Entropy as a measure of quantum optical correlation

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(Received 14 November 1988; revised manuscript received 11 May 1989)

Previous work on quantum correlations has focused attention on the influence of the correlation on observables. In this paper we introduce an observable-independent index of correlation based on the information content, or entropy, of that correlation. We analyze a pair of spins and verify that they are maximally correlated when the index of correlation is maximized. The index of correlation suggests that two optical-field modes are most strongly correlated when prepared in a two-mode squeezed state.

I. INTRODUCTION

Many of the most significant conceptual problems in quantum mechanics have involved quantum correlations. The most famous of these problems is the celebrated EPR paradox.[1] The paradox illustrates the fundamental significance of quantum correlations, in that measurement of an observable of one of a pair of correlated systems determines the result of a measurement of a corresponding property of the partner system. This "collapse" of the wave function occurs irrespective of the distance between the two systems. Another correlation-dependent phenomenon is the configuration interaction in multielectron atoms, where correlations induced by the Coulomb repulsion between the electrons play an important role in determining the atomic structure.[2]

Consider a pair of quantum systems labeled by the suffixes a and b. The reduced density matrix describing the properties of the a (b) system is obtained by tracing the total density matrix ρ over the b (a) system. Thus

$$\rho_{a(b)} = \mathrm{Tr}_{b(a)}\rho \ . \tag{1}$$

The a and b systems are correlated if the measurement of an observable of the a (b) system projects the b (a) system into a new state. This will be true for at least one of the observables of the a (b) system unless the statistical properties of the two systems are independent. In this case the two systems are uncorrelated. We can formalize this statement by considering two operators A and B acting on the a and b systems, respectively. If the expectation value of AB factorizes,

$$\langle AB \rangle = \langle A \rangle \langle B \rangle \ , \tag{2}$$

then we say that the observables A and B are uncorrelated. Such correlations are directly responsible for the nonclassical properties of the two-mode squeezed states of light.[3] If *all* possible operators, A and B, are uncorrelated then the two-sysem density matrix can be written in a factorized form as

$$\rho = \rho_a \otimes \rho_b \tag{3}$$

and the statistical properties of the two systems are independent. However, if such a factorization of the density matrix is not possible, then the systems are correlated.

The correlation between the systems is apparent in measurements of observable properties of the two systems. Nevertheless, an absolute statement concerning the correlation, or lack of it, requires a full knowledge of the two-sysem density matrix. Measurements of a specific property of the systems may show us that the systems are correlated but it cannot definitely show that they are uncorrelated. Moreover, such measurements cannot provide us with an absolute and *quantitative* measure of the correlations between the two systems. In this paper we propose an absolute and observable-independent index of the correlation between two quantum systems. This index is a measure of the information contained in the correlation and is based on the von Neumann entropy.[4]

In Sec. II we define the index of correlation and show that only a pure state for the combined system will saturate this parameter. The most strongly correlated state will be one in which the combined system is in a pure state but each of the subsystems displays thermal fluctuation properties. Such states are well known in quantum-statistical mechanics as thermofields.[5,6] In Sec. III we compare the index of correlation with a more conventional measure (the correlation coefficient[7]) for the simple case of a pair of spins. In Sec. IV we recall some of the properties of bosonic thermofield states which are formally identical to the two-mode squeezed state of light.[6,8] We discuss the implications of this identification for the optimal correlation of light fields. Although we do not treat here the fermionic thermofield states (formally identical to the two-atom squeezed state[9]), the results of this paper are applicable to such systems.

II. THE INDEX OF CORRELATION

We have seen how correlation is a fundamental property of a two-component quantum system and that this property is reflected in our ability, or inability, to factorize the complete density matrix. A natural measure of the information content in the density matrix is the entropy. If we know that the system is in a pure state then the entropy is zero. In this case we have as much information about the system as quantum mechanics allows us. However, if we know nothing about the system then it is equally likely to be found in any state and the entropy is maximum.

