Quantum theory of a laser with injected atomic coherence: Quantum noise quenching via nonlinear processes

Ning Lu and János A. Bergou*

Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131 and Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, West Germany

(Received 16 June 1988; revised manuscript received 21 February 1989)

A quantum theory of a two-level single-mode laser with injected atomic coherence is developed by generalizing the Scully-Lamb laser theory to a form appropriate for the analysis of a coherently pumped laser. We assume that the active atoms are prepared initially in a coherent superposition of the upper and lower levels, and we derive the master equation for the field density operator by treating the interaction of the laser field with many active atoms simultaneously. It is shown that the photon-number distribution can be exactly Poissonian. The laser operation is analyzed in terms of the Fokker-Planck equation for the laser field. Both the intensity and phase diffusion coefficients are phase sensitive and, for stable laser operation, become much smaller than those of an ordinary laser. Consequently, the injected atomic coherence reduces both the photon-number noise and phase noise simultaneously. The intensity diffusion coefficient can vanish exactly, and at the same time the phase diffusion coefficient can become very small. This leads to spontaneous-emission noise quenching in the photon-number distribution, and the laser field can become very close to a coherent state. A scheme to generate the proper form of the initial atomic coherence necessary for the quantum noise quenching is proposed and analyzed.

I. INTRODUCTION

The quantum theory of lasers was developed 20 years ago and the underlying physics of laser operation has been well understood since then.[1-4] In an ordinary laser, two-level active atoms are incoherently pumped to their upper levels and subsequently return to their lower levels through stimulated emission. Besides returning to its lower level via stimulated emission, an active atom may also return to the lower level via spontaneous emission. While the stimulated emission of the atoms contributes to the laser field in the laser cavity, the spontaneous emission of the atoms contributes to fluctuations in the photon number and phase of the laser field, which, in turn, leads to the uncertainity in the photon-number distribution and the linewidth of the laser, respectively. It is of great interest to reduce the intensity and phase noise arising from the spontaneous emission of the lasing atoms, and to further understand the nature and origin of quantum noise in the laser.[5] The phase of a laser field is randomly distributed over a 2π range with equal probability due to the lack of a preferred phase angle. By injecting an external optical field into a laser cavity,[6] one can induce a preferred phase angle and even lock the laser phase to a particular value. It has been shown that the linewidth of a laser subject to such a symmetry-breaking injected signal is reduced.[7] As to the variance in photon number of a laser field,[1] it is always larger than that of a coherent state[8] with the same mean photon number. When the laser is operated far above threshold, laser photon statistics approach a Poissonian distribution—the photon-number distribution of coherent states, or say, classical fields. Reducing the amplitude noise below that of a coherent state has recently been discussed and demonstrated experimentally in a pump-noise-suppressed laser.[9]

In this paper we discuss a novel method for the reduction of photon-number noise and phase noise in an ordinary two-level laser. Instead of incoherently pumping active atoms into their upper levels as in ordinary laser devices, one could pump them coherently, i.e., prepare a two-level lasing atom initially in a coherent superposition of its upper and lower states when it is injected into a laser cavity. We show that, with a proper relation among the initial atomic coherences of randomly injected atoms, it is possible to reduce the amplitude and phase diffusion coefficients of the laser field simultaneously. The laser with injected atomic coherence can even generate a field with exactly Poissonian photon statistics for a particular initial atomic population and coherence. That is, in this way, one can generate a field which is more classical than that produced by an ordinary laser. The reduction of phase noise in such a coherently pumped laser does not appear in a linear theory of the coherently pumped two-level laser. Thus phase noise reduction is achieved via nonlinear processes in the laser. A physical explanation is given in this paper as to the origin of the noise reduction. Briefly, since *both* the amplitude and phase fluctuations of a laser are due to the spontaneous-emission events of lasing atoms, the simultaneous intensity and phase noise reduction in a laser with injected atomic coherence are obviously related to the suppression of the number of spontaneous emission events. To see phase noise reduction, a nonlinear laser theory (at least third order in the atom-field coupling constant g) must be developed. Quantum noise reduction in lasers due to injected atomic coherence was first found in correlated-emission lasers[10,11] with injected atomic coherence,[12,13] in which atoms are prepared initially in a coherent superposition of the two upper levels of a V-type three-level system, and the noise reduction in the relative phase angle of two-mode laser fields appears even in a linear theory.

In order to achieve noise reduction in a coherently pumped two-level laser, it is vital that a proper phase relation is satisfied among all randomly injected lasing atoms, as mentioned before. More accurately, the relation is such that the phase difference between two randomly injected atoms equals the phase accumulation between the two injection times with rate ν, where ν is the laser operation frequency. (In terms of absolute phases, initial atomic phases of randomly injected atoms are different.) In other words, initial phases of the injected atoms should be the same relative to the instantaneous total laser phase (i.e., including the frequency part νt). To account for the effects of the initial atomic coherence, especially the phase difference of the injected atoms, the interaction between the laser field and the active atoms should be treated from an overall point of view when the Scully-Lamb model[1,2] of the laser is used. This is accomplished by generalizing the usual treatment, provided by the Scully-Lamb theory for an incoherently pumped laser, to a form appropriate for studying the coherently pumped laser.

In Sec. II, we present the general formalism for a two-level laser with injected atomic coherence and derive a master equation for the field density matrix. The photon statistics of the laser is discussed in Sec. III. Because atomic coherence is involved here the photon statistics are no longer as simple as in an ordinary laser. An alternative is to use a Fokker-Planck equation. In Sec. IV, by converting the master equation into a Fokker-Planck equation, we obtain photon-number and phase diffusion coefficients and study the steady-state laser operation. We show that both the intensity and phase diffusion coefficients are reduced in a steady state compared to those in an ordinary laser. The preparation of a proper initial coherence for the injected atoms is proposed and analyzed in Sec. V. Finally, we comment on the relation between the approach we develop here and the usual method used in the Scully-Lamb theory, and give a physical explanation for the desired initial atomic phase as well as for quantum noise reduction in Sec. VI.

II. MODEL AND THE FIELD MASTER EQUATION

In this section, we first generalize the usual method of the Scully-Lamb theory[1,2] to a form appropriate for dealing with laser problems involving injected atomic coherence, and then derive the master equation for the field-density matrix elements of a two-level coherently pumped laser.

We consider two-level active atoms consisting of an upper state $|a\rangle$ with energy $\hbar\omega_a$ and a lower state $|b\rangle$ with energy $\hbar\omega_b$. These active atoms are randomly injected into the laser cavity at a rate r_a to interact with the laser field. We model the quantum theory of a laser with injected atomic coherence in the following way: The jth atom is injected into the laser cavity at time t_j with initial atomic coherence $\rho^j_{ab}(t_j)\neq 0$, where the superscript j denotes the density matrix of the jth atom. Just as in the usual Scully-Lamb theory of the laser, we assume that (i) an atom "sees" the effects of other atoms only through the laser field, i.e., the evolution of an atom is independent of those of other atoms, and (ii) the cavity decay time γ^{-1} is much longer than the atomic lifetime. Instead of calculating the contribution to the laser field from one atom first and then obtaining the contribution from all injected atoms as in the usual treatment of the Scully-Lamb theory, we treat the atom-field interaction in a more rigorous way: the laser field interacts with many active atoms in the cavity simultaneously.[14] Notice that this is different from the method commonly used in the Scully-Lamb model, since the latter is only capable of studying the incoherently pumped laser. Thus the approach developed here is more general.

The total Hamiltonian in the Schrödinger picture for the laser field and the active atoms is

$$H=\hbar H_0+\hbar V = \hbar\left[\Omega a^\dagger a+\sum_j H_j^{\text{at.}}\right]+\hbar\sum_j \Theta(t-t_j)V_j \ , \tag{2.1}$$

with

$$H_j^{\text{at.}} \sum_{A=a,b} \omega_A |A^j\rangle\langle A^j| \ , \tag{2.2}$$

$$V_j = g\sigma_j^\dagger a + g a^\dagger \sigma_j \ , \tag{2.3}$$

where Ω is the cavity-mode frequency, a ($a^\dagger$) is the field annihilation (creation) operator, $H_j^{\text{at.}}$ is the free Hamiltonian of the jth atom, V_j is the interaction Hamiltonian of the jth atom with the laser field under the dipole and rotating-wave approximations, $g(>0)$ is the atom-field coupling constant, $\sigma_j=|b^j\rangle\langle a^j|$ and $\sigma_j^\dagger=|a^j\rangle\langle b^j|$ are the lowering and raising operators of the jth atom, respectively, and

$$\Theta(t-t_j)=\begin{cases}1, & t\geq t_j\\ 0, & t<t_j\end{cases} \tag{2.4}$$

is a step function used here to specify the initial time of the interaction of the jth atom with the laser field.

