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Kinematic compression and expansion of the velocity distributions of particles in gas flows

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Velocity distribution functions showing kinematic compression and expansion are derived in the context of gravitational forces. Whereas kinematic compression is well known in laser ion-beam spectroscopy, kinematic expansion is a new concept. In our analysis, collisionless and weak-collision cases are considered. A brief discussion shows that the weak-collision case may be appropriate in certain regions of the interstellar medium and that kinematic compression and expansion could have interesting observational consequences. In particular, as we show elsewhere, kinematic expansion may be important in the formation of broad and asymmetric quasistellar object emission lines.

I. INTRODUCTION

When ion beams are accelerated their resonance-absorption linewidths measured in the direction of the beam have been found to be greatly narrowed. This kinematic phenomenon has come to be termed "kinematic compression" of the Doppler width. Experimental absorption lines are typically found to be ten or more times narrower than would be estimated from the kinetic temperature of the ion source. Since the first observations of kinematic compression[1–5] the phenomenon has proved very valuable in laser-ion spectroscopy in improving the resolution of optical spectra to exhibit lifetime broadening and hyperfine structure (see Ref. 6 for a recent example). Kinematic compression was first predicted on theoretical grounds by Kaufman[7] with a view to its application in charge-transfer neutralized Cs-beam spectroscopy. The essential prediction of Kaufman of velocity bunching in accelerated ion beams has been both verified (in the same year) and exploited as indicated above.

In his article Kaufman presents a simple expression for the factor of reduction of velocity difference for two ions of mass m, one initially at rest in the direction of the beam and the other with an average velocity $(2kT/m)^{1/2}$, on acceleration of the ions through a potential difference. Kaufman also goes on to describe the effects of angular divergence of the beam, charge-transfer neutralization and photon recoil, considerations which need to detain us further in the present work for reasons which will become apparent below. Kinematic compression and an associated phenomenon proposed here of "kinematic expansion" are processes of a basic kinematic nature and deserve closer attention than has been devoted to them in this one article by Kaufman. Here we develop analyses for velocity distribution functions in accelerated and decelerated flows concentrating on the kinematics rather than the consequences for absorption or emission line shapes. In Sec. II B we derive a velocity distribution function for kinematic compression and expansion by solving the Boltzmann equation in the absence of collisions (Hamilton's equations). In Sec. II C we consider the effects of collisions on the velocity distribution function in the weak-collision limit.

The motivation for this work came through our interest in astrophysical problems of radiation transport[8] and we set out here to consider kinematic phenomena in an astrophysical context. It was first pointed out in Ref. 9 that in the astrophysical medium a gravitational field in accelerating a flowing gas could play the role played by an electric field in accelerating an ion beam, the gravitational potential taking the place of an electrical potential. In Ref. 9 a simple expression for the degree of kinematic compression was given which was essentially the same as that presented earlier by Kaufman[7] for the electrical analog. Collisions were not considered and there the matter has rested. In the astrophysical medium neither photon recoil nor specific charge-transfer processes need be treated for the purpose of this work. Angular divergence arises through two distinct phenomena. In the first place there is the natural divergence due to transverse velocity components of the atoms and molecules in the flow. This divergence is in principle automatically included in any analysis which involves the calculation of distribution functions since we use the Liouville theorem. Secondly, flow divergence may occur due to irregularities in the force field. These are ignored in our work.

II. A SIMPLE KINEMATIC THEORY

A. Model flows

Since in the present paper we wish to highlight the phenomena of kinematic compression and expansion, we choose a model system which we feel is sufficiently reasonable to be convincing but which avoids the complexity of a realistic astrophysical model. Our physical model for kinematic compression is one in which gas is attracted from an interstellar cloud by the gravitational field of a body which has itself no more than a very local atmosphere. For example, one may picture a stream of gas accelerated towards a white dwarf whose lack of extensive atmosphere avoids the development of shock waves save very close indeed to the white dwarf. In our model, acceleration takes place in a plane-parallel potential, a good approximation for distances from the white dwarf very much greater than its diameter and for a stream of gas of cross-sectional dimension comparable with the dimensions of the white dwarf. For kinematic expansion we imagine material thrown violently outwards from, for example, a planetary nebula, a highly evolved but still active giant star. The ejected material then undergoes deceleration in a plane-parallel gravitational field. We assume that shock waves do not intervene.