Entropy in quantum mechanics (as defined by von Neumann) is a straightforward generalization of the Boltzmann entropy in classical statistical mechanics. If ρ is the density operator for an isolated system then the entropy is defined by

$$S=-k_B\mathrm{Tr}(\rho\ln\rho) , \tag{4}$$

where k_B is Boltzmann's constant. The entropy as defined above is zero for a pure state and positive for a mixed state and so measures deviations from pure-state behavior. This quantity is also time independent as the dynamics of ρ are governed by a unitary transformation. However, if we consider two systems (labeled a and b as before) then the entropy of the $a(b)$ system is formed through the reduced density operator for the system by the following expression:

$$S_{a(b)}=-k_B\mathrm{Tr}_{a(b)}(\rho_{a(b)}\ln\rho_{a(b)}) . \tag{5}$$

This quantity is, in general, time dependent because of the tracing operation. Another consequence of the tracing operation is that if the combined system is in a pure state then the reduced systems will, in general, no longer be in pure states. In 1970 Araki and Lieb proved the following inequality:[10]

$$|S_a-S_b|\le S\le S_a+S_b . \tag{6}$$

One interesting consequence of this relation is that if the total system is in a pure state then $S=0$ and therefore the a and b systems have equal entropies. If, for example, we consider a two-level atom interacting with a quantized field mode in a perfect cavity, then the field can be described at all times by just two quantum states if the initial state of the atom-field system is pure.[11]

Entropy can be considered from an information theory point of view[12,13] and regarded as the amount of uncertainty contained within the density operator. A mixed state requires more information to fully specify than does a pure state. If we only observe the two systems independently then we lose *all* the information contained in the correlation. The results of our measurements will only allow us to construct the reduced density matrices. We can quantify this in a precise and unambiguous way by defining the index of correlation I_c as the amount of information lost in the tracing procedure:

$$I_c=S_a+S_b-S . \tag{7}$$

Failure to measure joint properties of the correlated systems results in a loss of this quantity of information. The index of correlation is always greater than or equal to zero and will be precisely zero only for uncorrelated states as described by (3). If we label our systems such that $S_a\ge S_b$, then from the Araki-Lieb inequality we find that I_c is bounded as

$$I_c\le 2S_b . \tag{8}$$

The maximum possible value of I_c will be

$$I_c^{\max}=2S_b . \tag{9}$$

However, the maximum value that S_b can take is just S_a. Consequently the maximum degree of correlation is obtained when $S_a=S_b$ and so, from (6), we find that $S=0$. That is, the total system is in a pure state. Thus we can say that for the a and b systems to be maximally correlated the total system comprising a and b must be in a pure state. The maximum value obtainable for the subsystem entropies will, of course, depend upon the nature of the subsystems in question. If the two systems have different numbers of states (as, for example, in the Jaynes-Cummings model[11]), then I_c will be maximized when we have a complete lack of knowledge concerning the system with the smaller number of states. That is, the smaller system is equally likely to be found in any of its states. However, when the information contained in the correlation is included it becomes possible to specify the pure state of the total system uniquely. For the remainder of this paper we will be concerned with correlated state of pairs of *similar* quantum systems, that is, systems with equal numbers of states.

It is often possible (and for unbounded systems necessary) to constrain the entropy by specifying a mean energy, or particle number, for the system. The state with maximum entropy, but finite mean energy, exhibits thermal fluctuations and is described by a density matrix of the general form,[14]

$$\rho_{\mathrm{th}}=Z^{-1}(\beta)e^{-\beta H} , \tag{10}$$

where $Z(\beta)$ is the partition function, H is the system Hamiltonian, and β is the inverse temperature ($\beta=1/k_BT$). The maximally correlated state of two similar quantum systems will be in a pure state in which each of the component systems (a and b) are thermal in character. Moreover, the Araki-Lieb inequality (6) shows that the entropies associated with the two subsystems are equal. Such states are well known in quantum-statistical mechanics as thermofields[5,6] and we shall briefly examine these states in Sec. IV.

In this section we have proposed an observable-independent measure of correlation I_c which we have defined as the amount of information contained within the correlation. In Sec. III we look at a simple correlated quantum system consisting of two spins and compare a more standard measure of correlation with our proposed index.

