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Isolated solitons in an ultrarelativistic electron-positron plasma of a pulsar magnetosphere

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The nonlinear propagation of intense electromagnetic radiation in an electron-positron plasma of a pulsar magnetosphere is investigated. Basic equations describing a macroscopic behavior of an ultrarelativistic plasma ($T \gg mc^2$) in a covariant form are used to treat the problem. Relativistic and time-derivative ponderomotive nonlinearities are considered. Earlier results of envelope solitons in the weak relativistic case are confirmed, while isolated solitons are found to exist in the ultrarelativistic plasma.

Pulsars are regarded as rotating magnetized neutron stars with a strong magnetic field ($\sim 10^{12}$ G). Theoretical models[1,2] has been developed to predict the production of electron-positron plasmas in the pulsar magnetosphere. According to the current polar-cap pulsar model,[3] the pulsar magnetosphere is composed of secondary electrons and positrons which result from pair production induced by high-energy curvature-radiation photons, emitted by primary positrons or electron beams coming from the pulsar surface. The problem of pulsar emission and pulsar magnetospheric structure stimulates the investigation of some physical processes in the electron-positron plasma, such as the propagation of electromagnetic waves and the nonlinear mechanism related to the pulsar emission.

In our previous paper[4] we showed the relativistic excitation of envelope solitons in an electron-positron plasma of a pulsar magnetosphere. Large-amplitude localized electromagnetic radiation is found to exist in the pulsar environment. In this Brief Report we extend our earlier investigation on the nonlinear propagation of intense electromagnetic radiation in a pulsar magnetosphere to include the ultrarelativistic effects. We consider the macroscopic behavior of an ultrarelativistic plasma ($T \gg mc^2$) in a covariant form. Relativistic and time-derivative ponderomotive nonlinearities are accounted for. Weak relativistic effects are found to excite the envelope solitons, while the ultrarelativistic nonlinearity generates isolated solitons in the plasma, which means the electric or magnetic fields appear as isolated spikes.

Basic equations describing the macroscopic behavior of an ultrarelativistic plasma in a curved space-time were derived in Ref. 5. The system consists of the Einstein equation, Maxwell equations, the conservation law for particles, and the equation of state of matter. We rewrite the formulation here.

The Einstein equation[6] is written as

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R=\frac{4\pi G}{c^4}T_{\alpha\beta}\ , \tag{1}$$

where $R_{\alpha\beta}$ is the Ricci tensor, $g_{\alpha\beta}$ the space-time metric tensor, R the scalar curvature given by $R=g^{\alpha\beta}R_{\alpha\beta}$, $G=6.67\times10^{-8}\ \mathrm{cm^3\,g^{-1}\,sec^{-3}}$ is the gravitational constant, and $T_{\alpha\beta}$ is the energy-momentum tensor. The Einstein equation itself presumes no form for $T_{\alpha\beta}$, but the Bianchi identity implies

$$T^{\alpha\beta}_{;\beta}=0\ , \tag{2}$$

where the semicolon denotes the covariant derivative for β.

$T^{\alpha\beta}$ is the energy-momentum tensor consisting of plasma and electromagnetic field,

$$T^{\alpha\beta}=T^{\alpha\beta(M)}+T^{\alpha\beta(\mathrm{EM})}\ , \tag{3}$$

$$T^{\alpha\beta(M)}=(p+\mathcal{E})u^\alpha u^\beta-pg^{\alpha\beta}\ , \tag{4}$$

$$T^{\alpha\beta(\mathrm{EM})}=\frac{1}{4\pi}(-F^{\alpha\lambda}F^{\beta}_{\lambda}+\tfrac{1}{4}F_{\sigma\rho}F^{\sigma\rho}g^{\alpha\beta})\ , \tag{5}$$

where p is the plasma pressure, $\mathcal{E}$ the mass-energy density, u^α the fluid's four-velocity, and $F_{\alpha\beta}$ is the covariant electromagnetic field tensor given by potential A_α,

$$F_{\alpha\beta}=A_{\beta;\alpha}-A_{\alpha;\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha\ , \tag{6}$$

where $\partial_\alpha=(\partial/\partial x)\alpha$ denotes the usual derivative.

