# Mean first-passage time of random walks on a random lattice

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The mean first-passage time (MFPT) of random walks on one-dimensional disordered lattice segments is considered. Disorder is modeled by prescribing random transition and sojourn probabilities to the lattice sites. We consider several models of disorder: models with symmetric and asymmetric random transition probabilities, random sojourn probabilities, and models with bond randomness. For these models we present exact results on the MFPT and disorder-averaged MFPT. We do not find any anomalous dependence of the disorder-averaged MFPT on the size of the lattice segment. The distribution of the MFPT resulting from the disorder is Gaussian for the models with symmetric and site symmetric transition probabilities and non-Gaussian for models with asymmetric transition probabilities.

## I. INTRODUCTION

The problem of diffusion in a one-dimensional lattice with random, nearest-neighbor transition probabilities has attracted much attention in recent times.[1] This is mainly because it provides one of the simplest models of disordered systems.[1,2] Properties such as diffusion constant, conductivity, etc., of disordered systems have been studied employing such random-walk models.[1–5] However, mean first-passage time (MFPT) problems in disordered systems have not received as much attention until very recently.[6,7] Indeed, MFPT is a very useful concept in random-walk theory and has found wide applications in a variety of fields.[8]

In this paper, we study disorder-averaged MFPT of random walks on segments of disordered lattices. The problem of the influence of disorder on the MFPT has been addressed by Noskowicz and Goldhirsch.[6] Employing recursion-relation procedures,[9,10] they have obtained upper and lower bounds for the MFPT. However, using the method of Zwerger and Kehr,[11] explicit expressions for the MFPT in terms of the transition probabilities on a disordered lattice can be obtained.[12] This method is discussed in Sec. II, and forms the basis for our subsequent derivations.

We consider in this paper four specific models of disorder. In the first, called the random-barrier model, the transition probabilities are symmetric. For this model, we obtain expressions for the MFPT and the disorder-averaged MFPT. We find that the asymptotic dependence of MFPT on the length of the lattice segment is like that of a pure diffusive process, but with a modified, disorder-dependent diffusion constant. The distribution of the MFPT due to disorder is still Gaussian for this model. The details are discussed in Sec. III.

The second model we consider is called the random-trap model. In this model, the sojourn probabilities at the lattice sites are random. The expressions for the MFPT and disorder-averaged MFPT are similar to the random-barrier model, and hence the conclusions are the same for both the models. The details are discussed in Sec. IV.

The third class of models considered is the one with asymmetric transition probabilities. These models have motivated a lot of work in the theory of probability.[13,14] The Sinai model[14] is a particular case of these models where a certain condition on the transition probabilities is fulfilled. In this model the mean and mean-square displacements exhibit anomalous dependence on time. We report in Sec. V our studies of this class of models. The exact result on the disorder-averaged MFPT shows that this quantity behaves like a negatively biased walk under the Sinai condition. We also establish upper and lower bounds for the typical MFPT by employing methods similar to the ones given in Ref. 6.

The fourth model we consider is the one with bond randomness.[4,5] The line joining two adjacent lattice sites constitutes a bond. The transition probabilities between the sites of a bond can be correlated, and those belonging to different bonds are uncorrelated. Our results on MFPT for this model are discussed in Sec. VI. In particular, we find that disorder-averaged MFPT behaves like that of ordinary and biased walks.

Finally, in Sec. VII, we briefly summarize the principal results and conclusions of the study and indicate possible future work.

## II. MEAN FIRST-PASSAGE TIME ON A SEGMENT

We consider random walks on a one-dimensional lattice segment of $N+1$ sites, as shown in Fig. 1. The lattice sites are denoted by the integers, $j=0,1,\ldots,N$. Disorder is modeled, as usual, by prescribing random nearest-neighbor transition probabilities to the lattice sites. Thus for any site $j$, $p_j$ denotes the probability for the random walk to jump to site $j+1$ (right jump), per step. The left-jump probability at the lattice site $j$ is denoted by $q_j$. In general, $q_j+p_j \le 1$, and we define $1-(q_j+p_j)$ as the probability for the random walk to stay at site $j$ per step, also called the sojourn probability.

 

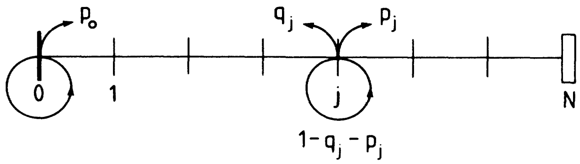


FIG. 1. Segment of a random lattice. Right and left jumps are indicated by arrows and sojourn by loops.