We also note that for kinematic compression in the gravitational context there is a problem which is not encountered in the electrical analog. In the electrical case it is easy to imagine ions or electrons entering a region of acceleration after passing through a shielded field-free region or being formed by photoionization in a region in which there exists an electric field. Equivalent gravitational systems cannot generally be postulated since gravitational fields cannot be shielded in the simple manner of electrical fields nor do we wish to postulate the sudden creation of matter within a gravitational field. There is also a further problem which is common to both electrical and gravitational systems. One normally thinks of a flow as a phenomenon in which the total flux of material is conserved. If matter is not to be lost in a flow from its inception, particles with Maxwell-Boltzmann velocities with components opposed to the direction of acceleration must (in the absence of collisions) be free to move in directions opposite to that of the flow, remaining in a retarding field which eventually reverses their direction of motion. For flux conservation for a flow notionally starting at rest the field must therefore extend arbitrarily far in all directions and must not be influenced by the presence of other massive bodies. In order to overcome these problems we introduce the additional feature that the cloud of interstellar gas has already achieved a velocity in the direction of the attracting body considerably in excess of $(2kT/m)^{1/2}$ by some means before we begin to consider phenomena of kinematic compression.

B. A distribution function for kinematic compression and expansion

We treat flow in only one dimension. Thus, in the initial velocity distribution function

$$p_{xyz}(v_x,v_y,v_z,x,y,z)=p_x(v_x,x)p_y(v_y,y)p_z(v_z,z)$$

and $p_x(v_x,x)$ and $p_y(v_y,y)$ remain unchanged during acceleration or deceleration. We consider, for the present, the collisionless limit in which particles follow uninterrupted ballistic trajectories. We assume nevertheless that particles in the flow before acceleration or deceleration have a Gaussian velocity distribution and that the gas flow is initially at a well-defined and unperturbed thermodynamic temperature. We emphasize that we wish to consider distribution function for particles which have undergone acceleration or deceleration over the same distance z. Qualitatively a particle which is faster than average in the direction of acceleration or deceleration spends less time in the gravitational field and is therefore accelerated or decelerated less than a slower than average particle. It is this effect which leads to velocity bunching as was noted by Kaufman.[7]

It follows from Liouville's theorem that the phase-space density of particles is constant as the flow velocity changes (see, for example, Ref. 10). We may therefore write

$$\frac{d}{dz}p(v(v_0,z),z)=0 \ , \tag{1}$$

where v_0 is the initial flow velocity (see Sec. II A). Thus, writing $p\equiv p(v(v_0,z),z)$,

$$\frac{\partial p}{\partial z}+\frac{\partial p}{\partial v}\left(\frac{\partial v}{\partial z}\right)_{v_0}=0 \ . \tag{2}$$

Writing the force per unit mass as $(\partial v/\partial t)_{v_0}=F$, positive for kinematic compression and negative for expansion, and $(\partial z/\partial t)_{v_0}=v$ we therefore find the collisionless Boltzman equation

$$v\frac{\partial p}{\partial z}+F\frac{\partial p}{\partial v}=0 \tag{3}$$

in the steady state ($\partial p/\partial t=0$). Equation (3) may be solved by standard techniques outlined, for example, in Ref. 11. Briefly we solve (3) subject to the condition that the initial distribution function is a Gaussian centered on the initial flow velocity v_0,

$$p(v,0)=(m/2\pi kT)^{1/2}\exp[-(m/2kT)(v-v_0)^2] \ . \tag{4}$$

The characteristic equation of Eq. (3) is $\frac{1}{2}v^2-Fz=\frac{1}{2}v_0^2$, a constant, from which it follows that the solution after some manipulation is of the form $p((v^2-2Fz)^{1/2})$. Replacing v with $(v^2-2Fz)^{1/2}$ in Eq. (4) and expanding the square yields