The Maxwell equations are written in the curved space-time as

$$\partial_\gamma F_{\alpha\beta}+\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}=0\ , \tag{7}$$

$$F^{\alpha\beta}_{;\beta}=\frac{4\pi}{c}j^\alpha\ , \tag{8}$$

where J^α is the four-current density given by

$$j^\alpha=\frac{\rho c}{\sqrt{-g}}\frac{dx^\alpha}{dx^0}\ , \tag{9}$$

in which g is the determinant of $g^{\alpha\beta}$ and ρ is the charge density.

The particle conservation law is given by

$$(nu^\alpha)_{;\alpha}=0\ , \tag{10}$$

where n is the proper number density.

For adiabatic changes in plasma the equation of state is considered as

$$P = \text{const} \times n^{\Gamma} , \tag{11}$$

where Γ is the ratio of specific heat.

The case of special relativity can be realized on a point in the curved space-time. At this point the gravity is completely canceled in a moving frame. In such a case $g^{\alpha\beta}$ changes to $\eta^{\alpha\beta}$ and the covariant derivative takes the usual form. From Eq. (2) with Eqs. (3), (4), and (5) one obtains

$$\partial_\beta T^{\alpha\beta(\text{EM})} = -\frac{1}{c} F^{\alpha\beta} j_\beta \tag{12}$$

and

$$\partial_\beta T^{\alpha\beta(M)} = \partial_\beta [(p+\mathcal{E}) u^\alpha u^\beta - p \eta^{\alpha\beta}] = \frac{1}{c} F^{\alpha\beta} j_\beta . \tag{13}$$

On the other hand, the four-velocity u^α ($\alpha = 1,2,3$) is given by

$$u^\alpha = \frac{v^\alpha}{(1-\beta^2)^{1/2}} , \tag{14}$$

$$u^0 = \frac{1}{(1-\beta^2)^{1/2}} , \tag{15}$$

where $\beta = v/c$. From the above expression we get

$$T^{(M)}_{\alpha\beta} = \frac{(p+\mathcal{E}) v_\alpha v_\beta}{c^2(1-\beta^2)} + p\delta_{\alpha\beta} \quad (\beta \neq 0) , \tag{16}$$

$$T^{(M)}_{\alpha_0} = \frac{(p+\mathcal{E}) v_\alpha}{c(1-\beta^2)} , \tag{17}$$

$$T^{(M)}_{00} = \frac{(p+\mathcal{E})}{(1-\beta^2)} - p , \tag{18}$$

where the relations $T_{0\alpha} = -T^{0\alpha}$, $T_{00} = T^{00}$, and $T_{\alpha\beta} = T^{\alpha\beta}$ are used.

From the component ($\alpha = 0$) of Eq. (13), we obtain the energy-conservation law in a three-dimensional-vector form,

$$\frac{\partial p}{\partial t} - \frac{\partial}{\partial t}[\gamma^2(p+\mathcal{E})] - \text{div}[\gamma^2(p+\mathcal{E})\mathbf{v}] + \mathbf{E}\cdot\mathbf{J} = 0 , \tag{19}$$

where

$$\gamma = (1-\beta^2)^{-1/2} , \tag{20}$$

$$\mathbf{J} = \rho\mathbf{v} = e\gamma n \mathbf{v} . \tag{21}$$

From the component ($\alpha \neq 0$) of Eq. (13), one obtains the equation of motion,

$$\frac{\gamma^2}{c^2}(p+\mathcal{E})\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right]\mathbf{v} = -\nabla p + \rho\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B} - \frac{\mathbf{v}}{c^2}\left[\frac{\partial p}{\partial t} + \mathbf{E}\cdot\mathbf{J}\right] , \tag{22}$$

where Eq. (19) and

$$\frac{1}{c} F^{\alpha\beta} J_\beta = \frac{1}{c} F^{0\beta} j_\beta + \frac{1}{c} \sum_{\beta=1}^{3} F^{\alpha\beta} j_\beta = \rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}$$

are used.