At the lattice site $j=0$, there is no left jump. We set $q_0=0$ and thus $1-p_0$ is the sojourn probability per step at $j=0$. The set $\{p_0,p_1,\ldots,p_{N-1},q_1,q_2,\ldots,q_{N-1}\}$ constitutes $2N-1$ random variables and each of these is restricted to the range $(0,\frac{1}{2})$ since we require $q_j+p_j\le 1$. The disorder is modeled by prescribing, in general, a joint distribution for the $2N-1$ random variables.

Consider a given realization of the set $\{q_j\colon j=1,N-1;p_j\colon j=0,N-1\}$ that defines a particular random lattice. The random walk starts at $j=0$. Let $t_{0,N}(q_1,q_2,\ldots,q_{N-1},p_0,p_1,\ldots,p_{N-1})$ denote the random variable called the first-passage time (FPT). It is defined as the number of steps required for the random walk to reach the site $N$ for the first time. The average of FPT over the ensemble of all possible random walks on a given realization of the random lattice is called the MFPT and is denoted by $\overline{t}_{0,N}(q_1,q_2,\ldots,q_{N-1},p_0,p_1,\ldots,p_{N-1})$. We shall be further interested in calculating $\langle\overline{t}_{0,N}\rangle$, called the disorder-averaged MFPT, the angular brackets denoting an average over the distribution of the random variables $(q_j;p_j)$. To this end we proceed as follows.

We define $\hat{G}_{i,j}(n)$ as the probability for the random walk to start at site $i$ and eventually reach site $j$ for the first time in $n$ steps. Let $G_{i,j}(z)$ be the corresponding generating function, also called the $z$ probability,[9,10] defined as

$$G_{i,j}(z)=\sum_{n=0}^{\infty} z^n \hat{G}_{i,j}(n)\ . \tag{1}$$

Formally the MFPT is given by

$$\overline{t}_{0,N}=\frac{d}{dz}\ln G_{0,N}(z)\bigg|_{z=1}\ . \tag{2}$$

The key step in our formulation[11] consists of deriving a continued-fraction recursion relation for $G_{i,i+1}(z)$—the $z$ probability for the first-passage time from site $i$ to $i+1$. As an auxiliary quantity, we need the $z$ probability for staying at a site. Let $\hat{\chi}_i(n)$ be the probability for the random walk to stay at site $i$, for $n$ steps consecutively. Thus

$$\hat{\chi}_i(n)=[1-(q_i+p_i)]^n\ . \tag{3}$$

The corresponding generating function or the $z$ probability is given by

$$\chi_i(z)=\{1-z[1-(q_i+p_i)]\}^{-1}\ . \tag{4}$$

A random walk at site $i$ can (i) stay there for any number of steps, with $z$ probability $\chi_i(z)$, (ii) jump to site $i-1$, with $z$ probability $zq_i$, (iii) make a first passage from site $i-1$ to site $i$, with $z$ probability $G_{i-1,i}(z)$, (iv) stay at site $i$ for any number of steps, with $z$ probability $\chi_i(z)$, and eventually, (v) jump from site $i$ to site $i+1$, with $z$ probability $zp_i$.

After the process (i), the random walk can carry out the processes (ii), (iii), and (iv), $m$ times $(m=0,1,2,\ldots,\infty)$ before the last process (v). This leads to

$$\begin{aligned}G_{i,i+1}(z)&=\chi_i(z)\{1+zq_iG_{i-1,i}(z)\chi_i(z)+[zq_iG_{i-1,i}(z)\chi_i(z)][zq_iG_{i-1,i}(z)\chi_i(z)]+\cdots\}zp_i\\&=\frac{zp_i\chi_i(z)}{1-zq_iG_{i-1,i}(z)\chi_i(z)}\ .\end{aligned} \tag{5}$$

Substituting for $\chi_i(z)$ from (4) we get

$$G_{i,i+1}(z)=\frac{zp_i}{1-z[1-(q_i+p_i)+q_iG_{i-1,i}(z)]}\ . \tag{6}$$

Note that $G_{0,1}(z)$ is known, and is given by

$$G_{0,1}(z)=zp_0/[1-z(1-p_0)]\ . \tag{7}$$

Thus we can successively obtain expressions for $G_{1,2}(z),G_{2,3}(z),\ldots,G_{N-1,N}(z)$ employing the recursion relation (6). It can be easily verified by setting $z=1$, in (6) and (7), that $G_{i,i+1}(z=1)=1$ for all $i=0,N-1$.