The total density operator for all atoms and the field obeys the equation of motion

$$\dot\rho=-i[H_0+V,\rho] \ . \tag{2.5}$$

The reduced density operator for the jth atom and the field is obtained from the total density operator ρ by tracing over all atoms except the jth one,

$$\rho_j^f=\text{Tr}_{A^1,A^2,\ldots,A^{j-1},A^{j+1},\ldots}\rho \ . \tag{2.6}$$

The reduced density operator for the laser field is obtained from ρ by tracing over all atoms,

$$\rho^f=\text{Tr}_{A^1,A^2,\ldots}\rho=\text{Tr}_{A^j}\rho_j^f \ . \tag{2.7}$$

Similarly, the reduced density operator for the jth atom is obtained from ρ by tracing over the field and all atoms but the jth one,

$$\rho^j=\mathrm{Tr}_f\mathrm{Tr}_{A^1,A^2,\ldots,A^{j-1},A^{j+1},\ldots}\rho=\mathrm{Tr}_f\rho_j^f \; . \tag{2.8}$$

For convenience we go to an interaction picture via the following unitary transformation:

$$\tilde{\rho}(t)=\exp\left[i\nu a^\dagger a t+i\sum_j H_j^{\text{at.}}(t-t_j)\right]\rho(t)\times\exp\left[-i\nu a^\dagger a t-i\sum_j H_j^{\text{at.}}(t-t_j)\right] , \tag{2.9}$$

where ν is the actual laser frequency and here, as well as throughout this paper, we use a tilde to denote a physical quantity in the interaction picture. Using Eqs. (2.5) and (2.9), one finds that the total density operator $\tilde{\rho}$ in the interaction picture satisfies the equation

$$\dot{\tilde{\rho}}=-i(\Omega-\nu)[a^\dagger a,\tilde{\rho}]-i[\tilde{V},\tilde{\rho}] , \tag{2.10}$$

where $\tilde{V}$ is the total interaction Hamiltonian in the interaction picture,

$$\tilde{V}(t)=\exp\left[i\nu a^\dagger a t+i\sum_j H_j^{\text{at.}}(t-t_j)\right]V\times\exp\left[-i\nu a^\dagger a t-i\sum_j H_j^{\text{at.}}(t-t_j)\right]=\sum_j\Theta(t-t_j)\tilde{V}_j , \tag{2.11}$$

and V_j is that for the jth atom and the laser field,

$$\tilde{V}_j=g\sigma_j^\dagger a e^{i(\Delta t-\omega t_j)}+ga^\dagger\sigma_j e^{-i(\Delta t-\omega t_j)} . \tag{2.12}$$

Here

$$\Delta=\omega-\nu=\omega_a-\omega_b-\nu \tag{2.13}$$

is the atom-field detuning.

As in the Schrödinger picture, the various reduced density operators in the interaction picture can be obtained by appropriately tracing over atomic and/or field variables,

$$\tilde{\rho}_j^f=\mathrm{Tr}_{A^1,A^2,\ldots,A^{j-1},A^{j+1},\ldots}\tilde{\rho} , \tag{2.14}$$

$$\tilde{\rho}^f=\mathrm{Tr}_{A^1,A^2,\ldots}\tilde{\rho}=\mathrm{Tr}_{A^j}\tilde{\rho}_j^f , \tag{2.15}$$

$$\tilde{\rho}^j=\mathrm{Tr}_f\mathrm{Tr}_{A^1,A^2,\ldots,A^{j-1},A^{j+1},\ldots}\tilde{\rho}=\mathrm{Tr}_f\tilde{\rho}_j^f . \tag{2.16}$$

Tracing over all atomic variables on both sides of Eq. (2.9), one finds the relation between the reduced field-density operator in the interaction picture and that in the Schrödinger picture

$$\tilde{\rho}^f(t)=e^{i\nu a^\dagger a t}\rho^f(t)e^{-i\nu a^\dagger a t} . \tag{2.17}$$

In particular, the two field-density operators coincide with each other at time $t=0$,

$$\tilde{\rho}^f(0)=\rho^f(0) . \tag{2.18}$$

Similarly, the relation between the reduced density operator for the jth atom in the interaction picture and that in the Schrödinger picture is found by tracing over the field and all atoms but the jth one on both sides of Eq. (2.9),

$$\tilde{\rho}^j(t)=\exp[iH_j^{\text{at.}}(t-t_j)]\rho^j(t)\times\exp[-iH_j^{\text{at.}}(t-t_j)] . \tag{2.19}$$

The coincidence of the two operators occurs at time t_j,

$$\tilde{\rho}^j(t_j)=\rho^j(t_j) . \tag{2.20}$$

Tracing over all atoms and using Eqs. (2.14) and (2.15), one finds the equation of motion for the reduced field-density matrix $\tilde{\rho}^f$ from Eq. (2.10) to be

$$\dot{\tilde{\rho}}^f=-i(\Omega-\nu)[a^\dagger a,\tilde{\rho}^f]-i\sum_j\Theta(t-t_j)\mathrm{Tr}_{A^j}[\tilde{V}_j,\tilde{\rho}_j^f]+\hat{\mathcal{L}}\tilde{\rho}^f . \tag{2.21}$$

We have added the last term to describe the cavity loss due to the interaction with a loss reservoir

$$\hat{\mathcal{L}}\tilde{\rho}^f=\tfrac{1}{2}\gamma(2a\tilde{\rho}^f a^\dagger-a^\dagger a\tilde{\rho}^f-\tilde{\rho}^f a^\dagger a) , \tag{2.22}$$

where γ is the cavity loss rate. In order to obtain an equation of motion for $\tilde{\rho}_j^f$ we use assumption (i) stated previously, i.e., the evolution of an atom in the laser field is independent of those of all other atoms. Consequently, we have

$$\dot{\tilde{\rho}}_j^f=-i\Theta(t-t_j)[\tilde{V}_j,\tilde{\rho}_j^f]-\tfrac{1}{2}(\Gamma^j\tilde{\rho}_j^f+\tilde{\rho}_j^f\Gamma^j) , \tag{2.23}$$

where

$$\Gamma^j=\sum_{A=a,b}\Gamma_A|A^j\rangle\langle A^j| \tag{2.24}$$

is the decay operator for the jth atom when there is no atomic collision.

Equations (2.21) and (2.23) are our two basic equations. Note that they differ from the usual method used in the Scully-Lamb theory in the form of $\tilde{V}_j$. When one first calculates the contribution of one atom injected at time t_j and then sums over injection times t_j, the exponential factor $e^{\pm i(\Delta t-\omega t_j)}$ in $\tilde{V}_j$ becomes $e^{\pm i\Delta(t-t_j)}$. It turns out that the form of $\tilde{V}_j$ with the factor $e^{\pm i\Delta(t-t_j)}$ happens to give the correct answer only when population pumping is involved. In fact, no unitary transformation of the interaction Hamiltonian V in the Schrödinger picture will lead to $\tilde{V}_j$ with the factor $e^{\pm i\Delta(t-t_j)}$. We shall return to this point in Sec. VI.

Summation over the randomly injected atoms in Eq. (2.21) can be replaced by integration over the injection time t_j, i.e., $\sum_j\to r_a\int_{-\infty}^{t}dt_j$, where r_a is the atomic injection rate. Substituting Eqs. (2.12) and (2.22) into Eq. (2.21) one arrives at the following equation for the field-density matrix element $\tilde{\rho}_{n,m}$:

$$\dot{\tilde{\rho}}_{nm} = -i(\Omega-\nu)(n-m)\tilde{\rho}_{nm} - ir_a\int_{-\infty}^{t} dt_j \Theta(t-t_j) g\Big[\sqrt{n+1}e^{i(\Delta t-\omega t_j)}\tilde{\rho}^{j}_{bn+1,am} + \sqrt{n}\,e^{-i(\Delta t-\omega t_j)}\tilde{\rho}^{j}_{an-1,bm} - \tilde{\rho}^{j}_{an,bm+1}\sqrt{m+1}e^{-i(\Delta t-\omega t_j)} - \tilde{\rho}^{j}_{bn,am-1}\sqrt{m}\,e^{i(\Delta t-\omega t_j)}\Big] + \gamma\sqrt{(n+1)(m+1)}\tilde{\rho}_{n+1,m+1} - \tfrac{1}{2}\gamma(n+m)\tilde{\rho}_{nm} , \tag{2.25}$$

where, without ambiguity, the superscript f has been dropped from the field-density matrix elements. The density matrix elements $\tilde{\rho}^{j}_{An,A'm}$ ($A,A'=a,b$) are to be found from Eq. (2.23). As in the usual laser theory,[1-4] this can be accomplished by using assumption (ii) mentioned above, i.e., $\gamma \gg \Gamma_a,\Gamma_b$, to approximate $\tilde{\rho}^{f}(t_j)$ by $\tilde{\rho}^{f}(t)$. To facilitate the calculation of $\tilde{\rho}^{j}_{An,A'm}$, we introduce c^{j}_{An} such that

$$\tilde{\rho}^{j}_{An,A'm} = c^{j}_{An}(c^{j}_{A'm})^{*}, \quad A,A'=a,b . \tag{2.26}$$

The equations of motion for c^{j}_{An} are readily found from Eq. (2.23) as

$$\frac{d}{dt}\begin{bmatrix} c^{j}_{a,n-1} \\ c^{j}_{b,n} \end{bmatrix} = \begin{bmatrix} -\Gamma_a & -ig\sqrt{n}\,e^{-i(\Delta t-\omega t_j)} \\ ig\sqrt{n}\,e^{-i(\Delta t-\omega t_j)} & -\Gamma_b \end{bmatrix}\begin{bmatrix} c^{j}_{a,n-1} \\ c^{j}_{b,n} \end{bmatrix} , \tag{2.27}$$

starting from $t=t_j$. Since the jth atom is injected at time t_j, the initial condition is $\tilde{\rho}^{f}_{j}(t_j)=\tilde{\rho}^{f}(t_j)\otimes\tilde{\rho}^{j}(t_j)$, and the corresponding solution for Eqs. (2.27) is (for simplicity, we take $\Gamma_a=\Gamma_b\equiv\Gamma$)

$$c^{j}_{a,n-1}(t) = e^{-(\Gamma-i\Delta)(t-t_j)/2}\{[\cos y_n - i(\Delta/\Omega_n)\sin y_n]c^{j}_{a,n-1}(t_j) - i(2g\sqrt{n}/\Omega_n)(\sin y_n)c^{j}_{b,n}(t_j)e^{-i\nu t_j}\} , \tag{2.28a}$$

$$c^{j}_{b,n}(t) = e^{-(\Gamma+i\Delta)(t-t_j)/2}\{[\cos y_n + i(\Delta/\Omega_n)\sin y_n]c^{j}_{b,n}(t_j) - i(2g\sqrt{n}/\Omega_n)(\sin y_n)c^{j}_{a,n-1}(t_j)e^{i\nu t_j}\} , \tag{2.28b}$$

where

$$y_n = \tfrac{1}{2}\Omega_n(t-t_j), \quad \Omega_n = (4ng^2+\Delta^2)^{1/2} . \tag{2.29}$$