$$\begin{aligned}p(v,z)={}&\mathcal{N}(m/2\pi kT)^{1/2}\\&\times\exp\{-(m/2kT)[v^2-2Fz\\&\qquad-2v_0(v^2-2Fz)^{1/2}+v_0^2]\} \ ,\end{aligned} \tag{5}$$

our desired distribution function where the additional normalization constant

$$\mathcal{N}=2\exp(-mv_0^2/2kT)/[1-\mathrm{erf}(mv_0^2/kT)^{1/2}] \ .$$

We may institute a partial check in the validity of Eq. (5) by considering the standard mass, momentum, and energy balance equations for a monatomic gas

$$\frac{d}{dz}\overline{v}_z\rho=0 \ , \tag{6a}$$

$$\frac{d}{dz}\overline{v_z^2}\rho=F\rho \ , \tag{6b}$$

$$\frac{d}{dz}\overline{v_z(v_x^2+v_y^2+v_z^2)}\rho=2F\overline{v}_z\rho \ , \tag{6c}$$

where ρ is the density, which we note is a function of z. The velocity v_z is the sum of two parts, the flow velocity

$\bar{v}_z$ and thermal motions in the z direction, Δv_z. Using $\Delta v_z^n = 0$ for $n = 1,3$ and the constancy of the flux $J(\equiv \bar{v}_z \rho)$, Eq. (6a), and writing $T_\parallel = \overline{v_z^2} - \bar{v}_z^2$, we have for the momentum equation (6b),

$$\bar{v}_z \frac{d}{dz}[(\bar{v}_z^2 + T_\parallel)J/v_z] = FJ \tag{7a}$$

from which it follows that

$$\tfrac{1}{2}\frac{d}{dz}\bar{v}_z^2 + \frac{d}{dz}T_\parallel - \tfrac{1}{2}(T_\parallel/\bar{v}_z^2)\frac{d}{dz}\bar{v}_z^2 = F \ . \tag{7b}$$

For the energy equation (6c) considering motion in only one dimension we have

$$\frac{d}{dz}\bar{v}_z^2 + 3\frac{d}{dz}T_\parallel = 2F \ . \tag{7c}$$

Eliminating F from (7b) and (7c) we obtain

$$(1/T_\parallel)\frac{d}{dz}T_\parallel = -(1/\bar{v}_z^2)\frac{d}{dz}\bar{v}_z^2 = (2/\rho)\frac{d\rho}{dz} \tag{8}$$

and hence $T_\parallel \rho^{-2} = T_0 \rho_0^{-2}$. In Figs. 1(a) and 1(b) we show the variation of $T_\parallel / T_0$ for kinematic compression and expansion calculated using (5) where we have found $T_\parallel$ using

$$T_\parallel = \overline{(v - \bar{v})^2} \ .$$

In each case the variation of $T_\parallel / T_0$ follows $T_\parallel \rho^{-2} = T_0 \rho_0^{-2}$ noting that ρ is proportional to $\bar{v}^{-1}$.

In the full collisional (thermodynamic) limit, acceleration (say) of the gas is equivalent to Joule-Thompson adiabatic expansion. Including terms involving $T_\perp$ $(\equiv \overline{v_x^2} = \overline{v_y^2})$ Eqs. (6) lead to $T\rho^{-2/3} = T_0 \rho_0^{-2/3}$, the standard expression for reversible adiabatic change. This limit is shown in Figs. 1(a) and 1(b) for comparison with the ballistic case.

In Figs. 2 and 3 we show examples of kinematically compressed and expanded velocity distribution functions. For kinematic compression (Fig. 2) we have chosen an example which shows the asymmetry which can develop in the distribution function using an initial velocity corresponding to $2.5kT_0$, where T_0 is the initial temperature, and a velocity increase by a factor of $\sqrt{5}$. For kinematic expansion (Fig. 3) we have chosen an initial energy of $10^6 kT_0$ dropping to $10^5 kT_0$ on deceleration, a velocity decrease through a factor of $\sqrt{10}$. For H_2 at 100 K (say) this corresponds to a velocity of ~ 900 km s^{-1} dropping to ~ 285 km s^{-1}. A further reduction of velocity to 45 km s^{-1} produces a very asymmetric distribution function (see inset to Fig. 3), a significant factor in quasistellar object (QSO) line-shape modeling.