The continuity equation is written as

$$\frac{\partial}{\partial t}(n\gamma) + \text{div}(n\gamma\mathbf{v}) = 0 . \tag{23}$$

For a two-species plasma (subscript s denotes species), the basic equations are

$$\frac{\partial}{\partial t}(\gamma_s n_s) + \text{div}(\gamma_s n_s \mathbf{v}_s) = 0 , \tag{24}$$

$$\rho_{m,s}(\partial_t + \mathbf{v}_s\cdot\nabla)\mathbf{v}_s = \rho_{q,s}\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_s\times\mathbf{B}\right] - \frac{\mathbf{v}_s}{c^2}\left[\rho_{q,s}\mathbf{E}\cdot\mathbf{v}_s + \frac{\partial p_s}{\partial t}\right] - \nabla p_s , \tag{25}$$

$$\frac{\partial p_s}{\partial t} - \frac{\partial}{\partial t}(\rho_{m,s}c^2) - \text{div}(\rho_{m,s}c^2\mathbf{v}_s) + \rho_{q,s}\mathbf{E}\cdot\mathbf{v}_s = 0 , \tag{26}$$

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} , \tag{27}$$

$$\text{div}\mathbf{E} = 4\pi\rho , \tag{28}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} , \tag{29}$$

$$\text{div}\mathbf{B} = 0 , \tag{30}$$

$$P_s = \text{const}\times n_s^{\Gamma_s} , \tag{31}$$

where

$$\rho_{m,s} = \frac{\gamma_s^2}{c^2}(p_s + \mathcal{E}_s), \quad \rho_{q,s} = q_s\gamma_s n_s ,$$

$$\mathbf{j} = q_p\gamma_p n_p \mathbf{v}_p + q_e\gamma_e n_e \mathbf{v}_e, \quad \rho = \gamma_p q_p n_p + \gamma_e q_e n_e ,$$

$$\gamma_s = (1 - v_s^2/c^2)^{-1/2} .$$

For the limiting case: $v_s \ll c$, $p_s \ll c^2$, the usual two-fluid equations in the nonrelativistic case are recovered.

Let us introduce the density momentum in the relativistic plasma:

$$\mathbf{P}_s = \frac{\gamma_s^2}{c^2}(p_s + \mathcal{E}_s)\mathbf{v}_s \equiv \rho_{m,s}\mathbf{v}_s . \tag{32}$$

The equation of motion is

$$\frac{d_s\mathbf{P}_s}{dt} = \rho_{q,s}\left[\mathbf{E} + \frac{1}{c}\mathbf{v}_s\times\mathbf{B}\right] - \nabla P_s , \tag{33}$$

where

$$\frac{d_s}{dt} = \partial_t + \mathbf{v}_s\cdot\nabla .$$

For the nonlinear propagation of a circularly polarized wave along the ambient magnetic field $B_0\hat{z}$ in an electron-positron plasma,

$$E_x \pm iE_y = \tilde{E}(z,t) = E^{\pm}(z,t)e^{-i\omega t + ikz} , \tag{34}$$

$$p_{sx} \pm ip_{sy} = \tilde{p}_s(z,t) = p_s^{\pm}(z,t)e^{-i\omega t + ikz} \tag{35}$$

the plus sign denotes the R polarization, the minus sign denotes the L polarization of the wave and $E^{\pm}$ and $p_s^{\pm}$ are the slowly varying complex amplitudes), the transverse oscillations of plasma are governed by the equation

$$\frac{d_s\tilde{p}_s^{\pm}}{dt} \pm i\Omega_s\tilde{p}_s^{\pm} = \rho_{q,s}\tilde{E}^{\pm} + \rho_{qs}v_{sz}\int \partial_z\tilde{E}^{\pm}dt , \tag{36}$$

where

$$\Omega_s = \frac{\rho_{q,s}B_0}{\frac{\gamma_s^2}{c}(p_s+\mathcal{E}_s)} \equiv \frac{\rho_{q,s}B_0}{\rho_{m,s}c} . \tag{37}$$

For the slowly varying amplitudes of the propagating waves, from Eq. (36) we get

$$p_s^{\pm} = \frac{i\rho_{q,s}E^{\pm}\left[1-\frac{kv_{sz}}{\omega}\right]}{\omega \mp \Omega_s - kv_{sz}} . \tag{38}$$

