Thermal quasiparticles with phase-space distribution

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A new Wigner phase-space distribution function is introduced for thermal quasiparticles currently in use in thermo field dynamics. The properties of this phase-space distribution are studied and compared with those of the corresponding zero-temperature distribution. Generalized uncertainty relations are implemented in order to study the explicit spreading in phase space as a function of temperature and quantum number of the thermal excitation, thereby permitting the investigation of quantum and thermal fluctuations.

I. INTRODUCTION

The phase-space representation of quantum mechanics is of current interest.[1] A function endowed with many properties of a phase-space probability distribution has been introduced by Wigner[2] and used to study a variety of systems.[3] Moreover, this function has been suggested of late as a candidate for the measurement of quantum states[4] and has recently been extended to the relativistic regime by studying a Lorenz covariant phase-space representation.[5] Being a quantum generalization of Boltzmann's N-particle distribution, the Wigner function provides an intuitive picture of complex processes.

The purpose of the present work is to extend the definition of the Wigner function to thermal quasiparticles. The Wigner function for nonpure states, including thermal ones, is of course well known. However, it has been defined and used only for states representable by a density matrix. As will become clear in this paper, a phase-space Wigner function may be very useful for the understanding of thermal excitations which do not correspond to the standard Hermitian density matrix. Such states are encountered in thermo field dynamics (TFD).[6]

In conventional quantum field theory, the dynamical behavior is described by observable free fields associated with stable particles. However, when quantum fields with thermal degrees of freedom are involved, the fields are no longer free and carry nonstable particles moving within a medium. These are the excitations investigated in this paper by a phase-space distribution. Such a study enhances the physical insight into the nature of these thermal "excitons" and their associated semifree fields.

In equilibrium TFD the degrees of freedom are doubled by introducing the *tilde conjugation rule.* This operator doubling in quantum statistics has been noticed in the past 20 years and seems to be more than a computational trick. A similar situation occurs in other formulations of real-time quantum field theories at finite temperature[7] such as the C^*-algebraic formalism,[8] the path ordering formalism of Schwinger and Keldysh,[9] and superoperator formalism of the Liouville equation.[10] Intuitively speaking, the operators without tildes take care of the usual quantum excitations, while the operators with tildes describe the thermal fluctuations. This will become clear when the various contributions to the generalized uncertainty relations applied to the thermal quasiparticles are discussed.

II. QUASIPARTICLES IN THERMO FIELD DYNAMICS

A Bose oscillator system is considered. The harmonic oscillator in phase-space resembles classical dynamics and is essential for the understanding of any phase-space description of quantum systems. However, it should be stressed from the outset that the harmonic case is chosen for clarity of presentation and is by no means necessary. The extension to systems having more dimensions is straightforward. The Hamiltonian is described by a and $a^\dagger$ satisfying

$$[a,a^\dagger]=1 \ . \tag{2.1}$$

In TFD the corresponding *tilde* degrees of freedom are introduced, providing the additional operators $(\tilde{a},\tilde{a}^\dagger)$ which satisfy

$$[\tilde{a},\tilde{a}^\dagger]=1 \ . \tag{2.2}$$

The aforementioned equations define the quantum algebra of these operators. Furthermore, it is assumed that operators with and without tildes commute, i.e.,

$$[a,\tilde{a}]=[a,\tilde{a}^\dagger]=[a^\dagger,\tilde{a}]=[a^\dagger,\tilde{a}^\dagger]=0 \ . \tag{2.3}$$

To the above doubled set of degrees of freedom there correspond orthonormalized eigenvectors of $a^\dagger a$ and $\tilde{a}^\dagger\tilde{a}$ with respective eigenvalues m and $\tilde{m}$ denoted by $|m,\tilde{m}\rangle$ and $\langle m,\tilde{m}|$. These serve to introduce the bra and ket temperature-dependent vacua, $\langle 0(\beta)|$ and $|0(\beta)\rangle$ with the inverse temperature $\beta=1/kT$, k being Boltzmann's constant, as

$$\begin{aligned}\langle 0(\beta)| &= \langle I|\rho(\beta)^{1-\alpha} \ ,\\ |0(\beta)\rangle &= \rho(\beta)^{\alpha}|I\rangle \ .\end{aligned} \tag{2.4}$$

Here α is a parameter $0\le\alpha\le 1$, $\rho(\beta)$ is the temperature-dependent density matrix, and

$$|I\rangle=\sum_m |m,\tilde m\rangle\ ,\qquad \langle I|=\sum_m \langle m,\tilde m|\ . \tag{2.5}$$