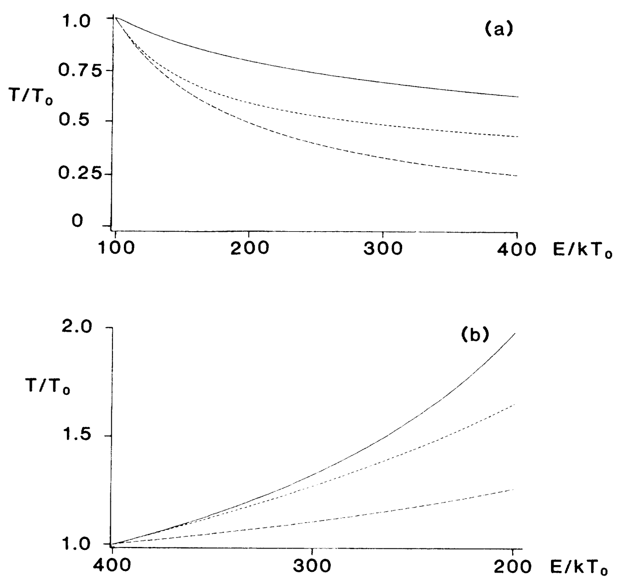

FIG. 1. (a) The variation of T/T_0 on acceleration of the gas flow for the thermodynamic case of adiabatic change (upper solid line), for collisionless kinematic compression (lower dashed line), and the weak collision case (intermediate dashed line, $\theta = 5$). In each case an initial velocity corresponding to an energy E of $100kT_0$ was used and we consider an acceleration yielding a velocity change of a factor of 2, that is, an energy change of a factor of 4. Note that T/T_0 falls correspondingly by a factor of 4 in the collisionless case. (b) The variation of T/T_0 on deceleration of the gas flow for the thermodynamic case (lower dashed line), for collisionless kinematic expansion (upper solid line) and the weak collision case (intermediate dashed line, $\theta = 5$). In each case an initial velocity corresponding to an energy of $400kT_0$ was used and we consider a deceleration yielding an energy change by a factor of 2. Note that T/T_0 rises correspondingly by a factor of 2 in the collisionless case.

C. Distribution functions in the presence of collisions

We now address the question: how can we treat physical conditions between the two limits of thermodynamic and collisionless? To answer this question the most sophisticated method would be to find accurate solutions of the Boltzmann equation. We do not attempt this in the

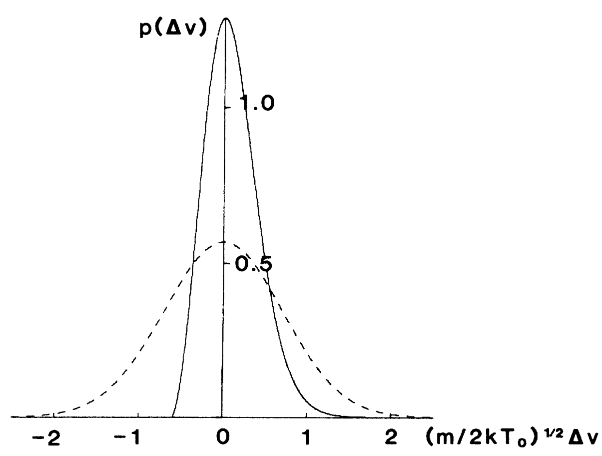

FIG. 2. A kinematically compressed velocity distribution function $p(\Delta v)$ (solid line) in the collisionless limit calculated using Eq. (5). Initial velocity corresponding to $2.5kT_0$; acceleration through a factor of $\sqrt{5}$. The dashed curve is the original Gaussian distribution. Δv is defined as the particle velocity v minus the flow velocity, $\bar{v}_z$.

