Assuming $\phi_l' = (n-l)\pi - \phi_l$ we obtain

$$|\cos\phi_l + \cos\phi_l'|^2 = \cos^2\phi_l[1 + \cos(l-n)\pi]^2 = \cos^2\phi_l\cos^2\frac{(l-n)\pi}{2} . \tag{5.22}$$

Substituting Eq. (5.22) into Eq. (5.21) we find that $P_{SN}(l)$ has pairwise oscillations and large-scale macroscopic oscillations. This agrees with the argument of the approximation discussed in Eq. (5.19).

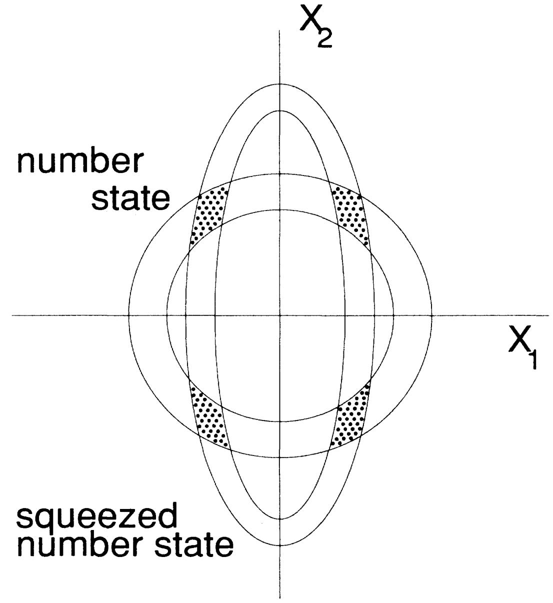

FIG. 5. Phase-space representation of the mean values and uncertainty contours for the squeezed number state together with the number state. The amplitude of the photon distribution is proportional to the dotted area.

The phase ϕ_l in Eq. (5.20) can be written as

$$\phi_l = \frac{l}{2}\pi - \alpha_l \quad (5.23)$$

where

$$\alpha_l \approx \frac{\zeta_l}{\sinh r} . \quad (5.24)$$

In Eq. (5.24) we have assumed $r \gg 1$. Using Stirling's formula[29] and $r \gg 1$, Eq. (5.19) becomes

$$P_{SN}(l) \approx \frac{2/(\zeta_l \cosh r)}{\sqrt{\pi(2l+1)}} \cos^2\frac{(n-l)\pi}{2} \cos^2\phi_l = \frac{2/\sqrt{\pi}}{(2l+1)\cosh r} \cos^2\frac{(n-l)\pi}{2} \times \left[\cos\frac{l\pi}{2}\cos\alpha_l - \sin\frac{l\pi}{2}\sin\alpha_l\right]^2 . \quad (5.25)$$

We have plotted $P_{SN}(l)$ for $n=20$ and $r=2$ in Fig. 4(a). When n and l are even,

$$P_{SN}(l) \propto \frac{1}{l}\cos^2\frac{\sqrt{2l+1}}{\sinh r} . \quad (5.26)$$

The envelope of oscillations is proportional to $1/l$ and as l grows the oscillations slow down. This is well illustrated in Fig. 4(a). On the other hand when n and l are odd,

$$P_{SN}(l) \propto \frac{1}{l}\sin^2\frac{\sqrt{2l+1}}{\sinh r} . \quad (5.27)$$

Equations (5.26) and (5.27) have been obtained for $n > l \gg 1$ and $r \gg 1$. When $r=5$ we compare the photon number distribution of $\hat{S}(r)|100\rangle$ with $\hat{S}(r)|99\rangle$ in Table I. For $\hat{S}(r)|100\rangle$, we see that $P_{SN}(l)$ decreases as l increases for the squeezed even-photon number state. $P_{SN}(l)$ for $\hat{S}(r)|99\rangle$ shows that $P_{SN}(l)$ increases as l increases for the squeezed odd-photon number state. As Eqs. (5.26) and (5.27) show $P_{SN}(l)$ is proportional to $\cos^2\alpha_l$ for the squeezed even-photon number state while $P_{SN}(l)$ for the squeezed odd-photon number state is proportional to $\sin^2\alpha_l$. For $r=5$ and relatively small l, $\sinh r \gg 1$ and α_l is small, thus $\cos^2\alpha_l \gg \sin^2\alpha_l$. Table I shows $P_{SN}(l)$ for the range $0 \le l \le 20$. For large l values, $P_{SN}(l)$ will oscillate with large frequencies.

Figure 6 shows the overlaps between squeezed number states and the photon number states in phase space. The contributions from the overlaps interfere and determine the value for the photon number distribution as in the Young two-slit experiment.[6] While Young's experiment shows the interference between the light fields from the two slits, the photon number distribution $P_{SN}(l)$ is similar to the four-slit interference because all four overlaps contribute to the photon number distribution. When the squeezed even-photon number state overlaps with the even-photon number state, the contributions from the overlaps interfere and construct a nonzero value photon number distribution $P_{SN}(l)$ with the phase $\phi \propto l_1\pi + \alpha_l$ where the integer $l_1 = l/2$. For the overlaps between the squeezed odd-photon number state and the odd-photon number state, the interference results in the phase $\phi \propto (l/2)\pi + \alpha_l$, where l is odd. When the squeezed odd-photon number or the squeezed even-photon number state, respectively, overlaps with the even-photon or the odd-photon number state the interference from the overlaps is destructive and gives $P_{SN}(l)=0$.

