Universality of off-shell two-body scattering amplitudes at negative and positive energies: Γ-matrix formalism

Sadhan K. Adhikari* and Kenneth L. Kowalski
Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106
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A generalized Heitler formalism is used to unify the description of various subtraction schemes that have appeared in the literature for calculating two-body off-shell amplitudes in terms of the solution Γ of an auxiliary nonsingular integral equation. This leads to a universal representation of the off-shell transition matrix for both positive and negative energies in terms of any one of a very general class of Γ matrices.

I. INTRODUCTION

It is well known that computational difficulties are caused by the $i\epsilon$ prescription in the kernel of the integral equation for the off-shell two-body transition matrix t_s when the parametric energy s has the form $s=|s|+i0$. A common method of circumventing these problems is to introduce an auxiliary nonsingular integral by means of a subtraction technique.[1] Since there are many ways of accomplishing this, it is not surprising that a number of proposals have appeared each of which seems to present certain advantages.[2-13] One of the most recent of these proposals is that of Stoof *et al.*,[14] who introduce a subtraction technique that appears to differ markedly from previous methods.

The extension of subtraction techniques to negative energies, $s<0$, where they are really not needed to remove scattering singularities, has provided another criterion for distinguishing among different methods. The $s<0$ region, of course is accessed kinematically when the two-body collision is part of a multiparticle process. Any two-particle bound states are manifested as poles in t_s when s is equal to the bound-state energy.

The primary objective of this paper is to show that for most purposes all of these subtraction techniques, whether at $s>0$ or $s<0$, are indistinguishable either as calculational devices or as vehicles for approximation techniques. We first show in Sec. II how all of the usual subtraction techniques can be regarded as following from a generalization of the Heitler formalism that is usually used to introduce the K_s matrix. Then the dynamical equations for the Γ_s matrices, which are generalizations of the K_s matrix, are determined for a general one-parameter class of subtraction functions. A new singular-kernel subtraction technique is introduced in Sec. III using a general class of two-parameter subtraction functions. This provides a general theoretical framework for the method of Ref. 14, as well as that of Ref. 15, within the context of our generalized Heitler formalism and the determination of the dynamical equations for the corresponding Γ_s matrices.

In both Sec. II and Sec. III we explicitly demonstrate how various rank-one approximations to the fully off-shell transition amplitude can be calculated using any one of the various Γ_s matrices. We also comment on the properties and expected accuracy of these approximations.

Along with the new development in Secs. II and III, we also clarify and unify previously obtained results. Our strategy for doing this is to highlight the universality of the Γ_s-matrix representation of the off-shell transition matrix that is a major consequence of our work. As we emphasize in Sec. IV, which summarizes our conclusions, it appears that the Γ_s-matrix formalism provides a coherent theoretical framework from which much of a large body of seemingly unrelated literature on off-shell scattering can be correlated and understood. It also allows us to point out some misconceptions in regard to the preferred use of particular subtraction schemes that have appeared in the literature. We hope we have identified a number of the major aspects of off-shell scattering that can now be considered to be entirely understood, thus isolating those aspects that are not.

II. UNIVERSAL Γ_S-MATRIX FORMALISM

Let us consider the integral equation for the off-shell two-body transition matrix $t_s(p,p')$ at parametric energy s in a given angular momentum state:

$$t_s(p,p')=V(p,p')+\int dq\, g(q^2,s)V(p,q)t_s(q,p')\ , \tag{1}$$

where

$$g(q^2,s)=\lambda q^2(s-q^2)^{-1}\ , \tag{2}$$

and where the integration limits here and throughout this paper are from 0 to ∞. For simplicity we suppose that the two particles are spinless[16,17] and we suppress the angular momentum index. The value of the constant λ depends upon the normalization convention used in the partial-wave decomposition. The non-negative real variables q, p, and p' denote the absolute values of the various wave vectors of relative motion. The parametric energy s can take any complex value except on the positive real axis where the appropriate limits must be taken; by positive s we always mean the limit $s\to|s|+i0^+$. We have denoted the functional dependence of the transition ma-

trix, as well as other quantities, on the parametric energy by means of a subscript s in order to distinguish it from the wave vectors referring to the initial and final momenta and the subtraction point. Finally, $V(p,p')$ is the partial-wave amplitude of the interparticle potential, which we take to be a real symmetric function of p and p'. Independently of that assumption on the potential, $t_s(p,p')$ also satisfies the integral equation obtained from Eq. (1) by interchanging the roles of V and t_s under the integral sign.