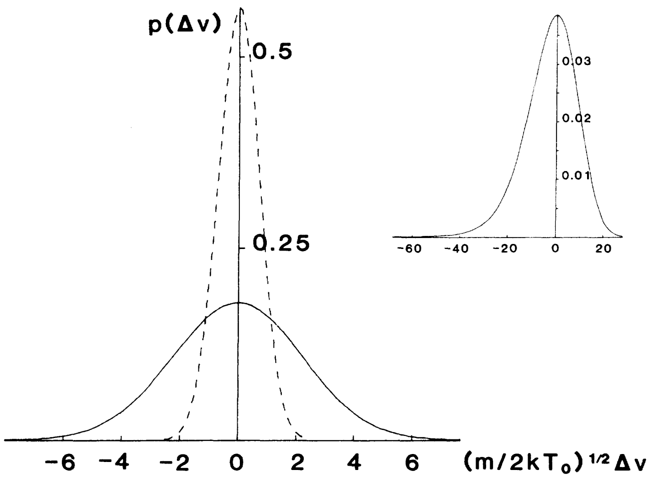

FIG. 3. A kinematically expanded velocity distribution function (solid line) in the collisionless limit calculated using Eq. (5). Initial velocity corresponding to $10^6 kT_0$; deceleration through a factor of $\sqrt{10}$. The dashed curve is the original Gaussian. In the inset we show the highly asymmetrical distribution function obtained with an overall deceleration through a factor of 20.

present work. The most simple approach—which we essentially adopt here—is to note that a few collisions are generally considered to be sufficient to establish a Gaussian velocity distribution (see, for example, Ref. 12). We can put this on a slightly more quantitative basis by making the very crude approximation that the collision term replacing the zero on the right-hand side (rhs) of Eq. (3) be represented by $[p_0(z)-p]/\tau$ (Refs. 13 and 14),where τ is a relaxation time and $p_0(z)$ is a local Gaussian distribution and p is the desired distribution function. To apply the concept of a relaxation time to a distribution of populations (rather than individual populations) implies a high degree of averaging. Nevertheless, any distribution will relax monotonically towards a local equilibrium on a time scale characterized by momentum transferring collisions. Thus τ may be crudely regarded as the inverse of a second-order rate coefficient for momentum transfer multiplied by a gas pressure. Two points arise: (i) the value of τ will increase (decrease) on acceleration (deceleration) of the gas flow because of the density changes outlined in Sec. II B, and (ii) we choose to formulate the local thermodynamic state as $T_\perp = T_\parallel$ so that $p_0(z)$ is given by a Gaussian with a temperature defined by the adiabatic expression.

The Boltzmann equation is not amenable to an analytic solution even in this very simple form. We therefore solve the equation numerically using the method of characteristics. The equation is hyperbolic and quasilinear since the rhs contains the unknown function p. We first transform into dimensionless variables $\eta=(m/2kT)^{1/2}v$, $\phi=mF_z z/kT$, and use $\theta=F_z\tau/(2kT/m)^{1/2}$ and $\eta_f=(m/2kT)^{1/2}v_f$, where T is the initial temperature of the gas. The equation may then be written

$$2\eta\frac{\partial p}{\partial \phi}+\frac{\partial p}{\partial \eta}=(1/\theta)\exp[-(\eta-\eta_f)^2]-p/\theta\ . \qquad (9)$$

Our initial condition is a Gaussian at $\phi=0$ so that we know a set of initial values of η,η_c, along which we can calculate characteristics. From (9) the equation of the characteristic is $\phi=\eta^2-\eta_c^2$. The equation for solution along a characteristic is governed by $d\phi/2\eta=dp/R$ [where R is the rhs of Eq. (9)] from which it follows that