We have, for the $z$ probabilities,

$$G_{0,N}(z)=\prod_{j=1}^{N} G_{j-1,j}(z)\ . \tag{8}$$

Let us differentiate both sides of this equation with respect to $z$ and set $z=1$. Denoting differentiation with respect to $z$ by the prime, we get,

$$\overline{t}_{0,N}\equiv G'_{0,N}(z=1)=\sum_{j=1}^{N} G'_{j-1,j}(z=1)\ . \tag{9}$$

We differentiate $G_{i,i+1}(z)$, given by (6), with respect to $z$, and set $z=1$, to obtain

$$G'_{i,i+1}(z=1)=\frac{1}{p_i}+\frac{q_i}{p_i}G'_{i-1,i}(z=1)\ . \tag{10}$$

This equation forms the central result of this paper. Starting from $G'_{0,1}(z=1)=p_0^{-1}$, obtained by substituting $i=0$ and $q_0=0$ in (10), we get successively expressions for $G'_{1,2}(z=1),G'_{2,3}(z=1),\ldots,$ and $G'_{N-1,N}(z=1)$. Finally summing these terms up as given by (9) we obtain an expression for the MFPT,

$$\bar{t}_{0,N}=\sum_{k=0}^{N-1}\frac{1}{p_k}+\sum_{k=0}^{N-2}\frac{1}{p_k}\sum_{i=k+1}^{N-1}\prod_{j=k+1}^{i}\frac{q_i}{p_j} . \quad (11)$$

Note that the above expression for the MFPT contains explicitly the basic jump probabilities $\{p_0,p_1,\ldots,p_{N-1},q_1,q_2,\ldots,q_{N-1}\}$, unlike the one given by Noskowicz and Goldhirsch.[6] An equivalent expression has been obtained by Van den Broeck[15] by a rather different method.

Let us substitute, in (11), $q_j=p_j=\gamma$ for $j$. (Note that, however, $q_0=0$.) This corresponds to simple random walk and we recover the well-known result,[10] $\bar{t}_{0,N}=N(N+1)/2\gamma$. As a second example consider biased random walks by setting $q_j=q$ $(j\neq 0)$ and $p_j=p$ with $q\neq p$ and $q+p=1$. We get, from (11),

$$\bar{t}_{0,N}=\frac{1+\alpha}{1-\alpha}N+\frac{\alpha(1+\alpha)}{(1-\alpha)^2}(\alpha^N-1) , \quad (12)$$

where $\alpha=(1-p)/p$. Letting $N\to\infty$, we obtain the asymptotic behavior for two cases, one with $\alpha<1$ ($p>q$, the bias toward the right, aiding the motion of the random walk) and the other with $\alpha>1$ ($p<q$, the bias toward the left opposing the motion of the random walk). These are

$$\bar{t}_{0,N}\sim\begin{cases}\dfrac{1+\alpha}{1-\alpha}N & \text{if } \alpha<1\\[2mm] \dfrac{\alpha(1+\alpha)}{(1-\alpha)^2}\alpha^N & \text{if } \alpha>1 .\end{cases} \quad (13)$$

The above results for biased random walks are well known.[9]

In what follows, we apply the formulation developed above and in particular Eq. (11) to different models of disordered systems.

## III. LATTICE WITH RANDOM SYMMETRIC TRANSITION PROBABILITIES

In this section we consider the random-barrier model which is usually considered in connection with continuous-time random walks. Here we construct a discrete time version of this model and to this end we proceed as follows.