We now show that the content of the so-called Γ_s-matrix formalism can be viewed as a generalization of the off-shell K_s or Heitler formalism. This leads to the simplest and entirely self-contained derivation of most of the results already in the literature and suggests many more possibilities than have been considered so far.

Let a quantity $\Gamma_s(p,p';k)$ be defined in terms of the transition matrix $t_s(p,p')$ by

$$t_s(p,p')=\Gamma_s(p,p';k) + \Gamma_s(p,k;k)\int dq\, g(q^2,s)\gamma(k,q)t_s(q,p') , \quad (3)$$

where we leave $\gamma(k,q)$ arbitrary for the moment and k is an arbitrary non-negative parameter with the dimension of a wave vector. Since the kernel of Eq. (3) is of rank one, we see that

$$t_s(p,p')=\Gamma_s(p,p';k)+\Gamma_s(p,k;k)I_s(k,p') , \quad (4)$$

where

$$I_s(k,p')=\frac{\int dq\, g(q^2,s)\gamma(k,q)\Gamma_s(q,p';k)}{1-\int dq\, g(q^2,s)\gamma(k,q)\Gamma_s(q,k;k)} . \quad (5)$$

Therefore we deduce that

$$t_s(k,k)=\frac{\Gamma_s(k,k;k)}{1-\int dq\, g(q^2,s)\gamma(k,q)\Gamma(q,k;k)} \quad (6)$$

and

$$t_s(p,k)=\frac{\Gamma_s(p,k;k)}{\Gamma_s(k,k;k)}t_s(k,k) . \quad (7)$$

If $t_s(p,p')$ is symmetric in p and p', then we find from Eqs. (4) and (7) that

$$I_s(k,p')=t_s(k,k)\frac{\Gamma_s(p',k;k)}{[\Gamma_s(k,k;k)]^2}-\frac{\Gamma_s(k,p';k)}{\Gamma_s(k,k;k)} . \quad (8)$$

If we insert this expression for $I_s(k,p')$ back into Eq. (4) we obtain the canonical expression

$$t_s(p,p')=f_s(p,k)t_s(k,k)f_s(p',k)+R_s(p,p';k) , \quad (9)$$

$$f_s(p,k)=\frac{\Gamma_s(p,k;k)}{\Gamma_s(k,k;k)} , \quad (10)$$

and the representation

$$R_s(p,p';k)=\Gamma_s(p,p';k)-\frac{\Gamma_s(p,k;k)\Gamma_s(k,p';k)}{\Gamma_s(k,k;k)} \quad (11)$$

of the residual function which was first derived in Ref. 12 directly from Eq. (1) and Eqs. (4)–(8).

As pointed out in Ref. 10, any symmetric off-shell amplitude such as $t_s(p,p')$ admits the universal decomposition (9) into a purely off-shell part and a factorizable piece with

$$f_s(p,k)\equiv\frac{t_s(p,k)}{t_s(k,k)} , \quad (12)$$

$$R_s(p,k;k)=R_s(k,p';k)=0 , \quad (13)$$

where k is the absolute magnitude of an arbitrary wave vector. Equation (9) is clearly an algebraic identity; it holds for arbitrary k, p, p', and s, and it is independent of whether or not $t_s(p,p')$ satisfies Eq. (1). These assertions are obvious from the fact that

$$R_s(p,p';k)\equiv t_s(p,p')-f_s(p,k)t_s(k,k)f_s(p',k) , \quad (14)$$

with f_s defined by Eq. (12). For $s>0$, we always take $k=\sqrt{s}$. Then $t_s(k,k)$ is the on-shell amplitude and the form (9) becomes more interesting because time-reversal invariance and off-shell unitarity for $t_s(p,p')$ imply that both $f_s(p,k)$ and $R_s(p;p';k)$ are purely real.[10,18] For $s<0$, $t_s(p,p')$ as defined by (1) is real and so, therefore, are f_s and R_s. The factorizable bound-state poles that $t_s(p,p')$ may have at discrete negative s will reside in $t_s(k,k)$ as it appears in the representation (9), rather than in f_s or R_s.[8,10,19–21]