III. TWO CORRELATED SPINS

A simple and well-known example of a correlated quantum state is the EPR state in which two spins (a and b) are prepared in a pure state described by the wave function

$$|\psi\rangle=\alpha^{1/2}|+\rangle_a|-\rangle_b+(1-\alpha)^{1/2}e^{i\Omega}|-\rangle_a|+\rangle_b , \tag{11}$$

where the states $|+\rangle$ and $|-\rangle$ correspond to the up and down orientations of the spin, in the z direction, respectively, and $0 \le \alpha \le 1$. The combined two-spin system is in a pure state, but the reduced density matrices for the two spins contain no information about the spin orientation. The index of correlation for this system is

$$I_c = -2k_B[\alpha \ln\alpha + (1-\alpha)\ln(1-\alpha)] \ . \tag{12}$$

This quantity is α dependent and is maximized when $\alpha = \frac{1}{2}$. We must remember that our proposed index of correlation is *observable independent* and that state (11) is not equally strongly correlated for all possible pairs of observables of the two systems. To demonstrate this we consider the more usual measure of the degree of correlation between two observables.[7] We label this quantity r and define it through the relation

$$r = \frac{\Delta(X,Y)}{\Delta X \Delta Y} \ , \tag{13}$$

where the quantities $\Delta(X,Y)$ and ΔX are defined by

$$\Delta(X,Y) = \langle XY \rangle - \langle X \rangle \langle Y \rangle \ , \tag{14}$$

$$\Delta X = (\langle X^2 \rangle - \langle X \rangle^2)^{1/2} \ . \tag{15}$$

For the two-spin system (11) the degree of correlation between spin measurements in arbitrary directions is given by

$$r(\theta,\phi,\theta',\phi') = \frac{\Delta(\sigma^a(\theta,\phi),\sigma^b(\theta',\phi'))}{\Delta\sigma^a(\theta,\phi)\Delta\sigma^b(\theta',\phi')} \ , \tag{16}$$

where $\theta,\phi,\theta',\phi'$, describe the orientations of the spin analyzers and the spin in a general direction is given by

$$\sigma(\theta,\phi) = \cos(\theta)\sigma_z + \sin(\theta)e^{-i\phi}\sigma_+ + \sin(\theta)e^{i\phi}\sigma_- \ . \tag{17}$$

Here σ_z is the spin-z operator and $\sigma_\pm$ are the spin-flip operators. The degree of correlation, $r(\theta,\phi,\theta',\phi')$, for these spin measurements is given by

$$r(\theta,\phi,\theta',\phi') = \frac{4\alpha(\alpha-1)\cos\theta\cos\theta' + 2\alpha^{1/2}(1-\alpha)^{1/2}\sin\theta\sin\theta'\cos(\phi-\phi'-\Omega)}{[\sin^2\theta + 4\alpha(1-\alpha)\cos^2\theta]^{1/2}[\sin^2\theta' + 4\alpha(1-\alpha)\cos^2\theta']^{1/2}} \ . \tag{18}$$

Clearly, the spin-z measurements ($\theta = \theta' = 0$) are perfectly anticorrelated with $r = -1$ for all α. However, we note that certain spin measurements will reveal no correlations at all. For example, if we choose $\Omega = 0$, and align the spin analyzers ($\theta = \theta'$ and $\phi = \phi'$), then there exists a cone of directions around the z axis given by $\tan^2\theta = 2\alpha^{1/2}(1-\alpha)^{1/2}$ for which $r = 0$ and the spin measurements will reveal no correlations. In order to compare our proposed index of correlation with this more usual measure of correlation we must integrate some even function of r over all possible orientations of the spin analyzers to form an orientation-independent, and therefore observable-independent, measure of correlation which can be directly compared with I_c. Let us consider r^2 as a suitable candidate function and form an observable-independent measure of correlation Q as described above,

$$Q = \frac{9}{16\pi^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} \sin\theta'\, d\theta' r^2(\theta,\phi,\theta',\phi') \ . \tag{19}$$

We explicitly evaluate this integral in the appendix and show that it is an even function of $2\alpha - 1$ with a single maximum at $\alpha = \frac{1}{2}$. Furthermore, $Q = 0$ when $\alpha = 0$ or 1. The index of correlation (12) also has these properties and we have therefore shown that I_c is consistent with the more usual measure of correlation.