For any lattice site $j$ $(\neq 0)$ we set $q_j=p_{j-1}$, i.e., the left-jump probability at site $j$ is equal to the right-jump probability at its previous nearest-neighbor site $j-1$. The symmetry in the transition probabilities reduces the number of random variables that define the random lattice, to $N$. Thus at site $j$ $(\neq 0)$ the left-jump probability per step is $q_j$, the right-jump probability per step is $q_{j+1}$ $(=p_j)$, and the sojourn probability per step is $1-(q_j+q_{j+1})$. At $j=0$, however $q_1$ $(=p_0)$ is the right-jump probability and $(1-q_1)$ is the sojourn probability per step. We restrict the range of the random variables $q_j$ to be $(0,\frac{1}{2})$, for all $j=1,N$, to have $q_j+q_{j+1}\leq 0$. We treat the set of $N$ random variables $\{q_j\colon j=1,N\}$ as independent and identically distributed with a common distribution, formally denoted by $\rho(q)$. In what follows we assume $\rho(q)$ to be such that $\langle q^{-n}\rangle=\int_0^{1/2}q^{-n}\rho(q)dq<\infty$, for all $n\geq 0$.

We can substitute $q_{j+1}=p_j$ in (11) and obtain an expression for $\bar{t}_{0,N}$ for the random-barrier model. A more transparent way, which leads to the same result, is to start with the recursion relation given by (10) and obtain

$$G'_{i-1,i}(z=1)=i/q_i . \quad (14)$$

This leads to a very simple expression for the MFPT,

$$\bar{t}_{0,N}\equiv\bar{t}_{0,N}(q_1,q_2,\ldots,q_N)=\sum_{j=1}^{N}\frac{j}{q_j} . \quad (15)$$

We can carry out the disorder averaging, and obtain $\langle\bar{t}_{0,N}\rangle=\langle q^{-1}\rangle N(N+1)/2$.

Thus asymptotically ($N\to\infty$), the disorder-averaged MFPT is proportional to $N^2$, a behavior similar to that of a simple random walk, but with an effective, disorder-dependent diffusion constant, $\tilde{D}\equiv N^2/2\langle\bar{t}_{0,N}\rangle=\langle q^{-1}\rangle^{-1}$, a result that agrees with the one obtained[2] by calculating the asymptotic behavior of mean-square displacement. Let $q^L$ and $q^H$ denote the lowest and the highest value that $q$ can take, respectively. Note that $q^L>0$ and $q^H\leq\frac{1}{2}$. It is easily seen from Eq. (15) that, asymptotically $(2q^H)^{-1}N^2\leq\bar{t}_{0,N}\leq(2q^L)^{-1}N^2$, which shows that the typical[6] and disorder-averaged MFPT have the same asymptotic $N$ dependence.

For the random-barrier model, the probability distribution of the MFPT resulting from the disorder can be characterized in more detail by exploiting the simple formula (15). The variance of $\bar{t}_{0,N}$ can be easily calculated and its asymptotic behavior is given by

$$\langle(\bar{t}_{0,N})^2\rangle-\langle\bar{t}_{0,N}\rangle^2\sim\tfrac{1}{3}(\langle q^{-2}\rangle-\langle q^{-1}\rangle^2)N^3 . \quad (16)$$

Hence the relative fluctuations (defined as the standard deviation divided by the mean) of $\bar{t}_{0,N}$ go as $N^{-1/2}$. Thus the distribution of $\bar{t}_{0,N}$ becomes more and more peaked as $N\to\infty$. By virtue of the central limit theorem, as $N\to\infty$, the distribution of $\bar{t}_{0,N}$ tends to a Gaussian. The third-order cumulant is asymptotically proportional to $N^4$,

$$\langle(\bar{t}_{0,N})^3\rangle_c\sim\tfrac{1}{4}(\langle q^{-3}\rangle-3\langle q^{-2}\rangle\langle q^{-1}\rangle+2\langle q^{-1}\rangle^3)N^4 . \quad (17)$$

Hence the relative magnitude, which is obtained by dividing with $\langle\bar{t}_{0,N}\rangle^3$ and taking the cubic root, is proportional to $N^{-2/3}$. This means that the relative skewness of the distribution becomes small in the limit of large $N$. The fourth-order cumulant is found to be asymptotically proportional to $N^7$,

$$\langle(\bar{t}_{0,N})^4\rangle_c\sim\tfrac{1}{2}(\langle q^{-1}\rangle^4-\langle q^{-2}\rangle\langle q^{-1}\rangle^2)N^7 . \quad (18)$$

Hence its relative magnitude decreases only proportional to $N^{-1/4}$ asymptotically. We expect analogous behavior for the relative magnitudes of the higher-order cumulants, i.e., our conjecture is that

$$\langle(\bar{t}_{0,N})^n\rangle_c^{1/n}/\langle\bar{t}_{0,N}\rangle\sim N^{-1/n} \quad (19)$$

for $n \geq 4$. Although the asymptotic form of the probability distribution of $\bar{t}_{0,N}$ is Gaussian, this limit is approached very slowly.