Unlike all previous derivations of Eqs. (9)–(11) the preceding one does not use dynamical scattering integral equations for either t_s or Γ_s. To say something about $\Gamma_s(p,p';k)$, presuming $t_s(p,p')$ is defined by (1), we need at least to constrain $\gamma(k,q)$. If $\gamma(k,q)$ is real and satisfies $\gamma(k,k)=1$, then off-shell unitarity for $t_s(p,p')$, for $s>0$, implies that $\Gamma_s(p,p';k)$ is real. For $s<0$, $t_s(p,p')$ is real and is $\Gamma_s(p,p';k)$ as well, so long as $\gamma(k,q)$ is real, whether or not $\gamma(k,k)=1$.

Generally, one obtains the integral equation for $\Gamma_s(p,p';k)$ from Eqs. (1) and (3):

$$\Gamma_s(p,p';k)=V(p,p')+\int dq\ A_s(p,q;k)\Gamma_s(q,p';k) , \quad (15)$$

which has a nonsingular kernel

$$A_s(p,q;k)=g(q^2,s)[V(p,q)-V(p,k)\gamma(k,q)] , \quad (16)$$

provided that for positive s we have $k=\sqrt{s}$ and $\gamma(k,k)=1$. For negative s the functional relation between k and s is arbitrary and certainly not determined by kernel nonsingularity constraints.[8,9,20,21]

The universality of the Γ_s-matrix formalism is manifested in the validity of Eqs. (9)–(11) independently of the γ function used to define Γ_s. The function $\gamma(k,q)$ need not even be such that $A_s(p,q;k)$ is nonsingular in order to achieve this. For example, let $\gamma(k,q)$ be a distribution-valued function of q with the properties

$$(s-q^2)^{-1}\gamma(k,q)=\mp i\pi\delta(k^2-q^2) , \quad (17a)$$

so that

$$[V(p,q)-V(p,k)\gamma(k,q)](s-q^2)^{-1} = P(s-q^2)^{-1}V(p,q) \quad (17b)$$

for $s=k^2\pm i0$ and

$$(s-q^2)^{-1}\gamma(k,q)=0 \tag{17c}$$

for s not on the positive-energy cut, where the symbol P denotes the Cauchy principal-value prescription. Then using Eqs. (17b) and (17c) in Eqs. (15) and (16) and Eq. (17a) in Eq. (3), we recover the K-matrix formalism with

$$\Gamma_s(p,p';k)=K_s(p,p') , \tag{17d}$$

where $K_s(p,p')$ is the completely off-shell K_s matrix:

$$K_s(p,p')=V(p,p')+P\int dq\, g(q^2,s)V(p,q)K_s(q,p') , \tag{17e}$$

and Eq. (3) becomes the usual off-shell Heitler equation[22]

$$t_s(p,p')=K_s(p,p')-i\frac{\pi}{2}k\lambda K_s(p,k)\Theta(k)t_s(k,p') , \tag{17f}$$

where Θ is the step function, $\Theta(x)=1$, for $x>0$ and $=0$ for $x<0$. Similarly, with $\gamma\equiv 0$ we recover the t_s-matrix formalism since then $\Gamma_s=t_s$.

A number of generalizations of Eq. (3) suggest themselves, especially if we wish to include the method of Ref. 14. One possibility is

$$t_s(p,p')=\Gamma_s(p,p';k,k_0)+\Gamma_s(p,k;k,k_0)\int dq\int dq' g(q^2,s)\gamma(k,k_0;q,q')t_s(q',p') . \tag{18a}$$

Using Eq. (18a) we again obtain Eqs. (9)–(11) for a symmetric $t_s(p,p')$. The subtraction technique that we develop in Sec. III results in a Heitler relation of the form (18a) but with

$$\gamma(k,k_0;q,q')=\bar{\gamma}(k,k_0,q)\delta(q'-k) , \tag{18b}$$

where k and k_0 are *two* arbitrary wave vectors. The formalism of Ref. 14 corresponds to taking $k_0>k$ and

$$\bar{\gamma}(k,k_0,q)=\Theta(k_0-q) . \tag{18c}$$

In Ref. 14 k_0 is denoted by Λ, so that the connection between the two notations with the choice (18c) is given by

$$\Gamma_s(p,p';k,k_0)\equiv\tau^{\Lambda}(p,p';E) , \tag{18d}$$

for $E\equiv s=k^2+i0$. It is clear that the Heitler-type equation (18a) encompasses more general possibilities than the scheme of Ref. 14.