We assume that the initial conditions for the injected atoms are

$$\rho^{j}(t_j) = \begin{bmatrix} \rho_{aa} & \bar{\rho}_{ab}e^{-i\nu t_j} \\ \bar{\rho}_{ba}e^{i\nu t_j} & \rho_{bb} \end{bmatrix}, \quad j=1,2,\ldots \tag{2.30}$$

where ρ_{aa}, ρ_{bb}, and $\bar{\rho}_{ab}=\bar{\rho}^{*}_{ba}$ are the same for all atoms. The coarse-grained time rate of change for the laser field is obtained by substituting Eqs. (2.28), via Eq. (2.26), into Eq. (2.25) and using Eqs. (2.20), (2.29), and (2.30). The master equation for the laser field is finally found after the integration to be

$$\begin{aligned}\dot{\tilde{\rho}}_{nm} = \{ & -\tfrac{1}{2}\alpha\rho_{aa}\tilde{\rho}_{nm}[n+1+m+1+i(n-m)\delta+g^2(n-m)^2/\Gamma^2] + \alpha\rho_{bb}\tilde{\rho}_{n+1,m+1}\sqrt{(n+1)(m+1)} \\ & + iS\bar{\rho}_{ab}\tilde{\rho}_{n,m+1}\sqrt{m+1}[1+g^2(m-n)/\Gamma(\Gamma-i\Delta)] \\ & - iS^{*}\bar{\rho}_{ba}\tilde{\rho}_{n+1,m}\sqrt{n+1}[1+g^2(n-m)/\Gamma(\Gamma+i\Delta)]\}/\xi_{nm} \\ + \{ & \alpha\rho_{aa}\tilde{\rho}_{n-1,m-1}\sqrt{nm} - \tfrac{1}{2}\alpha\rho_{bb}\tilde{\rho}_{nm}[n+m+i(m-n)\delta+g^2(n-m)^2/\Gamma^2] \\ & - iS\bar{\rho}_{ab}\tilde{\rho}_{n-1,m}\sqrt{n}\,[1+g^2(n-m)/\Gamma(\Gamma-i\Delta)] \\ & + iS^{*}\bar{\rho}_{ba}\tilde{\rho}_{n,m-1}\sqrt{m}\,[1+g^2(m-n)/\Gamma(\Gamma+i\Delta)]\}/\xi_{n-1,m-1} \\ & - i(\Omega-\nu)(n-m)\tilde{\rho}_{nm} + \gamma\sqrt{(n+1)(m+1)}\tilde{\rho}_{n+1,m+1} - \tfrac{1}{2}\gamma(n+m)\tilde{\rho}_{nm} ,\end{aligned} \tag{2.31}$$

with

$$\xi_{nm} = 1 + \frac{\beta}{2\alpha}(n+1+m+1) + \frac{\beta^2}{16\alpha^2}(1+\delta^2)(n-m)^2 , \tag{2.32}$$

where

$$\alpha=\frac{2r_a g^2}{\Gamma^2+\Delta^2},\quad \beta=\frac{8r_a g^4}{(\Gamma^2+\Delta^2)^2},\quad S=\frac{r_a g}{\Gamma+i\Delta},\quad \delta=\Delta/\Gamma\ . \tag{2.33}$$

α and β are the linear-gain coefficient and saturation parameter, respectively. Equation (2.31) reduces to the familiar field master equation[1] when $\bar{\rho}_{ab}=\bar{\rho}^*_{ba}=0$.

III. PHOTON STATISTICS

The equation of motion for the diagonal element $\tilde{\rho}_{nn}$ of the field-density matrix is obtained from Eqs. (2.31) and (2.32) by setting $m=n$,

$$\begin{aligned}\dot{\tilde{\rho}}_{nn}=&-[1+(n+1)\beta/\alpha]^{-1}[\alpha(n+1)(\rho_{aa}\tilde{\rho}_{nn}-\rho_{bb}\tilde{\rho}_{n+1,n+1})-\sqrt{n+1}(iS\bar{\rho}_{ab}\tilde{\rho}_{n,n+1}+\text{c.c.})]\\&+(1+n\beta/\alpha)^{-1}[\alpha n(\rho_{aa}\tilde{\rho}_{n-1,n-1}-\rho_{bb}\tilde{\rho}_{nn})-\sqrt{n}\,(iS\bar{\rho}_{ab}\tilde{\rho}_{n-1,n}+\text{c.c.})]+\gamma(n+1)\tilde{\rho}_{n+1,n+1}-\gamma n\tilde{\rho}_{nn}\ .\end{aligned} \tag{3.1}$$

This becomes the usual probability flow equation[1] when $\bar{\rho}_{ab}=0$. Notice that, besides the usual diagonal coupling between $\tilde{\rho}_{nn}$ and $\tilde{\rho}_{n+1,n+1}$, we now have additional coupling to off-diagonal density matrix elements $\tilde{\rho}_{n,n+1}$ and $\tilde{\rho}_{n+1,n}$. The flow of probability for finding n photons is plotted in Fig. 1 for the case $\sin(\phi-\theta+\arctan\delta)<0$, where the two phase angles ϕ and θ are defined by

$$\bar{\rho}_{ab}=|\bar{\rho}_{ab}|e^{i\theta}\ , \tag{3.2a}$$

and

$$\tilde{\rho}_{n,n-1}=|\tilde{\rho}_{n,n-1}|e^{i\phi}\ . \tag{3.2b}$$

In this case, atomic coherence increases the mean photon number $\langle n\rangle=n_0$. For the opposite case, i.e., $\sin(\phi-\theta+\arctan\delta)>0$, atomic coherence decreases the mean photon number.

In the steady state the photon-number distribution does not vary with time, i.e., $\dot{\tilde{\rho}}_{nn}=0$, and the phase ϕ is locked to a particular value ϕ_0 because $\tilde{\rho}_{n,n-1}$ does not vanish in the steady state. It is easy to see from Eq. (3.1) that this is satisfied when

$$\begin{aligned}(1+n\beta/\alpha)^{-1}[\alpha n(\rho_{aa}\tilde{\rho}_{n-1,n-1}-\rho_{bb}\tilde{\rho}_{nn})\\-\sqrt{n}\,(iS\bar{\rho}_{ab}\tilde{\rho}_{n-1,n}+\text{c.c.})]=\gamma n\tilde{\rho}_{nn}\ ,\end{aligned} \tag{3.3}$$

which is just detailed balancing of the photon-number flux. Due to the coupling to off-diagonal density matrix elements $\tilde{\rho}_{n-1,n}=\tilde{\rho}^*_{n,n-1}$, however, there exists no recurrence relation between $\tilde{\rho}_{nn}$ and $\tilde{\rho}_{n-1,n-1}$ here, in contrast to the usual laser case. A general discussion of the laser photon statistics as well as other laser operation problems can be carried out via the Fokker-Planck equation approach. This is the subject of Sec. IV. Nevertheless, we point out here a special case for which the photon statistics can be solved exactly. In the steady state when

$$\begin{aligned}&\rho_{aa}=1-\rho_{bb}=1-\frac{\gamma}{\alpha},\quad |\bar{\rho}_{ab}|=\sqrt{\rho_{aa}\rho_{bb}}\ ,\\&\phi=\phi_0=\theta-\arctan\delta-\tfrac{1}{2}\pi\ ,\end{aligned} \tag{3.4}$$

we find that

$$\tilde{\rho}_{nn}=e^{-n_0}\frac{n_0^n}{n!}\ , \tag{3.5a}$$

$$\tilde{\rho}_{n,n-1}=\tilde{\rho}^*_{n-1,n}=e^{-n_0}\frac{n_0^{n-1/2}e^{i\phi_0}}{\sqrt{n!(n-1)!}}\ , \tag{3.5b}$$

where

$$n_0=\frac{\alpha}{\gamma}\left[\frac{\alpha-\gamma}{\beta}\right] \tag{3.6}$$

is the mean photon number in this case, which happens to be the same as the average photon number in an ordinary laser with $\rho_{aa}=1$ and $\rho_{bb}=0$ [see Eq. (4.10)]. We see that the photon-number distribution is *exactly* Poissonian and the off-diagonal elements $\tilde{\rho}_{n,n-1}$ are the same as those in the coherent state $|\sqrt{n_0}e^{i\phi_0}\rangle$.

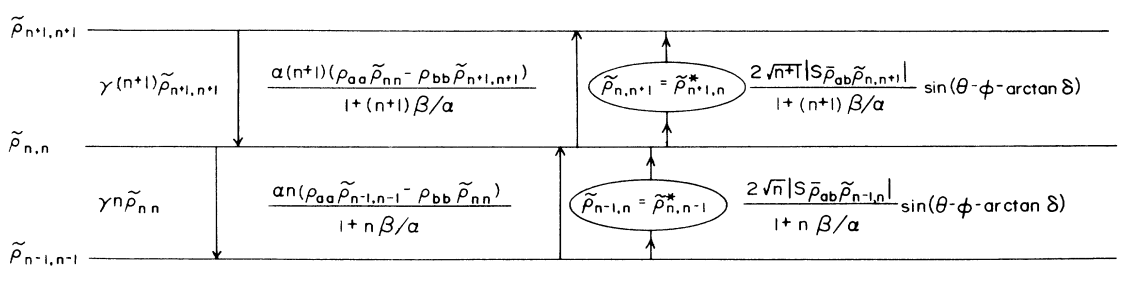

FIG. 1. Flow of probability for finding n photons in a coherently pumped laser for the case $\sin(\phi-\theta+\arctan\delta)<0$.