We have emphasized that I_c is an observable-independent measure of correlation and its significance can be demonstrated by examination of other properties of the state, Eq. (11). For example if we expand the state in terms of the eigenstates of the spin-x operators, we then obtain

$$\begin{aligned} |\psi\rangle &= \tfrac{1}{2}[\alpha^{1/2} + (1-\alpha)^{1/2}e^{i\Omega}](|x,+\rangle_a|x,+\rangle_b - |x,-\rangle_a|x,-\rangle_b) \\ &\quad + \tfrac{1}{2}[\alpha^{1/2} - (1-\alpha)^{1/2}e^{i\Omega}](|x,-\rangle_a|x,+\rangle_b - |x,+\rangle_a|x,-\rangle_b) \ . \end{aligned} \tag{20}$$

We see that the correlations between the x components of the two spins are clearly dependent on α; if $\alpha = 1$ the state becomes

$$\begin{aligned} |\psi(\alpha=1)\rangle &= \tfrac{1}{2}(|x,+\rangle_a + |x,-\rangle_a) \\ &\quad \times (|x,+\rangle_b + |x,-\rangle_b) \ . \end{aligned} \tag{21}$$

Measurement of the x component of one of the spins in this state provides no information about the x component of the other spin. A similar conclusion can be demonstrated if $\alpha = 0$. If, however, $\alpha^{1/2} = 1\sqrt{2}$ (and we choose $\Omega = 0$) then the state becomes

$$|\psi(\alpha^{1/2} = 1/\sqrt{2}, \Omega = 0)\rangle = \frac{1}{\sqrt{2}}(|x,+\rangle_a|x,+\rangle_b - |x,-\rangle_a|x,-\rangle_b) \tag{22}$$

and a measurement of the x component of one of the spins determines the x component of the other spin. A similar result holds for the y component of the two spins. This state, Eq. (22), displays strong correlations between the x, y, and z components of the two spins. The index of correlation is independent of specific observables but has successfully determined the condition for optimal correlation in the x, y, and z components of the spins.[15]

IV. THERMOFIELD STATES AND TWO-MODE SQUEEZING

In Sec. III we showed that optimally correlated states of similar quantum systems displayed single-system thermal properties but that the combined system was in a pure state. The existence of such states, for a range of quantum systems, has led to the development of a pure-state formalism to describe thermal ensemble averages in quantum mechanics and field theory. The typical thermofield state is a pure state in the state space spanned by two similar quantum systems. These two systems are known are the real and "fictitious" systems. Quantities associated with the fictitious system are conventionally denoted by a tilde. The so-called thermofield vacuum state [denoted by $|0(\beta)\rangle$] has the form[5,6]

$$|O(\beta)\rangle = Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}\rangle \ , \tag{23}$$

where the state $|n\rangle$ ($|\tilde{n}\rangle$) is the real (fictitious) space energy eigenstate with energy E_n. Clearly the expectation value of any operator acting on the real system alone will reproduce the thermal ensemble average

$$\langle O(\beta)|A|O(\beta)\rangle = Z^{-1}(\beta)\sum_n \langle n|A|n\rangle e^{-\beta E_n} \ . \tag{24}$$

Moreover, the reduced density matrices for the real and fictitious systems are thermal with equal entropies. The index of correlation between the real and fictitious systems prepared in the state $|0(\beta)\rangle$ is

$$I_c = 2k_B[\beta\langle H\rangle + \ln Z(\beta)] \ , \tag{25}$$

where H is the Hamiltonian for the system. This index of correlation is as large as possible whilst remaining consistent with the constraints on the energy. Therefore the thermofield vacuum state represents the most strongly correlated state of two similar quantum systems.