The wave equation is

$$\partial_z^2\tilde{E}^{\pm} - \frac{1}{c^2}\partial_t^2\tilde{E}^{\pm} = \frac{4\pi}{c^2}\partial_t\sum_s \frac{\rho_{q,s}}{\rho_{m,s}}\tilde{p}_s^{\pm} , \tag{39}$$

which in the linear response determines the dispersion relation

$$\frac{k^2c^2}{\omega^2} = 1 - \sum_s \frac{4\pi(\rho_{q,s})^2}{\rho_{m,s}} \frac{\left[1-\frac{kv_{sz}}{\omega}\right]}{\omega(\omega\mp\Omega_s - kv_{sz})} . \tag{40}$$

The longitudinal plasma motion is governed by the ponderomotive pressure, thermal pressure, and the ambipolar potential due to the charge separation, i.e.,

$$\frac{d_s p_{sz}}{dt} = -\rho_{q,s}\nabla_z\Phi + f_p - \nabla_z p_s . \tag{41}$$

Here, Φ is the ambipolar field and f_p is ponderomotive force. An expression for the time-derivative ponderomotive force is given in Ref. 7:

$$f_p = \frac{1}{16\pi}\left[(\epsilon_\perp - 1)\partial_z|E|^2 + \frac{k}{\omega}\left[\frac{1}{\omega}\frac{\partial}{\partial\omega}[\omega^2(\epsilon_\perp-1)]\right]\partial_t|E|^2\right] . \tag{42}$$

We consider $\omega/k \gg |v_{sz}|$ and $\Omega_e=-\Omega$, $\Omega_p=\Omega$; then, from the dispersion relation (40), we get

$$\epsilon_\perp = \frac{k^2c^2}{\omega^2} = 1 - \frac{2\omega_p^2}{\omega^2-\Omega^2} , \tag{43}$$

with

$$\omega_p^2 = \frac{4\pi e^2 n_0}{(p_0+\mathcal{E})/c^2} ,$$

and accordingly

$$f_p = -\frac{\omega_p^2}{\omega^2-\Omega^2}\left[\partial_z - \frac{2k\Omega^2}{\omega(\omega^2-\Omega^2)}\partial_t\right]\frac{|E^{\pm}|^2}{8\pi} . \tag{44}$$

For a linear phase shift of the wave

$$E^{\pm} = |E^{\pm}|e^{-i\theta t + i\kappa z} \quad (\theta,\kappa \text{ are constants}) , \tag{45}$$

it can be shown[8] that

$$|E|^2 = f(\xi), \quad \xi = z - v_0 t, \quad v_0 = \frac{\kappa c^2}{\omega}\left[1+\frac{\kappa}{k}\right] . \tag{46}$$

Equation (44) shows that the ponderomotive force is charge independent; therefore the ambipolar potential in the electron-positron plasma may be neglected. Furthermore, neglecting the mass inertia[8] (which means that the time variation of low-frequency electrostatic oscillation is less than the plasma frequency) in Eq. (41), we get

$$p_s = p_0 - \frac{\omega_p^2}{\omega^2-\Omega^2}\left[1+\frac{2kv_0\Omega^2}{\omega(\omega^2-\Omega^2)}\right]\frac{|E^{\pm}|^2}{8\pi} , \tag{47}$$

where p_0 is the initial (background) pressure density.

Under the WKB approximation the wave evolution is governed by the nonlinear Schrödinger equation

$$i\left[\partial_t + \frac{kc^2}{\omega}\partial_z\right]E^{\pm} + \frac{c^2}{2\omega}\partial_z^2E^{\pm} + \Delta E^{\pm} = 0 , \tag{48}$$

where

$$\Delta = -\frac{\omega\omega_p^2}{\omega^2-\Omega^2}\left[\frac{\delta n}{n_0} - \frac{\delta p}{2(p_0+\mathcal{E})}\right] \tag{49}$$

is the nonlinear frequency shift.