IV. FOKKER-PLANCK EQUATION

In this section, we transform the field master equation (2.31) into a Fokker-Planck equation by expanding the field-density operator $\tilde{\rho}^f$ in terms of the diagonal P representation. For the field-density matrix elements the expansions are[15]

$$\tilde{\rho}_{nm}=\int d^2\mathcal{E}\,P(\mathcal{E})e^{-|\mathcal{E}|^2}\frac{\mathcal{E}^n(\mathcal{E}^*)^m}{\sqrt{n!m!}} . \tag{4.1}$$

Assuming that the mean photon number is large, we neglect 1 compared to $|\mathcal{E}|^2$ and obtain the following equation of motion for $P(\mathcal{E})$:

$$\begin{aligned}\frac{\partial P(\mathcal{E})}{\partial t}=&\left[-\tfrac{1}{2}\alpha(\rho_{aa}-\rho_{bb})\left[(1-i\delta)\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+(1+i\delta)\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]+\alpha\rho_{aa}\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}\right.\\&\left.-\tfrac{1}{8}\beta(\rho_{aa}+\rho_{bb})(1+\delta^2)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]^2\right]M(\mathcal{E})\\&+\left\{iS\bar{\rho}_{ab}\left[\frac{\partial}{\partial\mathcal{E}}+\frac{2g^2}{\Gamma(\Gamma-i\Delta)}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]\mathcal{E}^*+\frac{g^2}{\Gamma(\Gamma-i\Delta)}\frac{\partial}{\partial\mathcal{E}}\left[\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*-\frac{\partial}{\partial\mathcal{E}}\mathcal{E}\right]\right]M(\mathcal{E})+\text{c.c.}\right\}\\&+\frac{\gamma}{2}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]P(\mathcal{E})+i(\Omega-\nu)\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]P(\mathcal{E}) ,\end{aligned} \tag{4.2}$$

with

$$M(\mathcal{E})=\left[1+\frac{\beta}{\alpha}|\mathcal{E}|^2-\frac{\beta}{2\alpha}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]+\frac{\beta^2(1+\delta^2)}{16\alpha^2}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}-\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]^2\right]^{-1}P(\mathcal{E}) . \tag{4.3}$$

Equation (4.2) contains derivatives of all orders[15] in $\mathcal{E}$ and $\mathcal{E}^*$ due to the presence of the inverse operator in $M(\mathcal{E})$. Expanding Eq. (4.2) up to a second order in the derivatives and again neglecting 1 compared to $|\mathcal{E}|^2$, we arrive at a Fokker-Planck equation. Its explicit expression is given in the Appendix.

This Fokker-Planck equation can be expressed in terms of intensity and phase variables, I and ϕ, through the relation[3] $\mathcal{E}=\sqrt{I}e^{i\phi}$ (again dropping 1 compared with I)

$$\frac{\partial P(I,\phi)}{\partial t}=\left[-\frac{\partial}{\partial I}d_I-\frac{\partial}{\partial\phi}d_\phi+\frac{\partial^2}{\partial I^2}D_{II}+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}+2\frac{\partial^2}{\partial I\partial\phi}D_{I\phi}\right]P(I,\phi) , \tag{4.4}$$

where

$$d_I=I\left[\frac{\alpha(\rho_{aa}-\rho_{bb})}{1+I\beta/\alpha}-\gamma\right]-\frac{2|S\bar{\rho}_{ab}|\sqrt{I}}{1+I\beta/\alpha}\sin(\phi-\theta+\arctan\delta) , \tag{4.5a}$$

$$d_\phi=\nu-\Omega-\frac{\alpha(\rho_{aa}-\rho_{bb})\delta}{2(1+I\beta/\alpha)}-\frac{|S\bar{\rho}_{ab}|}{\sqrt{I}(1+I\beta/\alpha)}\left[\cos(\phi-\theta+\arctan\delta)+\frac{\beta I}{\alpha}(1+\delta^2)^{1/2}\cos(\phi-\theta)\right] , \tag{4.5b}$$

$$D_{II}=\frac{\alpha I}{(1+I\beta/\alpha)^2}\left[\rho_{aa}+\rho_{bb}\frac{\beta I}{\alpha}+2|\bar{\rho}_{ab}|\left[\frac{\beta I}{\alpha}\right]^{1/2}\sin(\phi-\theta+\arctan\delta)\right] , \tag{4.5c}$$

$$D_{\phi\phi}=\frac{\alpha}{4I(1+I\beta/\alpha)}\left[\rho_{aa}+\frac{1+\delta^2}{2}\frac{\beta I}{\alpha}(\rho_{aa}+\rho_{bb})+|\bar{\rho}_{ab}|\left[(1+\delta^2)\frac{\beta I}{\alpha}\right]^{1/2}\sin(\phi-\theta)\right] , \tag{4.5d}$$

$$D_{I\phi}=\frac{\alpha}{4(1+I\beta/\alpha)^2}\left\{\frac{\beta I}{\alpha}(\rho_{aa}-\rho_{bb})\delta+\left[\frac{\beta I}{\alpha}\right]^{1/2}|\bar{\rho}_{ab}|\left[2\cos(\phi-\theta+\arctan\delta)+(1+\delta^2)^{1/2}\left[\frac{\beta I}{\alpha}-1\right]\cos(\phi-\theta)\right]\right\} . \tag{4.5e}$$

Here d_I and d_ϕ are the intensity and phase drift coefficients, respectively, D_{II} and $D_{\phi\phi}$ are the intensity and phase diffusion coefficients, respectively, whereas $D_{I\phi}$ is a cross diffusion coefficient. The laser system considered in this paper can be studied in terms of these drift and diffusion coefficients. In contrast to the usual laser case, all of these coefficients are functions of I and ϕ. For example, when $\phi=\theta-\arctan\delta-\frac{1}{2}\pi$ and $|\bar{\rho}_{ab}|=(\rho_{aa}\rho_{bb})^{1/2}$,

$$D_{II}=\frac{\alpha\rho_{aa}I}{(1+I\beta/\alpha)^2}\left[1-\left[\frac{\rho_{bb}}{\rho_{aa}}\frac{\beta I}{\alpha}\right]^{1/2}\right]^2 , \tag{4.6}$$

which vanishes at $I=(\rho_{aa}/\rho_{bb})(\alpha/\beta)$. The phase diffusion coefficient $D_{\phi\phi}$ can also become smaller than that in an ordinary laser but always remains positive. The actual values of the diffusion coefficients in the steady state are determined by the steady-state values of I and ϕ. In the P representation, the photon-number variance is

$$\langle(\Delta\hat{n})^2\rangle=\langle:(\Delta\hat{n})^2:\rangle+\langle\hat{n}\rangle=\langle(\delta I)^2\rangle+\langle I\rangle \ , \tag{4.7a}$$

and, in the presence of phase locking, the phase variance is

$$\langle(\Delta\phi)^2\rangle=\langle:(\Delta\phi)^2:\rangle+\frac{1}{4\langle\hat{n}\rangle}=\langle(\delta\phi)^2\rangle+\frac{1}{4\langle I\rangle} \ , \tag{4.7b}$$

where $\hat{n}=a^\dagger a$ is the photon-number operator, $::$ denotes the normal ordering of the operators, $\delta I=I-\langle I\rangle$, and $\delta\phi=\phi-\langle\phi\rangle$. From Eq. (4.4) one finds the equations of motion for the intensity and phase to be

$$\frac{d}{dt}\langle I\rangle=\langle d_I\rangle \ , \tag{4.8a}$$

$$\frac{d}{dt}\langle\phi\rangle=\langle d_\phi\rangle \ , \tag{4.8b}$$

and the equations of motion for the normally ordered photon-number variance and phase variance to be

$$\frac{d}{dt}\langle(\delta I)^2\rangle=2\langle d_I\delta I\rangle+2\langle D_{II}\rangle \ , \tag{4.9a}$$

$$\frac{d}{dt}\langle(\delta\phi)^2\rangle=2\langle d_\phi\delta\phi\rangle+2\langle D_{\phi\phi}\rangle \ . \tag{4.9b}$$

Assuming that the steady-state quasiprobability distribution $P(I,\phi)$ is sharply peaked at the mean photon number $\langle I\rangle=n_0$ and, if it exists, the locked phase value $\langle\phi\rangle=\phi_0$, then n_0 and ϕ_0 satisfy the deterministic equations $d_I(n_0,\ \phi_0)=0$ and $d_\phi(n_0,\ \phi_0)=0$, as indicated by Eqs. (4.8). In the following we study the cases $\bar{\rho}_{ab}=0$ and $\bar{\rho}_{ab}\neq 0$ separately.

A. Incoherently pumped laser, $\bar{\rho}_{ab}=0$

In order to compare with an ordinary two-level laser to see the effect of injected atomic coherence on laser operation, we first review the results for a laser with $\bar{\rho}_{ab}=0$. In this case, none of the drift and diffusion coefficients depends on ϕ [see Eqs. (4.5)], and population inversion $\rho_{aa}>\rho_{bb}$ is necessary for achieving laser operation. We first obtain the mean photon number[1,2] n_0 from $d_I(n_0)=0$,

$$n_0=\frac{\alpha}{\gamma}\frac{\alpha(\rho_{aa}-\rho_{bb})-\gamma}{\beta} \ . \tag{4.10}$$

Using Eq. (4.10) we then find the laser frequency[2,3] ν in a steady-state operation from $d_\phi(n_0)=0$ to be

$$\nu=\frac{\Gamma\Omega+\frac{1}{2}\gamma\omega}{\Gamma+\frac{1}{2}\gamma} \ , \tag{4.11}$$

and the steady-state diffusion coefficients from Eqs. (4.5c)–(4.5e) for $\Delta=0$ to be

$$\begin{aligned}D_{II}(n_0)&=\frac{\gamma(\gamma+\alpha\rho_{bb})n_0}{\alpha(\rho_{aa}-\rho_{bb})}\\ D_{\phi\phi}(n_0)&=\frac{\alpha(\rho_{aa}+\rho_{bb})+\gamma}{8n_0} \ ,\\ D_{I\phi}&=0 \ .\end{aligned} \tag{4.12}$$

Laser frequency pulling is apparent in Eq. (4.11), $D_{\phi\phi}(n_0)$ in Eqs. (4.12) gives half the natural linewidth of the laser[2,3] and $D_{II}(n_0)$ is related to the steady-state photon-number variance. The normally ordered photon-number variance $\langle:(\Delta\hat{n})^2:\rangle=\langle(\delta I)^2\rangle$ in the steady state can be obtained from Eq. (4.9a) by setting $d/dt=0$ and expanding d_I and D_{II} around $I=n_0$ up to first order in δI. Substituting the resulting expression for $\langle(\delta I)^2\rangle$ into Eq. (4.7a) and using Eqs. (4.10) and (4.12), we find that the total steady-state photon-number variance for $\Delta=0$ is[1]

$$\begin{aligned}\langle(\Delta\hat{n})^2\rangle&=n_0-\frac{D_{II}(n_0)}{\partial d_I(n_0)/\partial I}\\ &=\frac{\alpha\rho_{aa}}{\alpha(\rho_{aa}-\rho_{bb})-\gamma}n_0 \ ,\end{aligned} \tag{4.13}$$

which is larger than that of a Poisson distribution with the same mean photon number n_0.