If the systems of interest are bosonic field modes then the thermofield vacuum state has the simple form

$$|O(\beta)\rangle = (1-e^{-\beta\hbar\omega})^{-1/2}\sum_n e^{-\beta n\hbar w/2}|n,\tilde{n}\rangle \ . \tag{26}$$

This state may be obtained from the two-mode vacuum by means of a Bogoliubov transformation that mixes the real and fictitious systems

$$|O(\beta)\rangle = e^{\theta(\beta)(a^{\dagger}\tilde{a}^{\dagger} - \tilde{a}a)} \ , \tag{27}$$

where $\tanh[\theta(\beta)] = \exp(-\beta w/2)$. The unitarity of the required Bogoliubov transformation (and its fermionic counterpart) is the source of the utility of the thermofield formalism. It has been applied to a range of thermodynamic problems in quantum optics,[6,16] solid-state physics,[17] and quantum-field theory.[18] It has also been used to describe the production of particles by a black hole through the Hawking mechanism.[19]

Two-mode squeezed states are formed form the two-mode vacuum, or coherent states, by the action of a Bogoliubov transformation similar to that which appeared in Eq. (27).[6,8] However, unlike in the thermofield formalism, the two-mode squeezed states are correlated states of two real, and therefore accessible, modes. Two-mode squeezed states may be prepared in a range of ideal two-photon devices some of which have been realized experimentally.[20] The simplest of these devices is the nondegenerate parametric amplifier in which the signal and idler modes are driven by a classical pump. The Hamiltonian for this system is given by[21]

$$H = \hbar w_a a^{\dagger}a + \hbar w_b b^{\dagger}b - i(\lambda ab e^{iwt} - \lambda^* a^{\dagger}b^{\dagger}e^{-iwt}) \ , \tag{28}$$

where $a(b), a^{\dagger}(b^{\dagger})$ are the annihilation and creation operators for mode $a(b)$ and $w_{a(b)}$ is the mode frequency. λ is the coupling constant between the modes. If we specialize to the case of exact resonance $w = w_a + w_b$, then the Hamiltonian in the interaction picture is given by

$$H_I = -i(\lambda ab - \lambda^* a^{\dagger}b^{\dagger}) \tag{29}$$

and the time evolution operator is

$$e^{-iH_I t/\hbar} = e^{-\lambda t ab + \lambda^* t a^{\dagger}b^{\dagger}} \ . \tag{30}$$

This is a time-dependent Bogoliubov transformation and generates two-mode squeezed states characterized by a squeezing parameter $\xi = \lambda t$. The action of the two-photon device on the two-mode vacuum state is to produce a two-mode squeezed vacuum state

$$|\xi\rangle = (\cosh r)^{-1}\sum_n (\tanh r)^n e^{in\phi}|n\rangle_a|n\rangle_b \ , \tag{31}$$

where $\xi = re^{-i\phi}$. If we write

$$\tanh r = e^{-\beta_a\hbar w_a/2} = e^{-\beta_b\hbar w_b/2} \tag{32}$$

so that the product $\beta_i w_i$ is a constant, then the state $|\xi\rangle$ acquires the form of the thermofield vacuum state. From Sec. III we know that the thermofield vacuum states are optimally correlated. Therefore, based on our proposed index of correlation, the two-mode squeezed vacuum state appears to have a rather fundamental property: it is the most strongly correlated state of two modes of the electromagnetic field, subject to the constraint of a mean occupation number per mode. Its associated index of correlation is given by

$$I_c = 2k_B[\cosh^2 r\ \ln(\cosh^2 r) - \sinh^2 r\ \ln(\sinh^2 r)] \ . \tag{33}$$

The quantum correlation between the modes in a two-mode squeezed state is manifest in the fact that both modes contain precisely the same number of quanta. However, there is also phase information in the state which is manifest in the two-mode squeezing properties of the state.[3] This can be demonstrated by integrating out the angular dependence in the density matrix for the two-mode squeezed vacuum. The resulting phase-averaged state may be described by a density matrix $\bar{\rho}$ of the form

$$\bar{\rho}(r) = \frac{1}{2\pi}\int_0^{2\pi}|\xi\rangle\langle\xi|d\phi = (\cosh r)^{-2}\sum_n (\tanh r)^{2n}|n\rangle_a|n\rangle_b\, {}_b\langle n|_a\langle n| \ . \tag{34}$$

This density matrix still describes a correlated state with equal numbers of photons in each mode. However, because we have averaged over the phase of the squeezing parameter the system is no longer in a pure state and will no longer give squeezing. The index of correlation for this state is half that associated with the squeezed vacuum state (31). It appears that the phase and photon number each contain half the information about the correlation between the modes.