The above construction assures that the statistical average of an operator A is given by the vacuum expectation value

$$\langle 0(\beta)|A|0(\beta)\rangle=\mathrm{tr}[\rho(\beta)A]\ . \tag{2.6}$$

In particular,

$$n=\langle 0(\beta)|a^\dagger a|0(\beta)\rangle=\mathrm{tr}[\rho(\beta)a^\dagger a]=\frac{1}{e^{\beta\omega}-1}\ , \tag{2.7}$$

where ω is the oscillator frequency. The flexibility in choice of the parameter α is a manifestation of the cyclic nature of the trace formula $\mathrm{Tr}\rho A=\mathrm{Tr}\rho^{1-\alpha}A\rho^{\alpha}$. In this paper the alternatives $\alpha=\frac{1}{2}$ and 1 will be considered.

The thermal vacuum is used to define generalized creation $\xi^\dagger,\tilde\xi^\dagger$ and annihilation $\xi,\tilde\xi$ operators satisfying

$$\langle 0(\beta)|\xi^\dagger=\langle 0(\beta)|\tilde\xi^\dagger=0\ ,\qquad \xi|0(\beta)\rangle=\tilde\xi|0(\beta)\rangle=0\ . \tag{2.8}$$

These operators are related to the oscillator operators by means of a type of Bogoliubov transformation

$$\xi=(1+n)^{\alpha}a-n^{\alpha}\tilde a^\dagger\ ,\qquad \xi^\dagger=(1+n)^{1-\alpha}a^\dagger-n^{1-\alpha}\tilde a\ , \tag{2.9}$$

and their *tilde* conjugates, where n is given by Eq. (2.7). The rules for *tilde* conjugation for operators A and B, and c numbers c_1 and c_2 are given by

$$\begin{aligned}&(AB)\tilde{}=\tilde A\tilde B\ ,\\ &(c_1A+c_2B)\tilde{}=c_1^*\tilde A+c_2^*\tilde B\ ,\\ &\tilde A^\dagger=(A^\dagger)\tilde{}\ ,\\ &(\tilde A)\tilde{}=A\ ,\\ &|0(\beta)\rangle\tilde{}=|0(\beta)\rangle\ ,\\ &\langle 0(\beta)|\tilde{}=\langle 0(\beta)|\ .\end{aligned} \tag{2.10}$$

The thermal quasiparticle excitation is now defined as

$$|M\rangle=(1/M^{1/2})\xi^{\dagger M}|0(\beta)\rangle\ . \tag{2.11}$$

Note that this is not a usual pure state, as it depends on the temperature. Moreover, it cannot be represented by the usual Hermitian density operator, although it depends on the density matrix of the system contained in the thermal vacuum [Eq. (2.4)]. A similar form may be used for the definition of a *tilde* excitation.

III. THE WIGNER PHASE-SPACE DISTRIBUTION FOR THE THERMAL EXCITATION $|M\rangle$

Due to the doubling of degrees of freedom, the coordinate representation of the state $|M\rangle$ of Eq. (2.11)

$$\langle x,\tilde x'|M\rangle=\langle x,\tilde x'|(1/M^{1/2})\xi^{\dagger M}|0(\beta)\rangle \tag{3.1}$$

depends on both x and $\tilde x'$. It is desirable to confine oneself to the physical space x only. Thus, in the definition of the Wigner form $f_M(q,p)$ the *tilde* degrees of freedom are integrated, leaving a "reduced" representation in the phase space q,p:

$$\begin{aligned}f_M(q,p)=\frac{1}{2\pi^2}\int d\tilde q\,'d\tilde p\,'\int dy\,d\tilde y\,'e^{2i\tilde p'\tilde y'+2ipy}(&\langle q-y,\tilde q\,'-\tilde y\,'|M\rangle\langle M|q+y,\tilde q\,'+\tilde y\,'\rangle\\ &+\langle q+y,\tilde q\,'+\tilde y\,'|M\rangle\langle M|q-y,\tilde q\,'-\tilde y\,'\rangle)\ .\end{aligned} \tag{3.2}$$