The occurrence of bound-state poles at $s=s_B$ is associated with the condition[8,13,20,21]

$$\int dq\,\gamma(k,q)g(q^2,s_B)\Gamma_{s_B}(q,k;k)=1 , \tag{19a}$$

with the relationship

$$\psi_B(q)\sim q^{-2}g(q^2,s_B)\Gamma_{s_B}(q,k;k) \tag{19b}$$

between Γ_{s_B} and the bound-state wave function ψ_B in momentum space. Condition (19a) follows from Eq. (6). Some apparent exceptions to this last inference are possible. For example, a class of γ functions is known[23] for which the denominator in the right-hand side of Eq. (6) is independent of Γ_s at negative energies. In this instance, Γ_s has a bound-state pole at $s=s_B$ just as t_s does.

The residual function $R_s(p,p';k)$ is related to the resolvent kernel $\mathcal{R}_s(p,p';k)$ corresponding to $A_s(p,p';k)$ and to $f_s(p,k)$ by[9]

$$R_s(p,p';k)=\left[\mathcal{R}_s(p,p',k)-f_s(p,k)\mathcal{R}_s(k,p';k)\right]\times g(p'^2,s)^{-1} . \tag{20}$$

The resolvent kernel satisfies the integral equations[24]

$$\mathcal{R}_s=A_s+A_s*\mathcal{R}_s , \tag{21a}$$

$$\mathcal{R}_s=A_s+\mathcal{R}_s*A_s , \tag{21b}$$

that we have written in an abstract notation (the asterisk signifying integration) for brevity. The kernels of Eqs. (21) are exactly the same as in Eq. (15) so that one is basically confronted with the same integral equation, but with a more complicated inhomogeneous term, to calculate all of the components of the off-shell transition matrix. In fact, the resolvent kernel defines[24] what we mean by a solution of Eq. (15):

$$\Gamma_s=(I+\mathcal{R}_s)*V , \tag{22a}$$

and so

$$\mathcal{R}_s(p,p';k)=[\Gamma_s(p,p';k)-\Gamma_s(p,k;k)\gamma(k,p')]g(p'^2,s) . \tag{22b}$$

Therefore we also obtain from Eq. (20) the form (11) for the residual function. This establishes the equivalence of the two seemingly different treatments of this term and the general off-shell case given in Refs. 9 and 12.

The universality of the Γ_s-matrix formalism is manifested in the validity of Eqs. (9)–(11) independently of the choice of the γ function used to define Γ_s. The form (20), which explicitly incorporates the dynamical content of Eq. (1), is also universal in the same sense of being form invariant with respect to the choice of γ. When γ is such that A_s is nonsingular, then this universality clearly reflects the latitude one has in expressing the dynamical content of Eq. (1) in a way consistent with off-shell unitarity, for all real s, and time-reversal invariance. However, as we have seen, γ need not always be such that A_s is nonsingular in order to achieve this.

In order to emphasize the universality properties, we collect all of our results:

$$f_s(p,k)=\frac{\Gamma_s(p,k;k)}{\Gamma_s(k,k;k)}=\frac{K_s(p,k)}{K_s(k,k)}=\frac{t_s(p,k)}{t_s(k,k)}=\frac{\tau^{\Lambda}(p,k;E)}{\tau^{\Lambda}(k,k;E)} , \tag{23a}$$

and

$$R_s(p,p';k)=K_s(p,p')-\frac{K_s(p,k)K_s(k,p')}{K_s(k,k)}$$
$$=t_s(p,p')-\frac{t_s(p,k)t_s(k,p')}{t_s(k,k)}$$
$$=\Gamma_s(p,p';k)-\frac{\Gamma_s(p,k;k)\Gamma_s(k,p';k)}{\Gamma_s(k,k;k)}$$
$$=\tau^{\Lambda}(p,p';E)-\frac{\tau^{\Lambda}(p,k;E)\tau^{\Lambda}(k,p';E)}{\tau^{\Lambda}(k,k;E)}\ . \quad (23b)$$