V. SUMMARY

In this paper we have proposed an absolute measure of the strength of the correlation between two quantum systems. Our index of correlation is the amount of information associated with the correlation. Failure to measure joint properties of the correlated systems results in a loss of this information. We have shown that I_c is consistent with more conventional observable-based measures of correlation. However, we emphasize that it is independent of any specific observables.

If the systems are of a similar nature then our index of correlation suggests that the most strongly correlated state will be a pure state in which each subsystem exhibits thermal fluctuation properties. This class of states is known as the thermofield states. The bosonic thermofield vacuum is formally identical to the two-mode squeezed vacuum. Therefore, the two-mode squeezed vacuum state appears as the most strongly correlated of all two-mode states of light.

ACKNOWLEDGMENTS

S.M.B. thanks G.E.C. and the Fellowship of Engineering for financial assistance, S.J.D.P. thanks the United Kingdom Science and Engineering Research Council for financial assistance.

APPENDIX

After performing the integrations over ϕ and ϕ', the integral, Eq. (19), can be written as

$$Q=9\alpha(1-\alpha)\left[4\alpha(1-\alpha)\left[\int_0^{\pi}d\theta\frac{\cos^2\theta\sin\theta}{X(\theta)}\right]^2+\frac{1}{2}\left[\int_0^{\pi}d\theta\frac{\sin^3\theta}{X(\theta)}\right]^2\right], \tag{A1}$$

where $X(\theta)$ is given by

$$X(\theta)=\sin^2\theta+4\alpha(1-\alpha)\cos^2\theta\ . \tag{A2}$$

Making the following substitutions,

$$u=\cos^2\theta\ ,\qquad \tilde{\alpha}=2\alpha-1\ , \tag{A3}$$

we find that Q is given by the expression

$$Q=\tfrac{9}{4}\{[(1-\tilde{\alpha}^2)G_1(\tilde{\alpha})]^2+\tfrac{1}{2}(1-\tilde{\alpha}^2)[G_2(\tilde{\alpha})-G_1(\tilde{\alpha})]^2\}\ , \tag{A4}$$

where G_1 and G_2 are given by

$$G_1(\tilde{\alpha})=\frac{2}{\tilde{\alpha}^2}\left[\frac{1}{2\tilde{\alpha}}\ln\left[\frac{1+\tilde{\alpha}}{1-\tilde{\alpha}}\right]-1\right]\ , \tag{A5}$$

$$G_2(\tilde{\alpha})=\frac{1}{2\tilde{\alpha}}\ln\left[\frac{1+2\tilde{\alpha}}{1-2\tilde{\alpha}}\right]\ . \tag{A6}$$

We can expand these functions in power series of $\tilde{\alpha}$ obtaining

$$G_1(\tilde{\alpha})=\frac{2}{\tilde{\alpha}^2}\left[\frac{\tilde{\alpha}^2}{3}+\frac{\tilde{\alpha}^4}{5}+\frac{\tilde{\alpha}^6}{7}+\cdots\right]\ , \tag{A7}$$

$$G_2(\tilde{\alpha})=\frac{2}{\tilde{\alpha}^2}\left[\tilde{\alpha}^2+\frac{\tilde{\alpha}^4}{3}+\frac{\tilde{\alpha}^6}{7}+\cdots\right]\ . \tag{A8}$$

From these expressions and Eq. (A4) it is clear that Q is an even function of $\tilde{\alpha}$. Furthermore it is simple to show that Q has a maximum at $\tilde{\alpha}=0$, is zero at $\tilde{\alpha}=\pm1$, and that it has a positive gradient in the range $-1<\tilde{\alpha}<0$. These properties are shared by the index of correlation for this system, Eq. (12). Restating this, we have shown that if $Q(\alpha_1)$ is greater (less) than $Q(\alpha_2)$, then $I_c(\alpha_1)$ is greater (less) than $I_c(\alpha_2)$ for all α_1,α_2. Thus $I_c(\alpha)$ correctly ranks the EPR states, Eq. (11), according to their degree of correlation.

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