Equation (3.2) serves as a basic definition of this paper. The second term on the right-hand side (rhs) of this equation is introduced to assure that $f_M(q,p)$ is real. Integration of Eq. (3.2) using the definition of the "thermal quasiparticle" excitation (2.11) and the rules for *tilde* conjugation (2.10), the details are deferred to Appendix A, leads to the result

$$\begin{aligned}f_M(q,p)=\frac{1}{2\pi M}\sum_{m=0}^{M}\begin{bmatrix}M\\ m\end{bmatrix}\sum_{m'=0}^{M}\begin{bmatrix}M\\ m'\end{bmatrix}\int dy e^{2ipy}\{&\langle q-y|[(1+n)^{1-\alpha}a^\dagger]^m\rho^\alpha(-n^{1-\alpha}a^\dagger)^{M-m}\\ &\times(-n^\alpha a)^{M-m'}\rho^{1-\alpha}[(1+n)^\alpha a]^{m'}|q+y\rangle+\mathrm{H.c.}\}\ .\end{aligned} \tag{3.3}$$

In the case of $\alpha=\frac{1}{2}$ the two terms on the rhs of Eq. (3.2) or (3.3) are equal. If one insists on defining a density operator, Eq. (3.3) may suggest the density operator D_M defined as

$$D_M=\frac{1}{2M}\sum_{m=0}^{M}\begin{bmatrix}M\\ m\end{bmatrix}\sum_{m'=0}^{M}\begin{bmatrix}M\\ m'\end{bmatrix}\{[(1+n)^{1-\alpha}a^\dagger]^m\rho^\alpha(-n^{1-\alpha}a^\dagger)^{M-m}(-n^\alpha a)^{M-m'}\rho^{1-\alpha}[(1+n)^\alpha a]^{m'}+\mathrm{H.c.}\}\ . \tag{3.4}$$

Clearly this is not the usual density operator and it is a complicated function of the thermal density ρ. Nevertheless, this is the operator corresponding to the thermal quasiparticles. Note that it is the basic form of the Wigner function (3.2) and (3.3) which helps to define a generalized density matrix in (3.4). This generalized density operator will not be used in this paper but may be of interest in other contexts.

In the following, a few special cases of Eqs. (3.2) and (3.3) are investigated. First, the vacuum $|0(\beta)\rangle$ is considered, namely, the case of finite temperature with $M=0$, resulting in

$$f_0(q,p)=\frac{1}{\pi}\int dy e^{2ipy}\langle q-y|\rho|q+y\rangle \tag{3.5}$$

independent of α, this being the usual Wigner form of the density matrix ρ. Next, for the sake of clarity Eq. (3.3) is displayed for the particular case $M=1$,

$$f_1(q,p)=\frac{1}{\pi}\int dy e^{2ipy}\langle q-y|[(1+n)a^\dagger\rho a+n\rho^{1/2}a^\dagger a\rho^{1/2}-(1+n)^{1/2}n^{1/2}(a^\dagger\rho^{1/2}a\rho^{1/2}+\rho^{1/2}a^\dagger\rho^{1/2}a)]|q+y\rangle,\quad \alpha=\tfrac{1}{2} \tag{3.6}$$

and

$$f_1(q,p)=\frac{1}{\pi}\int dy e^{2ipy}\langle q-y|a^\dagger\rho a-\tfrac{1}{2}\rho a^\dagger a-\tfrac{1}{2}a^\dagger a\rho|q+y\rangle,\quad \alpha=1\ . \tag{3.7}$$

Equation (3.3) [and its particular cases (3.6) and (3.7)] at zero temperature $T=0$, where $n=0$ and $\rho=|0\rangle\langle 0|$, reduce to

$$f_M(q,p)=\frac{1}{\pi}\int dy e^{2ipy}\langle q-y|a^{\dagger M}|0\rangle\langle 0|a^M|q+y\rangle \tag{3.8}$$

which is the usual Wigner form of a pure state.[2]

In order to get insight into the nature of the phase-space representation of the thermal quasiparticles as given by Eq. (3.3), the cases $M=0,1,2$ for zero and finite temperature are displayed graphically. Figure 1 shows