## IV. LATTICE WITH RANDOM SOJOURN PROBABILITIES

In this section we consider a discrete version of the random-trap model. The class of random-trap models is important for modeling transport with temporary trapping of particles; see the review of Haus and Kehr.[1]

For this model we set at any lattice site $j$ ($\neq 0$), $q_j = p_j = \gamma_j$. The sojourn probability at any lattice site $j$ ($\neq 0$) is $1-2\gamma_j$. At $j=0$, the probability for a right jump is $\gamma_0$ and the sojourn probability per step is $1-\gamma_0$. $\{\gamma_j : j=0, N-1\}$ constitutes a set of random variables defining the lattice.

We can substitute $q_j = p_j = \gamma_j$ in Eq. (11), and obtain an expression for $\bar{t}_{0,N}$. However, it is easier to start with the recursion relation (10) to obtain

$$G'_{i-1,i}(z=1) = \sum_{j=0}^{i-1} \frac{1}{\gamma_j} \tag{20}$$

and thus

$$\bar{t}_{0,N} = \sum_{j=1}^{N} \frac{j}{\gamma_{N-j}} \ . \tag{21}$$

Equation (21) is very similar to (15) and the conclusions are the same as for the random-barrier model. For instance, the disorder-averaged MFPT is $\langle \bar{t}_{0,N} \rangle = \langle \gamma^{-1} \rangle N(N+1)/2$. We can say that both models are of the same universality class as far as the properties of $\bar{t}_{0,N}$ caused by the disorder are concerned and refer to the discussion in Sec. III.

## V. LATTICE WITH RANDOM ASYMMETRIC TRANSITION PROBABILITIES

For this model of disorder, we have at any lattice site $j$ ($\neq 0$), $q_j + p_j = 1$. Thus $\{p_0, p_1, \ldots, p_{N-1}\}$ constitutes a set of $N$ random variables and we take them as independent and identically distributed, with a common distribution $\rho(p)$. Note that $0 < p \leq 1$. Also note that at any lattice site $j \neq 0$, the sojourn probability per step is zero and at $j=0$, the sojourn probability per step is $1-p_0$. Let us denote $\alpha_j = (1-p_j)/p_j$ and rewrite (11) as

$$\begin{aligned} \bar{t}_{0,N} &\equiv \bar{t}_{0,N}(p_0, p_1, \ldots, p_{N-1}) \\ &= N + \sum_{k=0}^{N-1} \alpha_k + \sum_{k=0}^{N-2} \sum_{i=k+1}^{N-1} (1+\alpha_k) \prod_{j=k+1}^{i} \alpha_j \ . \end{aligned} \tag{22}$$

We can now perform the disorder average. Let $\langle \alpha \rangle$ denote the average of $(1-p)/p$ over $\rho(p)$. We get

$$\langle \bar{t}_{0,N} \rangle = \frac{1+\langle \alpha \rangle}{1-\langle \alpha \rangle} N + \frac{\langle \alpha \rangle^2 + \langle \alpha \rangle}{(1-\langle \alpha \rangle)^2} (\langle \alpha \rangle^N - 1) \ . \tag{23}$$

This expression for the disorder-averaged MFPT is exact and is valid for all $N$. Also note that $\langle \alpha \rangle \neq 1$ in the above. If $\langle \alpha \rangle = 1$, we see from (22) that $\langle \bar{t}_{0,N} \rangle = N(N+1)$.

Equation (23) is exactly the same as (12) for biased random walks if we set $\alpha = \langle \alpha \rangle$. Thus the disorder-averaged MFPT behaves exactly like the MFPT of a biased random walk, but with a disorder-dependent bias factor $\alpha = \langle \alpha \rangle$. Letting $N \to \infty$ in (23) we find the asymptotic behavior of $\langle \bar{t}_{0,N} \rangle$ as

$$\langle \bar{t}_{0,N} \rangle \sim \begin{cases} \dfrac{1+\langle \alpha \rangle}{1-\langle \alpha \rangle} N, & \langle \alpha \rangle < 1 \\ N^2, & \langle \alpha \rangle = 1 \\ \dfrac{\langle \alpha \rangle^2 + \langle \alpha \rangle}{(\langle \alpha \rangle - 1)^2} \langle \alpha \rangle^N, & \langle \alpha \rangle > 1 \ . \end{cases} \tag{24}$$