We consider the isothermal state of plasma,

$$P = nKT \quad \text{with } kT = \text{const} . \tag{50}$$

Then

$$\Delta = \frac{\omega\omega_p^4}{(\omega^2-\Omega^2)^2}\left[\frac{1}{KT} - \frac{1}{2(p_0+\mathcal{E})}\right]\left[1+\frac{2kv_0\Omega^2}{\omega(\omega^2-\Omega^2)}\right] \times \frac{|E^{\pm}|^2}{8\pi} . \tag{51}$$

Thus the evolution equation takes the form

$$i(\partial_t + v_g\partial_z)E^{\pm} + P'\partial_z^2E^{\pm} + Q|E^{\pm}|^2E^{\pm} = 0 , \tag{52}$$

where

$$P' = \frac{c^2}{2\omega} ,$$
$$Q = \frac{1}{8\pi}\frac{\omega\omega_p^4}{(\omega^2-\Omega^2)^2}\left[\frac{1}{KT} - \frac{1}{2(p_0+\mathcal{E})}\right] \times \left[1+\frac{2kv_0\Omega^2}{\omega(\omega^2-\Omega^2)}\right] . \tag{52a}$$

For $P'Q>0$, the wave is modulationally unstable and admits a solution[9]

$$E^{\pm} = E_0^{\pm}\,\text{sech}\left[\left|\frac{Q}{2P'}\right|^{1/2}E_0^{\pm}\xi\right]\exp(-iQE_0^{\pm 2}/2) , \tag{53}$$

where

$$E_0^{\pm} = (A/Q)^{1/2} ,$$

and $A=\kappa(V_g+P'\kappa)-\theta$ is the soliton amplitude and $\delta=|2P'/QE_0^2|^{1/2}$ is the soliton pulse width.

For $\omega^2>\Omega^2$ and weak relativistic plasma ($KT<P_0+\mathcal{E}$) an analysis shows that $E_0^\pm\sim A^{1/2}\sqrt{T}/n_0$, which means that the amplitude of the soliton is proportional to the square root of the temperature, and inversely proportional to the density. It predicts that a large-amplitude localized field is possible in the pulsar environment where the temperature is high and the plasma density is relatively low.

The pulse width is

$$\delta=\left|\frac{2P'}{QE_0^2}\right|^{1/2}\equiv\frac{1}{\left[k\kappa\left(1+\frac{\kappa}{2k}\right)-\frac{\omega Q}{c^2}\right]^{1/2}},$$

which means it is fully determined by the phase shift of the wave, not by the temperature and density.

Let us investigate the case of an ultrarelativistic plasma, i.e., $KT>P_0+\mathcal{E}$. For this case, as we see from Eq. (52a), $Q<0$, and we have the solution

$$|E^\pm|^2=E_0^{\pm 2}\csc^2(\mu\xi)\ , \tag{54}$$

with

$$E_0^\pm=(2A/Q)^{1/2},\quad \mu=(A/P')^{1/2}\ . \tag{54a}$$

The solution Eq. (54) describes a soliton which is cusped at the center, where $\partial_\xi E$ becomes infinite. Thus the electric or magnetic fields appear as isolated spikes.

[1]P. A. Sturrock, Astrophys. J. **164**, 529 (1971).

[2]M. A. Ruderman and P. G. Sutherland, Astrophys. J. **196**, 51 (1975).

[3]J. Arons and E. T. Scharleman, Astrophys. J. **231**, 854 (1979).

[4]U. A. Mofiz, G. M. Bhuiyan, Z. Ahmed, and M. A. Asgar, Phys. Rev. A **38**, 5935 (1988).

[5]J. Sakai and T. Kawata, J. Phys. Soc. Jpn. **49**, 747 (1980).

[6]L. D. Landau and E. M. Liftshitz, *Classical Theory of Field* (Pergamon, New York, 1975).

[7]K. Nishikawa, N. L. Tsintsadze, and M. Watanabe, Fiz. Plazmy **6**, 1302 (1980) [Sov. J. Plasma Phys. **6**, 713 (1981)].

[8]U. A. Mofiz and J. Podder, Phys. Rev. A **36**, 1811 (1987).

[9]B. B. Kadomtsev and V. I. Karpman, Usp. Fiz. Nauk **103**, 193 (1971) [Sov. Phys.—Usp. **14**, 40 (1971)].