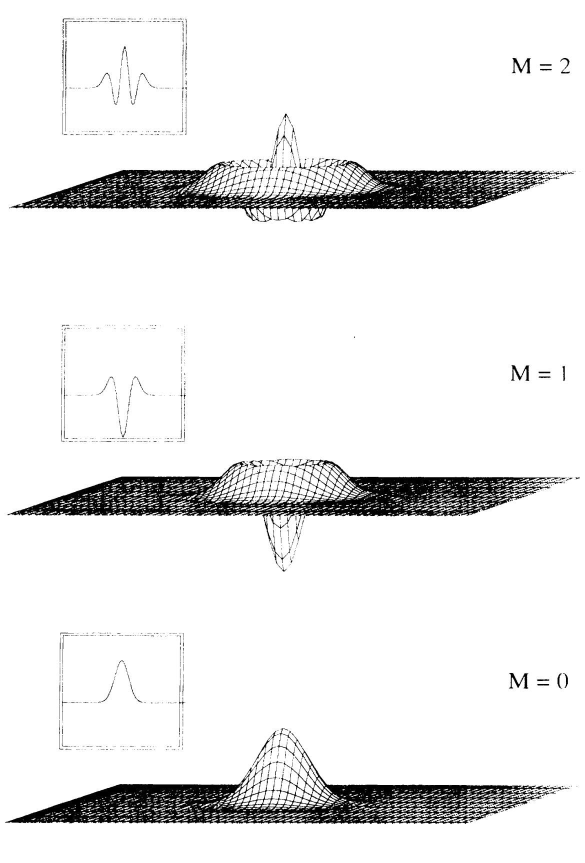

FIG. 1. The Wigner function $f_M(q,p)$ at zero temperature according to Eq. (3.8). The nodes of the pure state are manifested by the number of times the zero (q,p) plane is intersected in phase space. The inset showing a cut along the $p=0$ value clarifies the structure of the nodes.

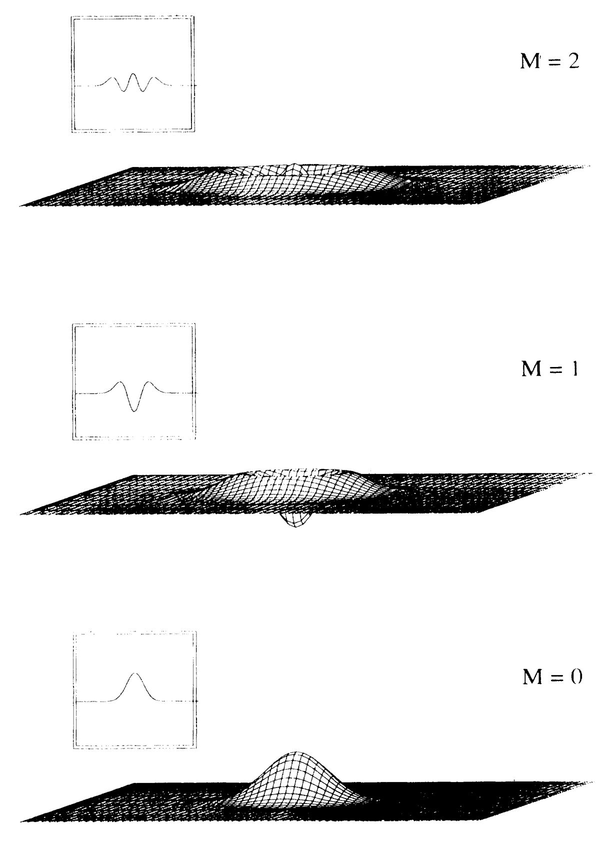

FIG. 2. The Wigner function $f_M(q,p)$ at finite temperature according to Eq. (3.2). The structure of the nodes is similar to that of Fig. 1 but smeared in phase space. The inset showing a cut along the $p=0$ value clarifies the structure of the nodes.

the zero-temperature Wigner form of these states according to Eq. (3.8). The usual nodes of the pure states are manifested by the number of times the zero plane is intersected. At finite temperature, $\beta=1.61$ ($n=\frac{1}{4}$), as displayed in Fig. 2 according to Eq. (3.3), the states maintain the same number of nodes and the same graphical shape but occupy a larger portion of phase space [see inset in the graphs showing the constant $p=0$ section of $f_M(q,p)$]. At yet a higher temperature, $\beta=0.693$ ($n=1$) displayed in Fig. 3, the states are even more spread out. Note that all scales in Figs. 1–3 are the same with a phase space covering the region q: $[-6,6.6]$, p: $[-7.61,7.61]$ in the units used throughout this paper (oscillator mass and frequency as well as Planck's constant divided by 2π equal to 1). The results show that the finite-temperature Wigner form is independent of α. This is not obvious on inspecting Eq. (3.3), or even when trying to compare Eqs. (3.6) and (3.7), but it is a very encouraging observation. It is conjectured that a proper Wigner function should not depend on α, this being suggested by the possibility of considering the Wigner function as a candidate of quantum observables.[4]