We remark that, in general, the Γ_s matrix is not symmetric in p and p', but both t_s and K_s are; so when time reversal is valid we have

$$t_s(p,p')=t_s(p',p)\ ,$$
$$K_s(p,p')=K_s(p',p)\ . \quad (24)$$

We note that the symmetry of K_s, along with Eq. (23b), implies that Eq. (9) holds with t_s replaced by K_s. It is the lack of symmetry of $\Gamma_s(p,p';k)$ that prevents it from being expressed in the canonical form (9). Instead,

$$\Gamma_s(p,p';k)=f_s(p,k)\Gamma_s(k,k;k)f_s^T(p',k)+R_s(p,p';k)\ , \quad (25)$$

where

$$f_s^T(p',k)=\frac{\Gamma_s(k,p';k)}{\Gamma_s(k,k;k)}\ , \quad (26)$$

which resembles some of the asymmetric forms considered in Refs. 2 and 3.

The full significance of the γ invariance underlying Γ_s-matrix universality is not clear. However, other choices besides Eqs. (17a)–(17c) and $\gamma\equiv 0$ also have distinctive properties that are well known, and should be pointed out in the context of the present new treatment of the subject. The subtraction function introduced by Kowlski, Feldman, and Noyes (KFN),[2–4,6]

$$\gamma_{\rm KFN}(k,q)=\frac{V(k,q)}{V(k,k)}\ , \quad (27)$$

is unique among all choices of γ in the resultant simplicity of the residual function R_s. That is, with this choice the kernel[25]

$$\Lambda_s(p,q;k)\equiv A_s^{\rm KFN}(p,q;k) \quad (28)$$

has the property

$$\Lambda_s(k,q;k)=0\ , \quad (29)$$

so therefore R_s differs from the resolvent kernel $\mathcal{R}_s^{\rm KFN}$ of Λ_s only by kinematic factors:

$$R_s(p,p';k)=\mathcal{R}_s^{\rm KFN}(p,p';k)g(p'^2,s)^{-1}\ . \quad (30)$$

Further, Λ_s is distinguished by the fact that it is the kernel for the integral equations satisfied by both the universal half-shell and residual functions:

$$f_s(p,k)=\frac{V(p,k)}{V(k,k)}+\int dq\ \Lambda_s(p,q;k)f_s(q,k)\ , \quad (31)$$

$$R_s(p,p';k)=V(p,p')-\frac{V(p,k)V(k,p')}{V(k,k)}+\int dq\ \Lambda_s(p,q;k)R_s(q,p';k)\ . \quad (32)$$

From Eqs. (23a) and (31) we see that the zeros of $t_s(k,k)$ and $K_s(k,k)$ are generally correlated with zeros in the Fredholm determinant of Λ_s. Finally, since with the choice (27) we have $\Gamma_s(k,k;k)=V(k,k)$, we see from Eq. (6) that the association of the integral relation (19a) with the occurrence of a bound state is obvious in this instance.

The subtraction function of Brown *et al.* (BFLS),[5]

$$\gamma_{\rm BFLS}=\left[\frac{q}{k}\right]^L\ , \quad (33)$$

is interesting because of the relationship of $\Gamma_s^{\rm BFLS}$ to the Jost function, at least for local potentials.[5,26] This is relevant to the known association of (19a) with bound-state poles and to the convergence of the iteration solution for $\Gamma_s^{\rm BFLS}$ in this case.[27]