B. Coherently pumped laser, $\bar{\rho}_{ab}\neq 0$

For simplicity, we consider the resonant case $\Omega=\omega$ in this section. From symmetry consideration we have $\nu=\Omega$ here as in the case of $\bar{\rho}_{ab}=0$ [see Eq. (4.11)]. Consequently, Eq. (4.5b) reduces to

$$d_\phi=-\frac{|S\bar{\rho}_{ab}|}{\sqrt{I}}\cos(\phi-\theta) \ . \tag{4.14}$$

Because of the injected atomic coherence, the laser phase ϕ is locked to a particular value ϕ_0 in the steady state. Using Eq. (4.14) we find the stable solution from $d_\phi(\phi_0)=0$ [cf. Eq. (4.8b)] and $\partial d_\phi(\phi_0)/\partial\phi<0$ to be

$$\phi_0=\theta-\tfrac{1}{2}\pi \ . \tag{4.15}$$

[Note that $\partial d_\phi(\phi_0)/\partial I=0$.] Knowing ϕ_0, the mean photon number n_0 in the steady state can be found from $d_I(n_0,\phi_0)=0$, i.e.,

$$(n_0\beta/\alpha)^{3/2}+[1-(\rho_{aa}-\rho_{bb})\alpha/\gamma](n_0\beta/\alpha)^{1/2}-2|\bar{\rho}_{ab}|\alpha/\gamma=0 \ . \tag{4.16}$$

This is a third-order algebraic equation for $(n_0\beta/\alpha)^{1/2}$. There exists a standard formula for its general solutions. It is easy to show that (i) Eq. (4.16) gives only one positive solution for $(n_0\beta/\alpha)^{1/2}$, and (ii) this n_0 satisfies $\partial d_I(n_0,\ \phi_0)/\partial I<0$ and, consequently, is always a stable solution. With the same parameters α, β, γ, ρ_{aa}, and ρ_{bb} satisfying $\alpha(\rho_{aa}-\rho_{bb})>\gamma$, n_0 found here is larger than n_0 in Eq. (4.10). We note that, for the laser to remain an active device, population inversion $\rho_{aa}>\rho_{bb}$ is not necessary when $|\bar{\rho}_{ab}|\neq 0$.[16] Namely, we can have "lasing without population inversion" in the coherently pumped laser. The reason is that, even though the laser gain is negative when $\rho_{aa}<\rho_{bb}$, the injected atomic coherence $\bar{\rho}_{ab}$ acts here as a driving force for the laser intensity, as can be seen from Eq. (4.5a) by noting that the drift coefficient for the laser amplitude $r=\sqrt{I}$ is related to the intensity drift coefficient through the relation $d_r=(2r)^{-1}\ d_I$ when $r\gg 1$. Using Eq. (4.15), the steady-state diffusion coefficients become [see Eqs. (4.5c)–(4.5e)]

$$D_{II}(n_0,\phi_0)=\frac{\alpha n_0}{(1+n_0\beta/\alpha)^2}\times[\rho_{aa}+\rho_{bb}(n_0\beta/\alpha)-2|\bar{\rho}_{ab}|(n_0\beta/\alpha)^{1/2}]\ , \tag{4.17a}$$

$$D_{\phi\phi}(n_0,\phi_0)=\frac{\alpha}{4n_0(1+n_0\beta/\alpha)}\times[\rho_{aa}+\tfrac{1}{2}(n_0\beta/\alpha)-|\bar{\rho}_{ab}|(n_0\beta/\alpha)^{1/2}]\ , \tag{4.17b}$$

$$D_{I\phi}(\phi_0)=0\ , \tag{4.17c}$$

where $\rho_{aa}+\rho_{bb}=1$ has been used in Eq. (4.17b). We see that the initial atomic coherence $\bar{\rho}_{ab}$ decreases both $D_{II}(n_0,\ \phi_0)$ and $D_{\phi\phi}(n_0,\ \phi_0)$. This implies that there is quantum noise reduction for both the intensity and phase in the laser with injected atomic coherence. It should be noted that $D_{\phi\phi}(n_0,\ \phi_0)$ as given by Eq. (4.17b) is smaller than that in an incoherently pumped laser ($|\bar{\rho}_{ab}|=0$), Eq. (4.12), with the same parameters α, β, γ, ρ_{aa}, and ρ_{bb}. In particular, when

$$\rho_{aa}=1-\rho_{bb}=1-\frac{\gamma}{\alpha},\quad |\bar{\rho}_{ab}|=\sqrt{\rho_{aa}\rho_{bb}}, \tag{4.18}$$

and accordingly [satisfying $d_I(n_0,\phi_0)=0$ and being stable],

$$n_0=\frac{\alpha}{\gamma}\left[\frac{\alpha-\gamma}{\beta}\right], \tag{4.19}$$

we find

$$D_{II}(n_0,\phi_0)=0,\quad D_{\phi\phi}(n_0,\phi_0)=\frac{\alpha-\gamma}{8n_0}\ . \tag{4.20}$$

Notice that conditions (4.15) and (4.18) are the same as those in Eq. (3.4), since $\Delta=0$ here. Equations (4.18) and (4.19) give an example for our previous statement that, for the laser to remain an active device, population inversion $\rho_{aa}>\rho_{bb}$ is not necessary when $\bar{\rho}_{ab}\neq 0$, since $\rho_{aa}<\rho_{bb}$ in Eq. (4.18) and $n_0>0$ in Eq. (4.19) if $\gamma<\alpha<2\gamma$. Comparing Eqs. (4.20) with Eqs. (4.12), one sees that we have both complete spontaneous-emission noise quenching in the laser intensity and large phase noise reduction. The phase diffusion coefficient $D_{\phi\phi}$ decreases from the typical value $(\alpha+\gamma)/8n_0$ to $(\alpha-\gamma)/8n_0$ here. For a near-threshold case $\alpha=1.02\gamma$, we have the reduction of phase diffusion coefficient by a factor of 100 compared to an incoherently pump laser. Physically one may understand this noise reduction by noting that the rate of spontaneous emission is only $\alpha\rho_{aa}=0.02\alpha$ ($\ll\alpha$). We note that the reduction of the phase diffusion coefficient due to injected atomic coherence, discussed here, greatly exceeds that achieved by injecting a squeezed vacuum into a laser cavity,[17] in which case $D_{\phi\phi}$ can be reduced at most by a factor of 2.

In the following we discuss the steady-state variances in photon number and phase. From Eq. (4.5a) (with $\Delta=0$), (4.14), and (4.15), we find

$$\frac{\partial d_I(n_0,\phi_0)}{\partial\phi}=\frac{\partial d_\phi(n_0,\phi_0)}{\partial I}=0\ . \tag{4.21}$$

Expanding d_I, D_{II}, d_ϕ, and $D_{\phi\phi}$ in Eqs. (4.9) around the stable operation point $I=n_0$, $\phi=\phi_0$ up to first order in δI and $\delta\phi$ and using Eq. (4.7) and (4.21), we obtain the steady-state variances

$$\langle(\Delta\hat{n})^2\rangle=n_0+\frac{D_{II}(n_0,\phi_0)}{|\partial d_I(n_0,\phi_0)/\partial I|}\ , \tag{4.22a}$$

$$\langle(\Delta\phi)^2\rangle=\frac{1}{4n_0}+\frac{D_{\phi\phi}(n_0,\phi_0)}{|\partial d_\phi(n_0,\phi_0)/\partial\phi|}\ . \tag{4.22b}$$

When the initial atomic condition is that given by Eq. (4.18), we find

$$\partial d_I(n_0,\phi_0)/\partial I=\partial d_\phi(n_0,\phi_0)/\partial\phi=-\gamma$$

from Eqs. (4.5a), (4.14), (4.15), and (4.19). Consequently, we obtain from Eqs. (4.20) and (4.22)

$$\langle(\Delta\hat{n})^2\rangle=n_0\ , \tag{4.23a}$$

$$\langle(\Delta\phi)^2\rangle=\frac{1+(\alpha/\gamma)}{8n_0}\ , \tag{4.23b}$$

$$\langle(\Delta\hat{n})^2\rangle\langle(\Delta\phi)^2\rangle=\tfrac{1}{8}[1+(\alpha/\gamma)]\ . \tag{4.23c}$$

Equation (4.23a) also means that the normalized second-order correlation function $g^{(2)}(0)=\langle:\hat{n}^2:\rangle/\langle\hat{n}\rangle^2=1$ and agrees with Eq. (3.5a). When $\alpha\gtrsim\gamma$, Eq. (4.23b) becomes $\langle(\Delta\phi)^2\rangle\gtrsim(4n_0)^{-1}$, which means that the laser phase noise is very close to that in a coherent state, and Eq. (4.22c) becomes $\langle(\Delta\hat{n})^2\rangle\langle(\Delta\phi)^2\rangle\gtrsim\frac{1}{4}$, which means that the laser field approaches the quantum limit for the minimum uncertainty product. Overall, the laser field approaches a coherent state $|\sqrt{n_0}e^{i\phi_0}\rangle$ or, say, a classical field when $\alpha\gtrsim\gamma$.

V. PREPARATION OF INITIAL ATOMIC COHERENCE

In Secs. III and IV, we have seen the importance of the proper initial atomic coherence in the realization of quantum noise quenching in the two-level laser with injected atomic coherence. In this section we discuss the preparation of such initial atomic coherence. This discussion in this section is quite general and is not limited to the two-level laser studied here. In fact, a proper initial atomic coherence, i.e., $\rho_{ac}^j(t_j)\propto e^{-i2\nu t_j}$ is also vital in the two-photon correlated-spontaneous-emission laser[16] in which cascade three-level atoms are initially prepared in a coherent superposition of the top and bottom levels.