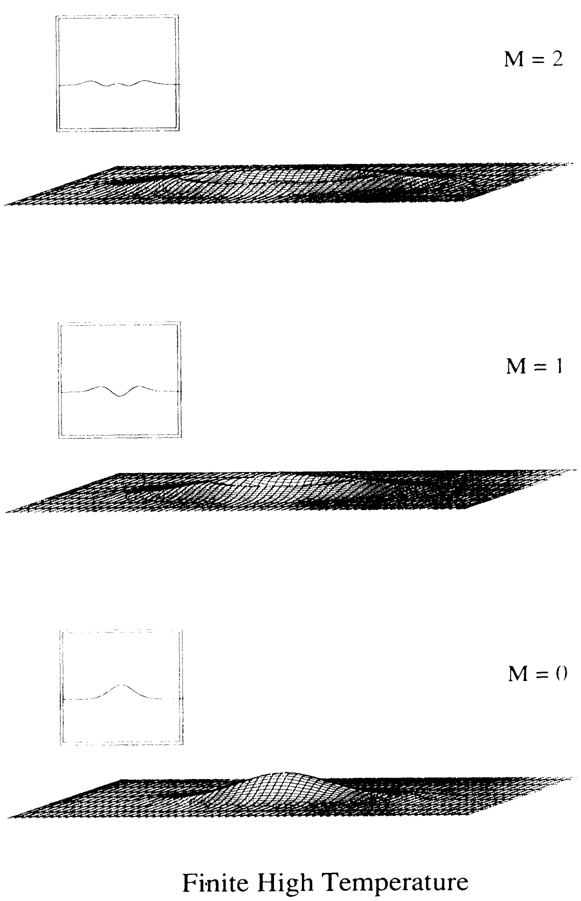

FIG. 3. Same as Fig. 2 but at higher temperature. The spreading in phase is apparent. The inset showing a cut along the $p=0$ value clarifies the structure of the nodes.

IV. GENERALIZED UNCERTAINTY RELATIONS

In order to further investigate the broadening in phase space, the generalized uncertainty relations[11] are implemented to the quasiparticle oscillator excitation $|M\rangle$ of Eq. (2.11). The oscillator frequency and mass are assigned the value 1. The uncertainty relations are considered for

$$\begin{aligned}\Delta q&=[\langle M|(q^2-\langle M|q|M\rangle^2)|M\rangle]^{1/2}\,,\\ \Delta p&=[\langle M|(p^2-\langle M|p|M\rangle^2)|M\rangle]^{1/2}\,.\end{aligned}\tag{4.1}$$

In Appendix B, the coordinate q and momentum p are expressed in terms of the quasiparticle operators ξ, $\tilde{\xi}$, $\xi^\dagger$, and $\tilde{\xi}^\dagger$. With this form the spreading in phase space, Eq. (4.1) leads to the uncertainty relations

$$\Delta q\,\Delta p=\tfrac{1}{2}+n+(1+n)M\,.\tag{4.2}$$

Here n is the average number of (2.7), the result (4.2) being independent of α. In obtaining this equation in Appendix B, use has been made of the commutation relations of the operators $\xi,\xi^\dagger$ and their *tilde* conjugate operators:

$$\begin{aligned}&[\xi,\xi^\dagger]=1\,,\\ &[\tilde{\xi},\tilde{\xi}^\dagger]=1\,,\end{aligned}\tag{4.3}$$

while, as in Eq. (2.3), all other commutators vanish. The annihilation relations of Eq. (2.8) have also been used extensively. Equation (4.2) reduces in the zero-temperature limit $n=0$ to

$$\Delta q\,\Delta p=\tfrac{1}{2}+M,\quad n=0\tag{4.4}$$

which is the well-known[12] result of the harmonic oscillator. For the case $M=0$, Eq. (4.2) coincides with the well-known[12] thermal uncertainty relation:

$$\Delta q\,\Delta p=\tfrac{1}{2}+n,\quad M=0\,.\tag{4.5}$$

Furthermore, in searching for the source of thermal fluctuations, it is illuminating to consider the mixed terms having both the dynamical and "thermal" (*tilde*) degrees of freedom:

$$\begin{aligned}&\Delta(q\tilde{q})=[\langle M|(q-\langle M|q|M\rangle)(\tilde{q}-\langle M|\tilde{q}|M\rangle)|M\rangle]^{1/2}\,,\\ &\Delta(p\tilde{p})=[\langle M|(p-\langle M|p|M\rangle)\\ &\qquad\times(\tilde{p}-\langle M|\tilde{p}|M\rangle)|M\rangle]^{1/2}\,,\\ &\Delta(q\tilde{q})\Delta(p\tilde{p})=[(1+n)n]^{1/2}(M+1)\,,\end{aligned}\tag{4.6}$$