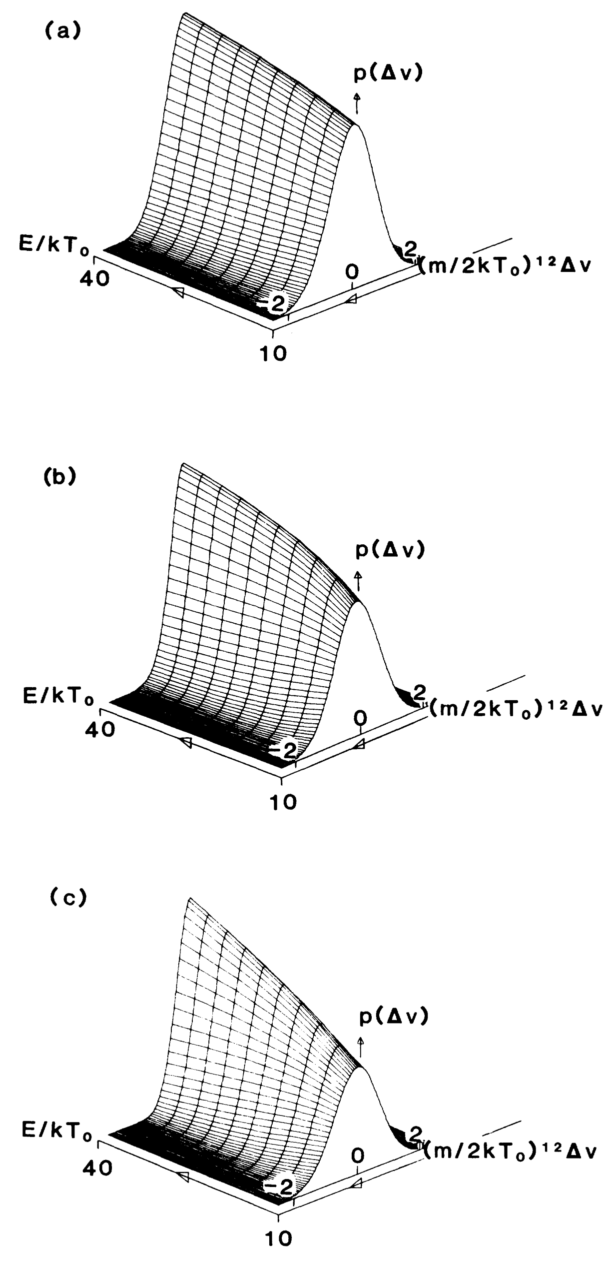

FIG. 4. The variation of the velocity distribution function $p(v,z)$ with z, for an initial velocity corresponding to an energy of $10kT_0$ with acceleration through a factor of 2, (a) the thermodynamic case, (b) kinematic compression for $\theta=5$, (c) kinematic compression in the collisionless limit. Figures have been drawn so that the maximum of $p(v,z)$ is the same height in each case. The initial distribution on the rhs of each figure is the same (Gaussian) in each example.

$$\frac{dp}{d\phi} = -p/2\theta(\phi+\eta_c^2)^{1/2} + \exp[(-\eta-\eta_f)^2]/2\theta(\phi+\eta_c^2)^{1/2} . \quad (10)$$

Equation (10) was solved by the Runge-Kutta method for a set of 50 suitably chosen characteristics. From the value of p so obtained we could then find, for a given z, values of

$$\overline{\overline{\eta^2}-\overline{\eta}^2} ,$$

equivalent to $T_{\parallel}$, and thus the ratios of $T_{\parallel}/T_0$ could be calculated. Results are shown in Figs. 1(a) and 1(b) for an initial $\theta=5$ for kinematic compression and expansion, respectively. It is evident that $\theta=5$ represents an intermediate case between thermodynamic and collisionless. Calculations show that for $\theta \lesssim 0.25$ the temperature change is indistinguishable from thermodynamic whereas for $\theta \gtrsim 100$ the temperature change is indistinguishable from collisionless.

For illustrative purposes it is useful to show the evolution of p with z. We represent p in terms of a deviation from a peak mean value that is, as a function of $\Delta\eta=\eta-\eta_f$. To perform this frame transformation we substitute $\eta=\Delta\eta+\eta_f$ into the characteristic $\phi=\eta^2-\eta_c^2$ to give