For the Sinai model[14] the prescription is that $\langle \ln(\alpha) \rangle = 0$, and the variance of $\ln(\alpha)$, denoted by $\sigma^2$, is finite. If $\langle \ln(\alpha) \rangle = 0$, then $\langle \alpha \rangle > 1$. Thus we find that the Sinai model[14] is analogous to biased random walks with an effective disorder-dependent bias factor $\langle \alpha \rangle > 1$, opposing the motion of the random walk. Thus $\langle \bar{t}_{0,N} \rangle$ does not exhibit anomalous behavior, as was found for the mean and mean-square displacement under the Sinai condition. We will further comment on this point in Sec. VII. However, for this model of a disordered system the average behavior does not reflect the typical behavior, as demonstrated by Noskowicz and Goldhirsch.[6] They showed that the typical MFPT goes asymptotically as $\exp(\sigma\sqrt{N})$, a behavior different from that of the disorder-averaged MFPT. The asymptotic behavior of the typical MFPT can be deduced by obtaining lower and upper bounds.[6] Since we have now the MFPT explicitly in terms of the basic jump probabilities, the arguments leading to the bounds are simpler and more transparent. We present these briefly in the Appendix.

A model with asymmetric transition probabilities can also be introduced by taking $q_i$ and $p_i$ at each site as independent random variables, with their values restricted to the interval $0 < p_i, q_i \leq \frac{1}{2}$, and $q_0 = 0$. This is exactly the general model introduced in Sec. II. In this case there is a sojourn probability per step $1-q_i-p_i \geq 0$ at each site. An explicit expression for the disorder-averaged MFPT is easily derived from (11). If the condition $\langle q \rangle \langle p^{-1} \rangle > 1$ is fulfilled the results are equivalent to the Sinai model. Indeed, this condition is met if $p$ and $q$ are from the same distribution.

## VI. LATTICE WITH BOND RANDOMNESS

In this model, the line joining two adjacent lattice sites is regarded as constituting a bond. Thus the $j$th bond is formed by the line connecting the site $j-1$ to the site $j$, where $j$ runs from 1 to $N$. The right-jump probability at site $j-1$ is correlated to the left-jump probability at the site $j$. Let us denote $\xi_j = q_j / p_{j-1}$ for the $j$th bond. The jump probabilities for the different bonds, however are uncorrelated. Since at any site $j$, the left- and right-jump probabilities are independent of each other and, since $q_j + p_j \leq 1$, we require that $q_j$ and $p_j$ be restricted to the range $(0, \frac{1}{2})$. Thus $\{\xi_j\}$ constitutes a set of $N$ random variables whose distribution can be obtained from the distribution of the pairs of random variables

$\{q_j, p_{j-1}: j=1,N\}$. This model of disorder, in the continuous-time description, has been considered by many authors.[16]

The MFPT for a given realization of this model is given by (11), which can be rewritten in the following convenient form:

$$\bar{t}_{0,N} = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \sum_{i=k+1}^{N-1} \frac{1}{p_i} \prod_{j=k+1}^{i} \xi_j \ . \quad (25)$$

Performing the disorder average and taking the asymptotic limit of $N \to \infty$, we obtain

$$\langle \bar{t}_{0,N} \rangle \sim \begin{cases} \langle p^{-1} \rangle \{1-\langle \xi \rangle\}^{-1} N, & \langle \xi \rangle < 1 \\ \langle p^{-1} \rangle N^2/2, & \langle \xi \rangle = 1 \\ \dfrac{\langle p^{-1} \rangle \langle \xi \rangle}{(1-\langle \xi \rangle)^2} \langle \xi \rangle^N, & \langle \xi \rangle > 1 \ . \end{cases} \quad (26)$$

We discuss the different cases separately.