as shown in Appendix B. In addition, the tilde counterpart of (4.2) is obtained as

$$\Delta\tilde{q}\,\Delta\tilde{p}=\tfrac{1}{2}+n(M+1)\,.\tag{4.7}$$

Adding the results of (4.2), (4.6), and (4.7) one arrives at the temperature-invariant form

$$(\Delta q)^2(\Delta p)^2-2[\Delta(q\tilde{q})]^2[\Delta(p\tilde{p})]^2+(\Delta\tilde{q})^2(\Delta\tilde{p})^2 = \tfrac{1}{2}+M+M^2 . \quad (4.8)$$

In the zero-temperature case $T=0$ the combination on the left-hand side (lhs) of Eq. (4.8) is indeed the rhs result, this in conformity with the temperature-invariance property. Equation (4.8) is a demonstration of the generalized temperature-invariant uncertainty relation. This equation resembles the usual uncertainty relations but contains two extra terms. The middle term introduces the mixing of variances with and without tildes, while the last term contains pure tilde variables. The derivation of this equation[11] relies on the invariant relation

$$pq-\tilde{p}\,\tilde{q}=PQ-\tilde{P}\,\tilde{Q} , \quad (4.9)$$

which may be verified by direct calculation using the definition of the quasiparticle coordinates and momenta defined as

$$Q=(1/2^{1/2})(\xi+\xi^{\dagger}) , \quad P=(1/i2^{1/2})(\xi-\xi^{\dagger}) . \quad (4.10)$$

The arguments[11] leading to Eq. (4.8) expressing the existence of a temperature invariant are summarized in Appendix C. In particular, the generalized uncertainty invariance identifies the interrelations between temperature and quantum fluctuations appearing in thermo field dynamics, and has been considered for thermal coherent states.[11] To see this in the present context, note that the mixed term Eq. (4.6) vanishes at zero temperature and is therefore considered as expressing pure thermal fluctuations. The first and last terms of Eq. (4.8), namely, the contributions in Eqs. (4.2) and (4.7), contain contributions of both quantum and thermal nature. The pure quantum fluctuations are expressed by (4.8) as a whole. This analysis indicates that the source of thermal fluctuations is expressed by the correlations between degrees of freedom with and without tildes. In summary, the thermal and state dependence of the spreading in phase space is explicitly given by Eqs. (4.2), (4.6), and (4.7).

Finally, it is worth noting that an energy spread may also be considered for these thermal quasiparticles. This may help in studying characteristic time scales via a time-energy uncertainty. The energy width defined with the Hamiltonian $H=\omega a^{\dagger}a$ ($\omega=1$) is obtained by a calculation similar to the procedure of Appendix B as

$$\langle M|HH|M\rangle-\langle M|H|M\rangle\langle M|H|M\rangle = n(n+1)+n(n+1)M \quad (4.11)$$

and the heat-capacity C (constant volume) of a quasiparticle M is given by

$$C=[n(n+1)+n(n+1)M]k\beta^2 , \quad (4.12)$$

where k is Boltzmann's constant.

V. CONCLUDING REMARKS

The phase-space distribution functions of quantum systems serve to increase physical insight, as they mimic classical mechanics phase-space probability distributions, albeit being negative. Moreover, they are useful for other practical purposes in many branches of modern physics.[3] In this paper, a new Wigner distribution for thermal quasiparticles has been defined and analyzed. The analysis has shown that these "quasiparticles" resemble the zero-temperature stable particles in their nodal and graphical structure, but occupy a larger portion of phase space. The resulting Wigner function is found to be independent of the parameter α of Eq. (2.4). The author has not succeeded in proving this independence rigorously, but all examples checked obey this finding. Furthermore, it is conjectured that a proper Wigner function should not depend on α, this being suggested by the possibility of considering the Wigner function as a candidate of quantum observables.[4]

By applying the generalized uncertainty relations, the spreading in phase space of the thermal excitations has been explicitly studied. This analysis has served to discuss the behavior of quantal and thermal fluctuations of these quasiparticles.

In thermo field dynamics,[6] the quasiparticles studied in this paper are used to construct thermal semifree fields and a thermal Fock space. The analysis given in this paper sheds light on the physical nature of these fields and their underlying phase-space properties.