A proper form of the initial atomic coherence, i.e., $\rho^j_{ab}(t_j) \propto e^{-i\nu t_j}$ is also needed (i) in a micromaser in order to obtain symmetry breaking via off-diagonal atomic injection[18] and (ii) in a (V-type three-level) correlated-spontaneous-emission laser with injected atomic coherence.[12,13]

To be specific, we will discuss the preparation problem in the context of a two-level laser with injected atomic coherence. For simplicity, we assume that the jth atom is first pumped to level a (or level b), and then travels through a preparation field between time $t_j - T - \tau$ and $t_j - T$ (see Fig. 2). The preparation field couples two atomic states with frequency ν, which is close to $\omega = \omega_a - \omega_b$. Finally, the jth atom enters the laser cavity at time t_j. The interaction Hamiltonian between the preparation field and the jth atom in the Schrödinger picture can be put in the form

$$U^j = -\tfrac{1}{2}\hbar(\chi e^{-i\nu t}\sigma_j^\dagger + \chi^* e^{i\nu t}\sigma_j) \ , \tag{5.1}$$

where χ is the corresponding Rabi frequency and, for simplicity, we treat the preparation field classically. If we define a density-matrix-element column vector for the jth atom as

$$\rho^j = \begin{bmatrix} \rho^j_{aa} \\ \rho^j_{bb} \\ \rho^j_{ab} e^{i\nu t} \\ \rho^j_{ba} e^{-i\nu t} \end{bmatrix} , \tag{5.2}$$

where $\rho^j_{\alpha\beta}$ are the density matrix elements of the jth atom in the Schrödinger picture, then ρ^j satisfies the equation

$$\frac{d}{dt}\rho^j = \underline{L}\rho^j, \quad j = 1,2,\ldots \tag{5.3}$$

where the matrix $\underline{L}$ is defined by

$$\underline{L} = \begin{bmatrix} -\Gamma_a & 0 & -\tfrac{1}{2}i\chi^* & \tfrac{1}{2}i\chi \\ 0 & -\Gamma_b & \tfrac{1}{2}i\chi^* & -\tfrac{1}{2}i\chi \\ -\tfrac{1}{2}i\chi & \tfrac{1}{2}i\chi & -\Gamma_{ab} - i(\omega-\nu) & 0 \\ \tfrac{1}{2}i\chi^* & -\tfrac{1}{2}i\chi^* & 0 & -\Gamma_{ab} + i(\omega-\nu) \end{bmatrix} , \tag{5.4}$$

with $\Gamma_{ab} = \tfrac{1}{2}(\Gamma_a + \Gamma_b)$. The solution of Eq. (5.3) is (assuming χ=const)

$$\rho^j(t_j - T) = e^{\underline{L}\tau}\rho^j(t_j - T - \tau) = e^{\underline{L}\tau} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \tag{5.5}$$

which shows that $\rho^j_{ab}(t_j - T)e^{i\nu(t_j - T)} = (e^{\underline{L}\tau})_{31}$ is independent of t_j and j. At the injection time t_j,

$$\begin{aligned} \rho^j_{ab}(t_j) &= \rho^j_{ab}(t_j - T)e^{-i\omega T} \\ &= (e^{\underline{L}\tau})_{31} e^{-i(\omega-\nu)T - i\nu t_j} \ , \end{aligned} \tag{5.6}$$

which is just the desired form of the initial atomic coherence.

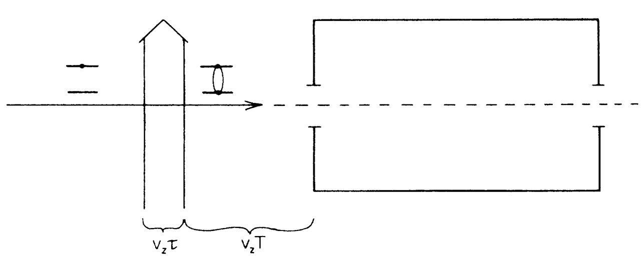

FIG. 2. A scheme to generate the proper form of the initial atomic coherence. Coherent superposition between levels a and b is produced by coherent excitation via a preparation field of frequency ν, leading to $\rho^j_{ab}(t_j) = \bar{\rho}_{ab} e^{-i\nu t_j}$.

VI. DISCUSSION

We address here the relation between the approach, developed in this paper, to treat lasers with injected atomic coherence and the usual approach used in the Scully-Lamb laser theory. During the last 20 years, only incoherent pumping was studied in laser physics, for which the Scully-Lamb laser theory is widely used. The method used to calculate the coarse-grained time rate of the change of laser field proceeds in two steps: (i) The contribution to laser field due to one atom (say, the jth atom) injected at random time t_j is first calculated. Usually this calculation is performed in an interaction picture. (ii) By summing over all injected atoms one obtains the total contribution. In step (i), when transforming from the Schrödinger picture to an injection picture, the atomic and field operators are transformed in the following way: $\sigma_j^\dagger \to \sigma_j^\dagger e^{i\omega(t-t_j)}$, $a \to a e^{-i\nu(t-t_j)}$, and $a^\dagger \to a^\dagger e^{i\nu(t-t_j)}$. In step (ii), an integration over t_j is performed. We point out here that the above transformation can be justified only for the system of a laser field interacting with *one* atom, but not for a system of a laser field interacting with many atoms injected at different times, which is just the case in actual laser systems. The field operator a in the interaction picture should take a form which does not change with t_j, e.g., $a \to a e^{-i\nu t}$, $a^\dagger \to a^\dagger e^{i\nu t}$; otherwise, in the process of atom-field interaction the relation between ρ^f and its counterpart in the interaction picture is not fixed. On the other hand, we note that the usual approach of the Scully-Lamb laser theory still gives the correct answer for an incoherently pumped laser, although an extra, incorrect, random phase factor $e^{i\nu t_j}$ is then associated with $e^{-i\nu t}$. The reason for this fact is that when there are only population terms in the initial atomic density matrix, the equation of motion for the reduced field-density operator $\tilde{\rho}^f$ contains only even-order terms in the coupling constant g, and a and $a^\dagger$ are paired in each of these terms, i.e., each of these terms contains equal numbers of a and $a^\dagger$. Consequently, the extra factor $e^{i\nu t_j}$ associated with $ae^{-i\nu t}$ is just canceled by the extra factor $e^{-i\nu t_j}$ associated with $a^\dagger e^{i\nu t}$ and the results happen to be correct. As a comparison, when there is initial atomic coherence, $\tilde{\rho}^f$ contains terms proportional to $\rho^j_{ab}(t_j)$. The field operators a and $a^\dagger$ are no longer paired in such terms. Consequently, the extra phase factor will not cancel and the resulting expression is not correct. Based on the above discussion we conclude that our approach developed here will give the same results as the Scully-Lamb theory if there is only incoherent pumping.

The quantum noise quieting and even quenching discovered in this work rely heavily on the proper form of the initial coherence given in Eq. (2.30). Compared to a laser with injected signal[6,7] of frequency ν, the dependence of the initial atomic coherence on the injection time t_j, $\rho^j_{ab}(t_j)\propto e^{-i\nu t_j}$ is similar to the time dependence of the injected signal $e^{-i\nu t}$. Also similar to the laser with injected signal, the injected atomic coherence here serves as a symmetry-breaking mechanism and acts as a driving force for the steady-state intensity. On the other hand, due to the nonlinearity of lasing atoms, the injected atomic coherence reduces both the laser intensity and phase diffusion coefficients in the steady state, which is absent in the laser with injected signal. In the following, we first examine what would happen if the initial atomic coherence takes other forms and then give an intuitive physical explanation for the initial atomic coherence given in Eq. (2.30).

Suppose that the initial atomic conditions are

$$\rho^j(t_j)=\begin{bmatrix}\rho_{aa} & \bar{\rho}_{ab}e^{-i\omega_0 t_j}\\ \bar{\rho}_{ba}e^{i\omega_0 t_j} & \rho_{bb}\end{bmatrix},\quad j=1,2,\ldots \tag{6.1}$$

where ρ_{aa}, ρ_{bb}, and $\bar{\rho}_{ab}=\bar{\rho}^*_{ba}$ are the same for all atoms and ω_0 can be zero or any other real value. Substituting Eqs. (2.28), via Eq. (2.26), into Eq. (2.25) and using Eqs. (2.20), (2.29), and (6.1), one obtains a master equation for the laser field which is similar to Eq. (2.31) but with the following changes in the $\bar{\rho}_{ab}$ and $\bar{\rho}_{ba}$ terms (including their denominators ξ_{nm} and $\xi_{n-1,m-1}$):

$$\begin{aligned}&\bar{\rho}_{ab}=\bar{\rho}^*_{ba}\to\bar{\rho}_{ab}e^{i(\nu-\omega_0)t}\,,\\&\Gamma\to\Gamma+i(\nu-\omega_0)\text{ in terms containing }\bar{\rho}_{ab}\,,\\&\Gamma\to\Gamma-i(\nu-\omega_0)\text{ in terms containing }\bar{\rho}_{ba}\,,\\&S\to\frac{r_a g}{\Gamma+i(\omega-\omega_0)}\,.\end{aligned} \tag{6.2}$$

When $\omega_0=0$, which means that all atoms have the same initial coherence, one sees that $S\to 0$, since $\omega\gg\Gamma$. Thus any possible quantum noise reduction caused by the injected atomic coherence is washed out in this case, as can be seen from Eq. (2.31). When ω_0 is close but not equal to ν, the phase angle θ in Eqs. (4.5) is replaced by $\theta+(\nu-\omega_0)t$, as implied by (6.2), and no stable phase locking is possible. Moreover, the diffusion coefficients D_{II} and $D_{\phi\phi}$ change periodically with time. Consequently, there is no stable quantum noise reduction when $\omega_0\neq\nu$.