$$\phi=\Delta\eta^2+2\Delta\eta\eta_f+\eta_f^2-\eta_c^2 . \quad (11)$$

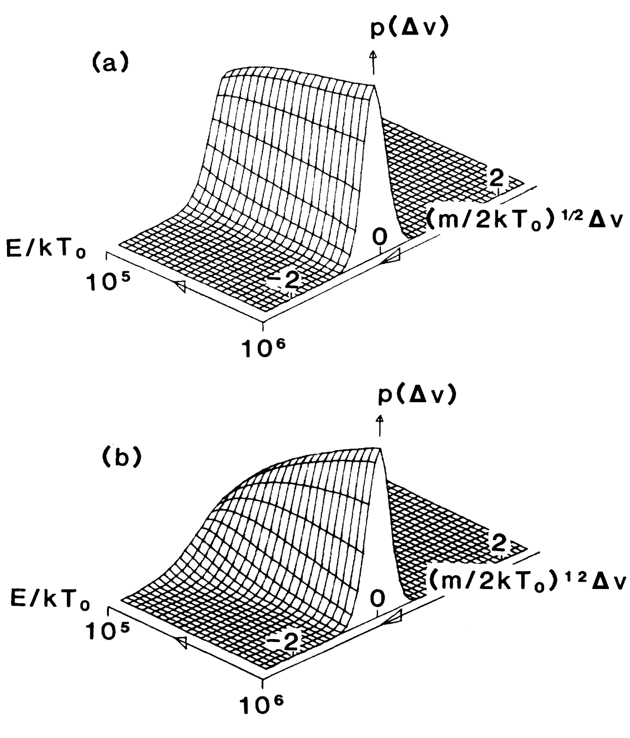

FIG. 5. The variation of the velocity distribution function $p(v,z)$ with z for a velocity corresponding to an initial energy of $10^6 kT_0$ and deceleration through a factor of $\sqrt{10}$, (a) kinematic expansion for $\theta=5$, (b) kinematic expansion in the collisionless limit.

For every value of ϕ we choose 50 values of $\Delta\eta$ between -2.5 and $+2.5$ and solve (11) to yield a set of values of η_c. The range of $\Delta\eta$ chosen extends the calculation 2.5 FWHM each side of the mean velocity in the distribution. Interpolating linearly between two values of p for values of η_c which bracket that found from Eq. (11) completes the transformation from η,ϕ to $\Delta\eta,\phi$ coordinates. Results of calculations for acceleration are shown in Figs. 4(a)–4(c) for the thermodynamic case, $\theta=5$ and for the collisionless case, respectively. Results are shown in Figs. 5(a) and 5(b) for deceleration. In Figs. 4(b) and 5(a) the distributions have been numerically normalized to satisfy

$$\int_{-2.5}^{2.5} p(\Delta\eta,\phi)d\Delta\eta=1 .$$

III. DISCUSSION AND CONCLUSIONS

We pose the following question: is it feasible that gas flows can be accelerated or decelerated in the interstellar medium such that the time for significant acceleration or deceleration is comparable with the time between momentum transferring collisions? If this is the case then the ballistic phenomena described here are significant for straightforward gaseous systems and the linewidths and shapes which an observer might discover would be subject to the influence which we have described in Secs. I and II over and above the purely thermodynamic phenomena. Let us take the example of a flow of gas with 10^{12} $H_2\,m^{-3}$ and a temperature of 100 K attracted towards some massive object, which we have suggested in Sec. II A might be a white dwarf of mass (say) equal to 1 solar mass ($\approx 2\times10^{30}$ kg). We set out to calculate how close a flow would have to be to such a massive object, mass M, given that $\theta=5$. The latter condition as we know ensures that we are in the partially ballistic regime. From the properties of the cloud one may estimate that $\tau \sim 5\times10^3$ s, implying $F=0.9$ ms^{-2}, which we can equate with GM/r^2. Thus r, the average distance of some part of the flow from the massive body has to be $\approx 1.2\times10^{10}$ m. We note that this distance is very much greater than the radius of a typical compact object of one solar mass. For example, a white dwarf may have a radius of $\sim 5\times10^6$ m (Ref. 15) and no significant atmosphere at distances of more than a few km from its surface. Therefore from this distance of $\sim 10^{10}$ m inwards, velocity bunching and kinematic compression could indeed begin to develop.