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APPENDIX A: DERIVATION OF THE EXPLICIT FORM OF $f_M(q,p)$ EQ. (3.3)

The integration over p' in Eq. (3.2) eliminates the integration over y', leading to

$$f_M(q,p)=\frac{1}{2\pi}\int d\tilde{q}\,'\int dy e^{2ipy}(\langle q-y,\tilde{q}\,'|M\rangle\langle M|q+y,\tilde{q}\,'\rangle+\langle q+y,\tilde{q}\,'|M\rangle\langle M|q-y,\tilde{q}\,'\rangle) . \quad (A1)$$

With the definitions of the thermal vacuum, the thermal quasiparticle state $|M\rangle$ of Eq. (2.11) takes the form

$$|M\rangle=\frac{1}{M^{1/2}}\xi^{\dagger M}\rho^{\alpha}|I\rangle . \quad (A2)$$

Using the definition of $\xi^\dagger$ of Eq. (2.9) and the fact that $a^\dagger$ and $\tilde{a}$ commute [see Eq. (2.3)],

$$|M\rangle=(1/M^{1/2})\sum_{m=0}^{M}\begin{bmatrix}M\\m\end{bmatrix}[(1+n)^{1-\alpha}a^\dagger]^m(-n^{1-\alpha}\tilde{a})^{M-m}\rho^\alpha|I\rangle \ . \tag{A3}$$

With the thermal state condition for an operator A in the dynamical space,

$$A\rho|I\rangle=\rho\tilde{A}^\dagger|I\rangle \ , \tag{A4}$$

one arrives at

$$|M\rangle=(1/M^{1/2})\sum_{m=0}^{M}\begin{bmatrix}M\\m\end{bmatrix}[(1+n)^{1-\alpha}a^\dagger]^m\rho^\alpha(-n^{1-\alpha}a^\dagger)^{M-m}|I\rangle \ . \tag{A5}$$

Expressing $|I\rangle$ of Eq. (2.5) in an alternative form,

$$|I\rangle=\int dq\,|q,\tilde{q}\rangle \ , \tag{A6}$$

the matrix elements appearing in Eq. (A1) take the form

$$\langle q-y,\tilde{q}'|M\rangle=(1/M^{1/2})\sum_{m=0}^{M}\begin{bmatrix}M\\m\end{bmatrix}\langle q-y|[(1+n)^{1-\alpha}a^\dagger]^m\rho^\alpha(-n^{1-\alpha}a^\dagger)^{M-m}|q'\rangle \ . \tag{A7}$$

Note that the *tilde* coordinates disappeared. Equation (A7) is now in a conventional matrix representation. A similar calculation leads to

$$\langle M|q+y,\tilde{q}'\rangle=(1/M^{1/2})\sum_{m'=0}^{M}\begin{bmatrix}M\\m'\end{bmatrix}\langle q'|(-n^\alpha a)^{M-m'}\rho^{1-\alpha}[(1+n)^\alpha a]^{m'}|q+y\rangle \ . \tag{A8}$$

Multiplying (A7) and (A8) and integrating over q' as dictated by (A1) and repeating this for the second term of (A1) one arrives at

$$f_M(q,p)=\frac{1}{2\pi M}\sum_{m=0}^{M}\begin{bmatrix}M\\m\end{bmatrix}\sum_{m'=0}^{M}\begin{bmatrix}M\\m'\end{bmatrix}\int dy e^{2ipy}\{\langle q-y|[(1+n)^{1-\alpha}a^\dagger]^m\rho^\alpha(-n^{1-\alpha}a^\dagger)^{M-m}$$
$$\times(-n^\alpha a)^{M-m'}\rho^{1-\alpha}\rho^{1-\alpha}[(1+n)^\alpha a]^{m'}|q+y\rangle+\text{H.c.}\} \tag{A9}$$

which is the celebrated Eq. (3.3).

APPENDIX B: DERIVATION OF THE GENERALIZED UNCERTAINTY RELATIONS [(4.2), (4.6), (4.7)]

Recalling the definition (2.9) one has

$$\xi=(1+n)^\alpha a-n^\alpha\tilde{a}^\dagger \ ,$$
$$\xi^\dagger=(1+n)^{1-\alpha}a-n^{1-\alpha}\tilde{a}^\dagger ,$$

from which

$$\tilde{\xi}=(1+n)^\alpha\tilde{a}-n^\alpha a^\dagger \ ,$$
$$\tilde{\xi}^\dagger=(1+n)^{1-\alpha}\tilde{a}-n^{1-\alpha}a^\dagger \ . \tag{B1}$$