A simple physical explanation to the required initial atomic coherence listed in Eq. (2.30) may be given in terms of the vector model of the optical Bloch equations, in which the laser field is treated classically. Considering steady-state laser operation in which the phase ϕ $(a|\mathcal{E}\rangle=\mathcal{E}|\mathcal{E}\rangle$, $\mathcal{E}=\sqrt{I}e^{i\phi})$ is locked to $\phi=\phi_0$, we may write the interaction Hamiltonian $\hbar V^{\rm cl}_j$ of the jth atom with the laser field in the Schrödinger picture as

$$V^{\rm cl}_j=g\sqrt{n_0}e^{i(\phi_0-\nu t)}\sigma^\dagger_j+g\sqrt{n_0}e^{i(\nu t-\phi_0)}\sigma_j\,, \tag{6.3}$$

which is the semiclassical version of Eq. (2.3) and $g\sqrt{n_0}$ is half the Rabi frequency. The jth atom interacting with the laser field in the cavity obeys the optical Bloch equations[19]

$$\frac{d}{dt}\mathbf{B}^j=\mathbf{\Omega}_B\times\mathbf{B}^j-\Gamma\mathbf{B}^j\,, \tag{6.4}$$

where

$$\mathbf{\Omega}_B=\begin{bmatrix}2g\sqrt{n_0}\cos\phi_0\\-2g\sqrt{n_0}\sin\phi_0\\\Delta\end{bmatrix} \tag{6.5a}$$

is the driving (laser) field vector and

$$\mathbf{B}^j=\begin{bmatrix}u^j\\v^j\\w^j\end{bmatrix}=\begin{bmatrix}\rho^j_{ab}e^{i\nu t}+\rho^j_{ba}e^{-i\nu t}\\i(\rho^j_{ab}e^{i\nu t}-\rho^j_{ba}e^{-i\nu t})\\\rho^j_{aa}-\rho^j_{bb}\end{bmatrix} \tag{6.5b}$$

is the Bloch vector for the jth atom. Substituting Eq. (2.30) into Eq. (6.5b) and using Eq. (3.2a), one obtains

$$\mathbf{B}^j(t_j)=\begin{bmatrix}2|\bar{\rho}_{ab}|\cos\theta\\-2|\bar{\rho}_{ab}|\sin\theta\\\rho_{aa}-\rho_{bb}\end{bmatrix}\,, \tag{6.6}$$

which is independent of j (see Fig. 3). In other words, all injected atoms with the initial condition (2.30) have the same initial atomic condition (or, say, orientation relative to $\mathbf{\Omega}_B$) in the Bloch vector space. Thus all $\mathbf{B}^j(t)$ have the same trajectories in the Bloch vector space and all atoms give the same contribution to the laser field. If, instead, the initial conditions (6.1) (with $\omega_0\neq\nu$) are satisfied, then different atoms have different initial orientations and consequently different trajectories in the Bloch vector space. The effects of the initial atomic coherence on the laser operation (e.g., phase locking and noise reduction), coming from each individual atom, tend to cancel in this case.

Besides accounting for the correct form of the initial atomic coherence shown in Eq. (2.30), the Bloch equations also provide an intuitive picture of the evolution of

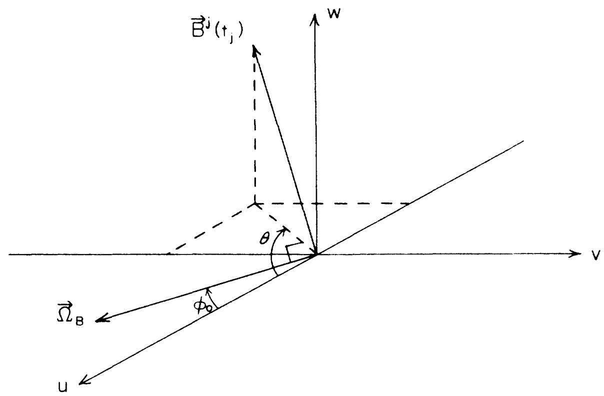

FIG. 3. Initial Bloch vector $\mathbf{B}^j(t_j)$ for the jth atom and steady-state driving field vector $\mathbf{\Omega}_B$ for the laser field at stable-locked phase $\phi=\theta-\frac{1}{2}\pi$ for $\Delta=0$ in the Bloch vector space.

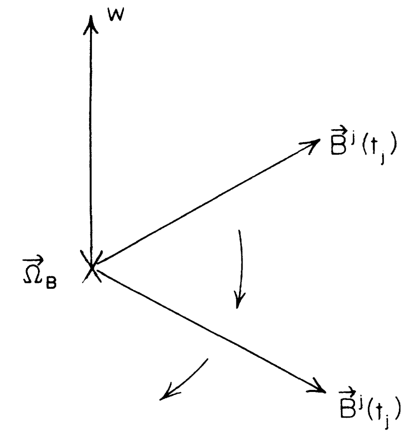

FIG. 4. The plane containing the w axis and $\mathbf{B}^j(t_j)$ in the Bloch vector space for $\Delta=0$. The plane is perpendicular to the driving field vector $\mathbf{\Omega}_B$, since $\phi_0=\theta-\frac{1}{2}\pi$. The symbol $\times$ denotes that the driving vector $\mathbf{\Omega}_B$ points into the paper.

the atomic density matrix elements [see Eq. (6.4)] and of the relation between the atomic phase θ and the locked phase ϕ_0 given in Eq. (4.14). We note that, owing to atomic decay, a physical process happening earlier has a larger probability than that happening later and thus dominates. When the laser field is resonant with the atomic transition $\Delta=0$, the driving vector $\mathbf{\Omega}_B$ is in the uv plane (see Fig. 3) and is perpendicular to the Bloch vector $\mathbf{B}^j(t_j)$ when $\phi_0=\theta\pm\frac{1}{2}\pi$, since

$$\mathbf{\Omega}_B\cdot\mathbf{B}^j(t_j)=4g\sqrt{n_0}|\bar{\rho}_{ab}|\cos(\phi_0-\theta)\ .$$

Consider $\phi_0=\theta-\frac{1}{2}\pi$ first (see Fig. 3), which is just Eq. (5.15). The driving vector $\mathbf{\Omega}_B$ makes the Bloch vector $\mathbf{B}^j$ $(j=1,2,\dots)$ rotate downward to $u=v=0$, $w<0$ (i.e., to $\rho^j_{aa}=0$). Stimulated emission takes place for the active atoms and the laser intensity is increased, which corresponds to a stable phase locking. For the other value $\phi_0=\theta+\frac{1}{2}\pi$, however, the Bloch vectors $\mathbf{B}^j$ $(j=1,2,\dots)$ are first rotated upward to $u=v=0$, $w>0$ (i.e., $\rho^j_{bb}=0$). Thus stimulated absorption occurs and the laser intensity decreases, which corresponds to an unstable phase locking. For other ϕ_0 values, $u=v=0$ cannot be reached. Consequently, they are unpreferred phase angles for the laser field. Our discussion here explains why the laser phase is locked to ϕ_0, as given in Eq. (4.15) for $\Delta=0$.

As opposed to ordinary lasers, lasing without inversion can be realized in lasers with injected atomic coherence. This can be physically understood from the Bloch equations. For simplicity, we look at the stable phase-locking case $\phi_0=\theta-\frac{1}{2}\pi$ for $\Delta=0$ (see Fig. 4). Regardless of the sign of $\rho_{aa}-\rho_{bb}$, the Bloch vectors $\mathbf{B}^j$ are rotated toward $u=v=0$, $w<0$ first to emit photons and thus the laser intensity is increased by atoms. In addition, it is apparent in the $\Delta=0$ case from the Bloch equations that when $\phi=\theta+\frac{1}{2}\pi$ (e.g., in an amplifier) atomic coherence decreases the photon numbers [see Eq. (4.5a)], and when $\phi=\theta,\theta+\pi$, the atomic coherence does not affect the laser intensity (as well as D_{II} and $D_{\phi\phi}$).

It seems that quantum noise reduction in both D_{II} and $D_{\phi\phi}$ may also be intuitively understood from the Bloch equations. For simplicity, we still consider the resonant case, $\Delta=0$. Equations (4.5) and (4.17) show that D_{II} and $D_{\phi\phi}$ are reduced when $\phi=\theta-\frac{1}{2}\pi$. From previous discussions we already know that stimulated emission takes a smaller fraction of a Rabi oscillation circle (compared to half of the Rabi oscillation circle in the usual $\rho^j_{aa}=1$ case). Since spontaneous emission is closely related to stimulated emission, the chance for spontaneous emission is reduced so that both the photon-number and phase diffusion coefficients D_{II} and $D_{\phi\phi}$ are reduced. On the other hand, both D_{II} and $D_{\phi\phi}$ with $\phi=\theta+\frac{1}{2}\pi$ are larger than those for $|\bar{\rho}_{ab}|=0$, since stimulated emission takes a larger fraction of the Rabi oscillation circle in this case (compared to the usual $\rho^j_{aa}=1$ case) so that the rate of spontaneous-emission events is increased.

Before summarizing our results in this work, we point out that in the derivation of our master equation for the reduced field-density operator, Eq. (2.31), we introduced the coarse-grained time-rate approximation—the laser field does not change appreciably on a time scale of atomic lifetime in the good cavity limit $\gamma\ll\Gamma$, i.e., $\tilde{\rho}^f(t_j)\simeq\tilde{\rho}^f(t)$. If, instead of the density matrix approach presented here, we perform a Langevin-equation linear theory analysis[20] (up to g^2 terms, valid for $\alpha\ll\gamma$), then (i) we find an extra term proportional to $|\bar{\rho}_{ab}|^2$ in the steady-state amplitude diffusion coefficient, which reduces the fluctuations in the amplitude quadrature of the laser field, and (ii) the amplitude and phase drift coefficients and the steady-state phase diffusion coefficient remain the same as in this paper. In a more elaborate nonlinear theory analysis,[21] which goes to all orders in the atom-field coupling constant g, our main discoveries (simultaneous noise reductions in the photon number and the laser phase, and the laser field can become very close to a coherent state) of the present paper are confirmed. In particular, when $\alpha\gtrsim\gamma$, corrections to the photon-number and phase variances given in Eqs. (4.23) are very small.