TABLE I. Photon number distributions for the squeezed number states. For $S(5)|100\rangle$, $P_{SN}(l)=0$ when l is odd. For $\hat{S}(5)|99\rangle$, $P_{SN}(l)=0$ when l is even.

$S(5)\|100\rangle$		$S(5)\|99\rangle$	
l	$P_{SN}(l)$	l	$P_{SN}(l)$
0	0.001 06	1	0.000 019
2	0.000 51	3	0.000 029
4	0.000 37	5	0.000 035
6	0.000 30	7	0.000 041
8	0.000 25	9	0.000 045
10	0.000 22	11	0.000 049
12	0.000 19	13	0.000 053
14	0.000 17	15	0.000 056
16	0.000 15	17	0.000 058
18	0.000 14	19	0.000 060

The squeezed thermal state is the Bose-Einstein weighted sum of the squeezed number states. When the input average photon number is small, i.e., $\bar{n} \le 1$, the only

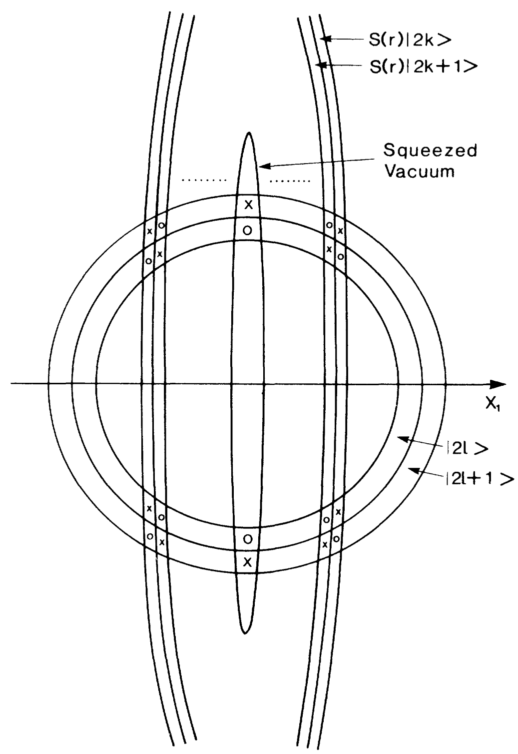

FIG. 6. Photon number states and squeezed number states in dimensionless phase space. When the squeezed even- (odd-) photon number state overlaps with the even- (odd-) photon number state, the contributions of the overlaps (denoted by ○) interfere and construct a nonzero value photon number distribution $P_{SN}(l)$. Otherwise the contributions from the overlaps (denoted by ×) destructively interfere and give $P_{SN}(l)=0$. k and l are positive integers.

important contribution of the sum is from the squeezed number state $n \approx 0$ (squeezed vacuum). Thus the photon number distribution for the squeezed thermal state of $\bar{n} \leq 1$ is similar to the squeezed vacuum and oscillates between zero and nonzeroes in a pairwise pattern. However when $\bar{n}$ is large we cannot disregard the contributions from the squeezed number states with n large. Eventually the contributions from appropriately large photon number states are more important. When n is appropriately large and $l < n$, the approximation (5.25) is applicable and the photon distribution will show a similar pattern to Table I. The sum of odd- and even-number photon states will show the pairwise oscillation as in Fig. 4(b).

VI. CONCLUSIONS

We have discussed properties of the squeezed number and the squeezed thermal states in the paper. The second-order correlation functions for the squeezed thermal and the squeezed number states are calculated and compared with that of the squeezed coherent state. The well-defined intensity of the photon number state gets blurred at the expense of the phase squeeze and the sub-Poissonian statistics of the photon number state becomes super-Poissonian. The squeezed thermal state is always super-Poissonian. For the squeezed thermal state, as $r \rightarrow \infty$, the second-order correlation function $g^{(2)} \rightarrow 3$.

The Glauber P representation is calculated for the squeezed thermal state. When a quadrature variance is less than the minimum uncertainty limit, the squeezed thermal state of the super-Poissonian statistics does not have a well-behaved P representation and cannot be described by classical theory. The Wigner functions and Q representations are obtained for the various squeezed states. The Gaussian thermal state Wigner function is stretched by the action of the squeezing.

The explicit expression of the photon number distribution is obtained for the squeezed number state, based on which the photon number distribution for the squeezed thermal state is calculated and plotted. Pairwise oscillations of the photon number distribution for the squeezed thermal state are explained with the overlaps between the photon number state and the squeezed state in phase space. The photon number distribution for the squeezed number state has large-scale oscillations as well as pairwise oscillations, which is due to the presence of four overlap areas, which makes two phase parameters important in the photon number distribution.

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