It follows that

$$a=(1+n)^{1-\alpha}\xi+n^\alpha\tilde{\xi}^\dagger \ ,$$
$$a^\dagger=(1+n)^\alpha\xi^\dagger+n^{1-\alpha}\tilde{\xi} \ ,$$
$$\tilde{a}=(1+n)^{1-\alpha}\tilde{\xi}+n^\alpha\xi^\dagger \ ,$$
$$\tilde{a}^\dagger=(1+n)^\alpha\xi^\dagger+n^{1-\alpha}\xi \ . \tag{B2}$$

Inserting (B2) into the definitions

$$q=(1/2^{1/2})(a+a^\dagger)$$

and

$$p=(1/i2^{1/2})(a-a^\dagger) \tag{B3}$$

one obtains

$$\langle M|q|M\rangle=\langle M|p|M\rangle=0 \ . \tag{B4}$$

It is now possible to calculate

$$\langle M|q^2|M\rangle=\tfrac{1}{2}(1+n)\langle M|(\xi\xi^\dagger+\xi^\dagger\xi)|M\rangle$$
$$+\tfrac{1}{2}n\langle M|(\tilde{\xi}\tilde{\xi}^\dagger+\tilde{\xi}^\dagger\tilde{\xi})|M\rangle \ , \tag{B5}$$

where all mixed terms of ξ and $\tilde{\xi}$ and other cross combinations have vanished. In addition, terms containing ξ^2, $\xi^{\dagger 2}$ and their *tilde* conjugates have vanished. Clearly this result is independent of α. Equation (B5) together with

$$\xi\xi^\dagger+\xi^\dagger\xi=2\xi^\dagger\xi+1$$

[see (4.5)] and $\langle M|\xi^\dagger\xi|M\rangle=M$ lead to

$$\langle M|q^2|M\rangle=\tfrac{1}{2}+n+(1+n)M \ . \tag{B6}$$

A similar calculation leads to

$$\langle M|p^2|M\rangle=\tfrac{1}{2}+n+(1+n)M \ . \tag{B7}$$

Equations (B4), (B6), and (B7) result in Eq. (4.2). A similar calculation for the mixed term leads to

$$\langle M|q\tilde{q}|M\rangle=(1+n)^{1-\alpha}n^{\alpha}(M+1) \ , \tag{B8}$$

$$\langle M|p\tilde{p}|M\rangle=(1+n)^{\alpha}n^{1-\alpha}(M+1) \ , \tag{B9}$$

the product of (B8) and (B9) being independent of α leading to Eq. (4.6). Repeating a similar calculation for $\langle M|\tilde{q}^2|M\rangle$ and $\langle M|\tilde{p}^2|M\rangle$ one arrives at

$$\begin{aligned}\langle M|\tilde{q}^2|M\rangle&=\tfrac{1}{2}+n(M+1) \ ,\\ \langle M|\tilde{p}^2|M\rangle&=\tfrac{1}{2}+n(M+1) \ ,\end{aligned} \tag{B10}$$

which lead to Eq. (4.7).

APPENDIX C: DERIVATION OF THE TEMPERATURE INVARIANT REF. 11 (4.8)

Recalling the invariant of Eq. (4.9) one has

$$I=pq-\tilde{p}\,\tilde{q}=PQ-\tilde{P}\,\tilde{Q} \ . \tag{C1}$$

One now defines another invariant in the form

$$J=(I-i)I \tag{C2}$$

which leads by direct substitution to

$$J=p^2q^2+\tilde{p}^{\,2}\tilde{q}^{\,2}-2p\tilde{p}q\tilde{q}=P^2Q^2+\tilde{P}^{\,2}\tilde{Q}^{\,2}-2P\tilde{P}Q\tilde{Q} \ . \tag{C3}$$

In Ref. 11 it has been shown that this invariance carries over to the spreadings, thus obtaining

$$\begin{aligned}&(\Delta q)^2(\Delta p)^2-2[\Delta(q\tilde{q})]^2[\Delta(p\tilde{p})]^2+(\Delta\tilde{q})^2(\Delta\tilde{p})^2\\ &\quad=(\Delta Q)^2(\Delta P)^2-2[\Delta(Q\tilde{Q})]^2[\Delta(P\tilde{P})]^2+(\Delta\tilde{Q})^2(\Delta\tilde{P})^2\end{aligned} \tag{C4}$$

which is a statement of the temperature invariance and is the form used in Eq. (4.8).

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