In summary, by generalizing the Scully-Lamb laser theory to a form appropriate for studying laser problems involving injected atomic coherence, we developed a quantum theory of the two-level single-mode coherently pumped laser. The generalization is made by treating the interaction of the laser field with many injected atoms simultaneously. We derived the master equation for the reduced field-density operator and study the laser photon statistics. We found that the photon-number distribution is *exactly* Poissonian for a particular initial atomic condition. We converted the field master equation into a Fokker-Planck equation and obtained drift and diffusion coefficients. Steady-state laser operation is discussed and it is shown that the laser phase symmetry is broken when there is injected atomic coherence, similar to the case of injecting an external field. The laser phase is actually locked to a particular value depending on the phase of the initial atomic coherence. The diffusion coefficients of both the laser intensity and phase are phase sensitive. In the steady state, they take the values at the locked phase angle and it turns out that both diffusion coefficients are reduced compared to the case of no initial atomic coherence. Namely, the injected atomic coherence reduces both photon-number noise and phase noise. The photon-number diffusion coefficient can vanish exactly for a special choice of the initial atomic variables and the resonant laser transition, in which case the photon statistics is Poissonian. When the photon-number diffusion co efficient vanishes, the phase diffusion coefficient becomes $(\alpha-\gamma)/8n_0$, where α, γ, and n_0 are the linear gain coefficient, cavity loss rate, and mean photon number, respectively. Compared to the usual phase diffusion constant $(\alpha+\gamma)/8n_0$ of an ordinary laser, this represents a very remarkable phase noise reduction. To achieve this noise reduction, it is vital that the initial phases of all injected atoms satisfy a proper relation $\rho_{ab}^j(t_j)\propto e^{-i\nu t_j}$ (ν is the laser frequency and t_j is the injection time of the jth atom). The photon-number and phase variances are also discussed and it is found that the coherence of the laser field can approach that of a coherent state when the photon-number distribution is Poissonian. In this case, the laser system becomes a quantum-noise-limited active device. A practical scheme to generate the proper form of initial atomic coherence necessary for this quantum noise quenching is proposed and analyzed. Its feasibility is demonstrated here. Finally, a comment on the relation between the approach developed here and the usual method used in the Scully-Lamb laser theory is made and it is shown that when coherent pumping in the laser is involved the present approach is necessary in order to obtain the correct results.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge helpful discussions with Professor M. O. Scully and Mr. C. Benkert. We gratefully acknowledge the support of the U.S. Office of Naval Research, and we are indebted to L. Pedrotti for a careful reading of the manuscript.

APPENDIX: THE FOKKER-PLANCK EQUATION IN TERMS OF $\mathcal{E}$ and $\mathcal{E}^*$

The Fokker-Planck equation in terms of complex variables $\mathcal{E}$ and $\mathcal{E}^*$ is obtained from Eq. (5.2) by first expanding $M(\mathcal{E})$ [see Eq. (5.3)] up to first order in the derivatives, yielding

$$M(\mathcal{E})=\frac{P(\mathcal{E})}{1+|\mathcal{E}|^2\beta/\alpha}+\frac{\beta}{2\alpha(1+|\mathcal{E}|^2\beta/\alpha)}\left[\frac{\partial}{\partial\mathcal{E}}\mathcal{E}+\frac{\partial}{\partial\mathcal{E}^*}\mathcal{E}^*\right]\frac{P(\mathcal{E})}{1+|\mathcal{E}|^2\beta/\alpha}\ , \tag{A1}$$

and then neglecting 1 compared to $|\mathcal{E}|^2$, leading to

$$\frac{\partial P(\mathcal{E})}{\partial t}=\left[\frac{\partial}{\partial\mathcal{E}}d_{\mathcal{E}}-\frac{\partial}{\partial\mathcal{E}^*}d_{\mathcal{E}^*}+2\frac{\partial^2}{\partial\mathcal{E}\partial\mathcal{E}^*}D_{\mathcal{E}^*\mathcal{E}}+\frac{\partial^2}{\partial\mathcal{E}^2}D_{\mathcal{E}\mathcal{E}}+\frac{\partial^2}{\partial\mathcal{E}^{*2}}D_{\mathcal{E}^*\mathcal{E}^*}\right]P(\mathcal{E})\ . \tag{A2}$$

Here

$$d_{\mathcal{E}}=\frac{\mathcal{E}}{2}\left[\frac{\alpha(\rho_{aa}-\rho_{bb})(1-i\delta)}{1+|\mathcal{E}|^2\beta/\alpha}-\gamma+2i(\nu-\Omega)\right]-i\left[S\bar{\rho}_{ab}\left[1+\frac{2g^2|\mathcal{E}|^2}{\Gamma(\Gamma-i\Delta)}\right]+S^*\bar{\rho}_{ba}\frac{2g^2\mathcal{E}^2}{\Gamma(\Gamma+i\Delta)}\right]\frac{1}{1+|\mathcal{E}|^2\beta/\alpha}\ , \tag{A3a}$$

$$d_{\mathcal{E}^*}=d_{\mathcal{E}}^*\ , \tag{A3b}$$

$$D_{\mathcal{E}^*\mathcal{E}}=\frac{4\alpha\rho_{aa}+\beta(\rho_{aa}+\rho_{bb})(1+\delta^2)|\mathcal{E}|^2}{8(1+|\mathcal{E}|^2\beta/\alpha)}-\frac{\beta(\rho_{aa}-\rho_{bb})|\mathcal{E}|^2}{4(1+|\mathcal{E}|^2\beta/\alpha)^2}+\left[\frac{iS\bar{\rho}_{ab}\mathcal{E}^*}{2(1+|\mathcal{E}|^2\beta/\alpha)}\left[\frac{g^2}{\Gamma(\Gamma-i\Delta)}+\frac{\beta/2\alpha}{1+|\mathcal{E}|^2\beta/\alpha}\right]+\text{c.c.}\right]\ , \tag{A3c}$$

$$D_{\mathscr{E}\mathscr{E}}=\frac{\beta(\rho_{aa}-\rho_{bb})(i\delta-1)\mathscr{E}^2}{4(1+|\mathscr{E}|^2\beta/\alpha)^2}-\frac{\beta(\rho_{aa}+\rho_{bb})(1+\delta^2)\mathscr{E}^2}{8(1+|\mathscr{E}|^2\beta/\alpha)}$$
$$+\frac{iS\bar{\rho}_{ab}\mathscr{E}}{1+|\mathscr{E}|^2\beta/\alpha}\left[\left[1+\frac{2g^2|\mathscr{E}|^2}{\Gamma(\Gamma-i\Delta)}\right]\frac{\beta/2\alpha}{1+|\mathscr{E}|^2\beta/\alpha}-\frac{g^2}{\Gamma(\Gamma-i\Delta)}\right]+\frac{iS^*\bar{\rho}_{ba}g^2\mathscr{E}^3\beta/\alpha}{\Gamma(\Gamma+i\Delta)(i+|\mathscr{E}|^2\beta/\alpha)^2}, \quad \text{(A3d)}$$

$$D_{\mathscr{E}^*\mathscr{E}^*}=D^*_{\mathscr{E}\mathscr{E}} . \quad \text{(A3e)}$$

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*On leave from the Central Research Institute for Physics, H-1525 Budapest 114, P.O. Box 49, Hungary.

[1] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967); M. O. Scully, D. M. Kim, and W. E. Lamb, Jr., Phys. Rev. A **2**, 2529 (1970).
[2] M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).
[3] M. Lax, in *Statistical Physics, Phase Transition and Superconductivity,* edited by M. Chrétien, E. P. Gross, and S. Deser (Gordon and Breach, New York, 1968), Vol. II; M. Lax and W. H. Louisell, Phys. Rev. **185**, 568 (1969); W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[4] H. Haken, *Laser Theory,* Vol. 25, Part 2c of *Handbuch der Physik,* edited by L. Genzel (Springer-Verlag, Berlin, 1970).
[5] M. O. Scully and S. Stenholm, Phys. Scr. **T21**, 119 (1988).
[6] V. DeGiorgio and M. O. Scully, Phys. Rev. A **2**, 1170 (1970).
[7] W. W. Chow, M. O. Scully, and E. W. Van Stryland, Opt. Commun. **15**, 6 (1975).
[8] R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).
[9] Y. Yamamoto and S. Machida, Phys. Rev. A **34**, 4025 (1986); S. Machida, Y. Yamamoto, and Y. Itaya, Phys. Rev. Lett. **58**, 1000 (1987).
[10] M. O. Scully, Phys. Rev. Lett. **55**, 2802 (1985).
[11] M. O. Scully and M. S. Zubairy, Phys. Rev. A **35**, 752 (1987).
[12] J. Bergou, M. Orszag, and M. O. Scully, Phys. Rev. A **38**, 768 (1988).
[13] K. Zaheer and M. S. Zubairy, Phys. Rev. A **38**, 5227 (1988).
[14] M. Scully, in *Proceedings of the International School of Physics, Enrico Fermi Course XLII,* edited by R. J. Glauber (Academic, New York, 1969).
[15] F. Casagrande and L. A. Lugiato, Phys. Rev. A **14**, 778 (1976); J. Gea-Banacloche, M. O. Scully, and M. S. Zubairy, Phys. Scr. **T21**, 81 (1988).
[16] M. O. Scully, K. Wódkiewicz, M. S. Zubairy, J. Bergou, N. Lu, and J. Meyer ter Vehn, Phys. Rev. Lett. **60**, 1832 (1988).
[17] J. Gea-Banacloche, Phys. Rev. Lett. **59**, 543 (1987).
[18] J. Krause, M. O. Scully, and H. Walther, Phys. Rev. A **34**, 2032 (1986).
[19] See, for example, L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
[20] C. Benkert, M. O. Scully, J. Bergou, and N. Lu (unpublished).
[21] N. Lu (unpublished).