### Case (i): $\langle \xi \rangle < 1$

It is seen that when $\langle \xi \rangle < 1$, the disorder-averaged MFPT behaves like the MFPT of a biased walk with an effective disorder-dependent bias to the right aiding the motion of the random walk. We can obtain an expression for the velocity, $V \equiv N/\langle \bar{t}_{0,N} \rangle = (1-\langle \xi \rangle)\langle p^{-1} \rangle^{-1}$, a result in agreement with that obtained by Derrida.[4] This problem has also been considered by Bernasconi and Schneider.[5] These authors prescribe that the pair $(p_j, q_{j+1})$ takes values $(u,0)$ with probability $c$ and $(\lambda v, v)$ with probability $1-c$. Actually Bernasconi and Schneider[5] have considered continuous-time models and hence the parameters $u$, $v$, and $\lambda v$ are the transition rates which can take any positive values. But since we are dealing with discrete time, we require these parameters to be restricted to the range $(0, \frac{1}{2})$. The case with $\lambda > 1-c$ corresponds to $\langle \xi \rangle < 1$ and we get

$$V = uv(\lambda - 1 + c)/(c\lambda v + u - uc) \ ,$$

a result in agreement with that of Bernasconi and Schneider.[5]

### Case (ii): $\langle \xi \rangle = 1$

For this case we find that $\langle \bar{t}_{0,N} \rangle$ behaves like the MFPT of a pure diffusive process with an effective disorder-dependent diffusion constant $\tilde{D} = \langle p^{-1} \rangle^{-1}$. Thus the disorder-averaged MFPT does not exhibit any anomalous behavior. This case corresponds to the case with $\lambda = 1-c$, considered by Bernasconi and Schneider,[5] for which they find that the disorder-averaged mean displacement behaves anomalously.

### Case (iii): $\langle \xi \rangle > 1$

For this case, $\langle \bar{t}_{0,N} \rangle$ behaves like that of a biased random walk with a disorder-dependent bias to the left, acting against the motion of the random walk. Again we see that $\langle \bar{t}_{0,N} \rangle$ does not exhibit any anomalous behavior. This case corresponds to the case with $\lambda < 1-c$ considered by Bernasconi and Schneider[5] for which they find that the disorder-averaged mean displacement behaves anomalously.

We have not considered the distribution of $\bar{t}_{0,N}$, resulting from the disorder, for this model. We expect it to be similar to that of the lattice segment with asymmetric transition probabilities. In particular, for case (iii), we expect $\bar{t}_{0,N}$ to show a behavior analogous to the Sinai model; cf. Sec. V.

## VII. CONCLUSION

We have presented in this paper a method to calculate the MFPT of random walks on one-dimensional random lattices. We reported expressions for the MFPT explicitly in terms of the basic transition probabilities. We considered four important models of disorder and for these we obtained exact analytical expressions for the disorder-averaged MFPT.

For the random-barrier and the random-trap models we find that the disorder-averaged MFPT behaves like the MFPT of a pure diffusive process. For the Sinai lattice we find that the disorder-averaged MFPT does not exhibit any anomalous behavior. Indeed, it behaves like the MFPT of a biased random walk with a bias acting against the motion of the walk. For the model with bond randomness we find again that the disorder-averaged MFPT behaves like the MFPT of an ordinary walk [case (ii) in Sec. VI] or a biased walk [cases (i) and (iii) in Sec. VI].

It would be useful and interesting to extend the formulation presented in this paper to the calculation of higher moments of first-passage time (random-walk average) and perhaps the distribution itself. Recently, Van den Broeck[15] proposed a formulation, very different from ours, and obtained an equivalent expression for the MFPT in terms of basic transition probabilities. His formulation enables calculation of higher moments of first-passage time (random-walk average). Study of higher moments in general, and the distribution in particular, would be helpful in characterizing the behavior of the MFPT.

The main theme of our investigation was the disorder-averaged MFPT, $\langle \bar{t}_{0,N} \rangle$, for the various models of disordered segments. Also some properties of the distribution of $\bar{t}_{0,N}$ resulting from the disorder were studied. For the random-barrier and random-trap models the distribution is asymptotically a Gaussian. Here the disorder determines more subtle features such as the behavior of the fourth- and higher-order cumulants for large $N$. In contrast, for the Sinai model, the distribution of $\bar{t}_{0,N}$ is not Gaussian. This is already evident from the fact that the typical and average MFPT behave quite differently as functions of the segments lengths; see Ref. 6 and the Appendix. Clearly the study of the distribution of the MFPT for this class of disordered models is a challenging problem, and is currently under investigation. Also the results on the mean and mean-square displacements for the Sinai model should be reexamined. The question is to which extent does the reported anomalous behavior of these quantities reflect typical and average behavior.