As another consequence of universality we see that the rank-one Kowalski-Noyes[6,8,10] (KN) approximation to the t_s matrix

$$t_s^{\rm KN}(p,p';k)\equiv f_s(p,k)t_s(k,k)f_s(p',k)\ , \quad (34)$$

which is obtained by neglecting $R_s(p,p';k)$ in Eq. (9), is universal in the sense of being independent of the choice of the subtraction function $\gamma(k,q)$ that may be used to calculate f_s. As negative energies k is not determined by kernel nonsingularity constraints.[8,13] However, we would like to point out that the "moving" subtraction point, $k=\sqrt{-s}$ for $s<0$, that was suggested in Refs. 8, 10, and 20, has the advantage of defining a negative-energy shell for each s for which $t_s^{\rm KN}$ represents the transition matrix exactly both on shell and half shell for a semi-infinite range of parametric energy because of Eq. (13), just as in the positive energy case. That is,

$$t_s(p,\sqrt{-s})=t_s^{\rm KN}(p,\sqrt{-s};\sqrt{-s})\ , \quad (35a)$$
$$t_s(\sqrt{-s},p')=t_s^{\rm KN}(\sqrt{-s},p';\sqrt{-s})\ , \quad (35b)$$

for all p, p', and $s<0$. The optimum choice for the "functional relationship" between k and s for $s<0$ is an important unknown problem concerning off-shell behavior. The present discussion has shown that this question is entirely separate from the choice of a particular γ, and therefore a particular Γ_s. This independence is not always emphasized.[13]

III. SINGULAR-KERNEL SUBTRACTION SCHEMES

We now identify a new type of subtraction scheme which leads to auxiliary integral equation with a nonsingular *integral*, but with a kernel that is still singular. Let $\overline{\gamma}(k,k_0,q)$ be some arbitrary real function of the three wave vectors k, k_0, and q, where k and k_0 can be regarded as entirely arbitrary. Set

$$\hat{I}_s(k,k_0)\equiv\int dq\ g(q^2,s)\overline{\gamma}(k,k_0,q)\ . \quad (36)$$

Then we can rewrite Eq. (1) as

$$t_s(p,p')=V(p,p')+V(p,k)\hat{I}_s(k,k_0)t_s(k,p') + \int dq\, g(q^2,s)[V(p,q)t_s(q,p') - V(p,k)\bar{\gamma}(k,k_0,q) \times t_s(k,p')] , \quad (37)$$

and we see that the last term on the right-hand side of (37) is an integral with an integrand with no singularity resulting from $g(q^2,s)$ for $s=k^2+i0$ provided that $\bar{\gamma}(k,k_0,k)=1$.

The standard way of using (36) and (37) involves choosing $\bar{\gamma}(k,k_0,q)$ so that $\hat{I}_s(k,k_0)$ vanishes. For example, in the counterpart of Eq. (37) for the K_s matrix defined by Eq. (17e), we could take $\bar{\gamma}(k,k_0,q)=(k/q)^2$ and exploit the fact that

$$P\int dq \frac{1}{k^2-q^2}=0 . \quad (38)$$

In this case we simply have an integral equation [Eq. (43), below] with an integral term that is formally nonsingular so no new auxiliary equation is needed.[15]

When $\hat{I}_s(k,k_0)\neq 0$ the solution of (37) can be expressed in terms of the solution of the auxiliary integral equation

$$\Gamma_s(p,p';k,k_0)=V(p,p')+\int dq\, g(q^2,s)[V(p,q)\Gamma_s(q,p';k,k_0)-V(p,k)\bar{\gamma}(k,k_0,q)\Gamma_s(k,p';k,k_0)] , \quad (39)$$

whose kernel is a highly singular, distribution-valued quantity.

As mentioned in Sec. II, the formalism of Ref. 14 is recovered if we take $k_0>k$ and

$$\bar{\gamma}(k,k_0,q)=\Theta(k_0-q) . \quad (40)$$

In terms of the notation of Ref. 14, $\Gamma_s(p,p';k,k_0)$ is denoted by Eq. (18d) for $s\equiv E=k^2+i0$, although our treatment makes the extension of the work of Ref. 14 to negative energy obvious. The relationship (18b) between $\gamma(k,k_0;q,q')$ and $\bar{\gamma}(k,k_0;q)$ leads to the generalized Heitler relation (18a) connecting t_s and the solution of Eq. (39). We also conclude from Eqs. (23) that as far as calculating either the on- or off-shell transition matrix is concerned, the technique of Ref. 14 offers no particular advantage.