*Note added in proof.* Related results on MFPT were obtained by Doussal[17] and by Matan and Havlin.[18]

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## APPENDIX: UPPER AND LOWER BOUNDS FOR $\bar{t}_{0,N}$

Here we obtain asymptotic behavior of upper and lower bounds for the typical MFPT.

### Upper bound

Let $\hat{\alpha}$ denote the highest value that $\alpha$ can take. We assume that $\hat{\alpha} < \infty$, which implies that $p_j > 0$, for all $j$. Let $\Pi(a,b) = \prod_{j=a}^{b} \alpha_j$. Then, from (22), we see that

$$\bar{t}_{0,N} \le N(1+\hat{\alpha}) + \sum_{k=0}^{N-2} \sum_{i=k+1}^{N-1} [\Pi(k+1,i) + \Pi(k,i)] \ . \quad \text{(A1)}$$

Let $x = \ln[\Pi(0,N-1)]$. It is clear that $x$ is the displacement of an $N$-step random walk, whose statistics are determined by those of $\ln\alpha$. Note that by the Sinai condition $\langle x \rangle = 0$. The variance of $x$ is $N\sigma^2$, where $\sigma^2 < \infty$ is the variance of $\ln\alpha$. For $N \to \infty$, $x$ is normally distributed, by the central limit theorem. It is easily seen that

$$P(x > \sigma N^{(1/2)+\epsilon}) = \tfrac{1}{2} - \text{erf}(N^{\epsilon}) \equiv \delta \ , \quad \text{(A2)}$$

where erf( ) denotes the error function defined as

$$\text{erf}(r) = \frac{1}{\sqrt{2\pi}} \int_0^r \exp\left[-\frac{y^2}{2}\right] dy \ . \quad \text{(A3)}$$

Thus, for any $\epsilon$, however small, we can render $\delta$ as small as desired, by taking $N$ sufficiently large. Hence, the highest value of $\ln[\Pi(a,b)]$ does not exceed $\exp(\sigma N^{(1/2)+\epsilon})$ for any $a,b$ such that $b - a \le N$. Since $\epsilon$ is arbitrary, we conclude that the upper bound of $\bar{t}_{0,N}$ goes asymptotically as $\exp(\sigma\sqrt{N})$.

### Lower bound

It is easy to see, from (22), that

$$\bar{t}_{0,N} > \prod_{j=0}^{N-1} \alpha_j \ . \quad \text{(A4)}$$

Following Noskowicz and Goldhirsch,[6] we consider the lattice $(0,N)$, to be made up of $S$ subsegments $\{0,n\},\{n,2n\},\ldots,\{(S-1)n,Sn\}$, such that $nS = N$. It is obvious that $\bar{t}_{0,N} > \sum_{k=0}^{S-1} \bar{t}_{kn,(k=1)n}$. Let us use the inequality (A4) for each subsegment, and get

$$\bar{t}_{0,N} > \sum_{k=0}^{S-1} \prod_{j=kn}^{(k+1)n-1} \alpha_j \ . \quad \text{(A5)}$$

The $S$ products in the above summation can be regarded as $S$ independent random variables. Let $Q$ denote the probability that at least one of the $S$ products is greater than $\exp(\sigma N^{(1/2)-\epsilon})$, where $\epsilon > 0$ is arbitrarily small. To calculate $Q$, we proceed as follows.

Let $x_k = \sum_{j=kn}^{(k+1)n-1} \ln\alpha_j$. When $n$ is large $x_k$ is normally distributed with mean 0 and variance $n\sigma^2$. Then,

$$P(x_k > \sigma N^{(1/2)-\epsilon}) = \tfrac{1}{2} - \text{erf}(S^{(1/2)-\epsilon} n^{-\epsilon}) \ . \quad \text{(A6)}$$

It follows immediately that

$$Q = 1 - [\tfrac{1}{2} + \text{erf}(S^{(1/2)-\epsilon} n^{-\epsilon})]^S \ . \quad \text{(A7)}$$

For a given $n$ (large) and $\epsilon$ (small), we can choose $S$ such that $nS \gg S^{1/2\epsilon}$. Thus we see that $Q$ can be made as close to unity as desired, for a given arbitrarily small $\epsilon$ and a given large $n$, by taking $S$ sufficiently large. It follows that asymptotically the lower bound of $t_{0,N}$ goes as $\exp(\sigma\sqrt{N})$.

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