Now, of course, $f_s(p,k)$ and $R_s(p,p',k)$ are independent of the parameter $k_0=\Lambda$. However, a calculational scheme is proposed in Ref. 14 for obtaining a nonuniversal rank-one approximation for $t_s(p,p')$ by approximating $R_s(p,p,k)$ in a Λ-dependent fashion by dropping the $\tau^{\Lambda}(p,p';E)$ term in $R_s(p,p';k)$ given in Eq. (23b). The freedom in Λ is then exploited to minimize the error made in doing this. Let us call this approximation to the residual term $R_s^{\Lambda}(p,p';k)$.

Since $R_s^{\Lambda}(k,p';k)\neq 0$, $R_s^{\Lambda}(p,k;k)\neq 0$, we see that the resultant rank-one transition operator $T^{\mathrm{SA}}(p,p';E)$ proposed in Ref. 14 violates not only off-shell unitarity, but on-shell unitarity as well. On the other hand, $T^{\mathrm{SA}}(p,p';E)$ does not possess the singularities that the t^{KN} approximation has at points where $t_s(k,k)=0$. Also, at least for the cases explored numerically in Ref. 14, the approximation seems reasonably accurate.

Evidently the same type of approximation can be achieved by dropping $\Gamma_s(p,p';k)$ in Eq. (23b). Then in order to minimize the γ-dependent error involved in doing this in the manner of Ref. 14, one can take, with $k_0>k$,

$$\gamma(k,k_0,q)=\Theta(k_0-q)\gamma(k,q) , \quad (41)$$

in place of $\gamma(k,q)$ in Eq. (15) and follow the same minimization procedure used in Ref. 14. This approach has the advantage of working with the numerically stable nonsingular kernel Γ_s equation. It also presents one with the additional opportunity of choosing a $\gamma(k,q)$ function that may reduce the error even further, an option not present in Ref. 14.

It is interesting to note that if we make the k_0-independent choice

$$\bar{\gamma}(k,k_0,q)=(k/q)^2 , \quad (42)$$

in Eq. (39), then $\Gamma_s(p,p';k,k_0)$ is a quantity $K_s^c(p,p')$ that satisfies

$$K_s^c(p,p')=V(p,p')+\int dq\, g(q^2,s)\left[V(p,q)K_s^c(q,p')-V(p,k)\left[\frac{k}{q}\right]^2 K_s^c(k,p')\right] . \quad (43)$$

For s real and positive, $s=k^2$, the function $K_{k^2}(p,p')$ as defined by Eq. (17e) and $K_{k^2}^c(p,p')$ are identical and Eq. (43) is identical to the equation used by Haftel and Tabakin.[15] Otherwise, $K_s^c(p,p')$ represents an entirely different continuation to arbitrary s than does $K_s(p,p')$.[28] In particular, for s real and negative, $K_s(p,p')=t_s(p,p')$, but $K_s^c(p,p')\neq K_s(p,p')$. For complex s in place of Eq. (17f) we have the following Heitler equation:

$$t_s(p,p')=K_s^c(p,p')-i\frac{\pi}{2}k\lambda K_s^c(p,k)t_s(k,p') . \quad (44)$$

IV. CONCLUSIONS

We have demonstrated by means of a generalization of the Heitler formalism how γ-function invariance underlies the universality properties of the off-shell two-body transition matrix. We have also shown how the class of γ functions can be enlarged to include distribution-valued subtraction functions. We conclude that the only possible advantage of one choice of γ over another lies in the convergence and stability properties of the numerical solutions of the Γ-matrix equations. The universality of

the Γ-matrix formalism is also shown to lead to a universal rank-one approximation to the off-shell transition matrix for both positive and negative energies.

In addition, we have identified the general class of subtraction procedures into which the example used by Stoof *et al.*[14] falls, and we showed how the auxiliary amplitudes involved in this case also satisfy the universality property. In addition, we examined the nonuniversal rank-one approximation T^{SA} proposed in Ref. 14 and found that T^{SA} violates both off-and on-shell unitarity. It was suggested that this type of approximation may be made more accurately by using cutoff subtraction functions in the Γ_s-matrix formalism.

The preceding results simplify, unify, and extend many different seemingly distinct approaches to off-shell scattering.

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*Permanent address: Departamento de Fisica, Universidade Federal de Pernambuco, 50.739 Recife, Pernambuco, Brazil.

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