A further consideration relates to the possibility, in principle, of observing the effects of kinematic compression (or expansion) rather than merely demonstrating that velocity bunching to whatever extent can take place. We need to establish that there can be sufficient optical depth in an absorption line in a distance sufficiently short that the large acceleration necessary for kinematic compression does not shift the relevant line too far out of resonance. For our example of H_2 in the preceding paragraph we choose the $R(1)$ line of the (0-0) Werner bands $C\,^1\Pi_u$–$X\,^1\Sigma_g^+$ at around 1000 Å. This line is well known in absorption[16] in the interstellar medium. Using the observations of Hessler[17] and the calculations of Wolniewicz[18] and Dalgarno and Stephens[19] we may estimate an Einstein B value of $\sim 3\times10^{12}$ $m^2\,J^{-1}\,s^{-1}$ for the $R(1)$ absorption line. The number density in $J=1$ of H_2 is 0.62 of N_{H_2} $(=10^{12}\ m^{-3})$ at 100 K.[16] The line-shape function $\phi=[\sqrt{\pi}\times(\text{Doppler width})]^{-1}$ is 6.2×10^{-11} s giving an absorption coefficient of $0.62(h\nu/4\pi)B\phi N_{H_2}=1.83\times10^{-5}\ m^{-1}$. We therefore expect unit optical depth over a distance of 5.46×10^4 m. In order to shift the resonant frequency a Dopper width (=9100 MHz) we require an acceleration yielding a velocity change of 310 ms^{-1} over this distance. It may readily be shown that acceleration of this magnitude can be achieved only if r falls to 6.2×10^7 m whereas we note that kinematic compression onsets at a distance of ~ 200 times greater. Thus, in this important case, Doppler shifting would not cause compression to be unobservable. We recognize that the resolution requirements for the observation of the effects of compression may be severe, lying in excess of 500 000 in the above example. However, we also note that much of what has been said of compression applies equally to kinematic expansion which by its nature places less stringent demands on resolution.

It is worthwhile to check that our calculations showing kinematic compression for $\theta=5$ are consistent with very few collisions taking place in the time required for significant acceleration in the system described above. The calculations shown in Fig. 4(b) correspond to an initial η (η_i) of 5, that is a velocity of 4.55 $km\,s^{-1}$ (for H_2 at 100 K) and a final η $[=(\eta_1^2+\phi)^{1/2},\ \phi=300]$ of ~ 18 corresponding to a velocity of 16.4 $km\,s^{-1}$. The time required to undergo this velocity change at a constant acceleration of 0.9 ms^{-2} is *ca.* 1.3×10^4 s. Noting that much the greater part of the kinematic compression takes place for $\phi=150$ it is clear that for an average τ of $\sim 5\times10^3$ s there is time for only one or two collisions to take place as the flow undergoes significant acceleration. Thus our calculations for $\theta=5$ are consistent with our physical intuition.

Our brief order-of-magnitude estimates raise the question of whether any observations presently in the literature appear to exhibit the kinematic effects which we describe. We have not conducted a wide search but we note that white dwarf atmospheres involving H_α and H_β absorption exhibit anomalous line shapes which have been attributed to non-local-thermodynamic-equilibrium effects.[20] Values of θ span the appropriate range in the outer boundary of white dwarf atmospheres and kinematic compression and expansion may play a role in a full interpretation of the line shapes found in these observations. Whatever the observational status of these kinematic phenomena, our results present an intriguing picture of masses of gas subject to gravitational forces of sufficient strength to cause appreciable acceleration or deceleration on a time scale sufficiently short that collisions cannot keep up and cannot maintain a Maxwell-Boltzmann distribution of molecular velocities.

In conclusion we should mention three further aspects of this work presently under development. First, the kinematic phenomena which we have discussed here have potentially observable consequences through emission and absorption line shapes of astrophysical atoms and molecules. Using the Doppler formula our results may readily be transformed into frequency rather than velocity distributions. A full line shape would be formed by the superposition of increasingly compressed or expanded and Doppler shifted profiles. Second, we plan to solve the Boltzmann equation through Monte Carlo simulations using a many-particle model. Third, we are presently considering the influence of kinematic expansion in the formation of emission lines in quasars. We should like to draw attention to the fact that the particles whose velocity distributions we have been discussing are of an unspecified physical nature and need not be atoms or molecules. We are therefore at liberty to apply the theory of kinematic expansion to ensembles of gas clouds which form part of the standard model of quasar atmospheres.[21–23] We find that the very broad asymmetric emission lines of C^{3+} and the broad Lyman-α lines which are the signature of quasars can be very accurately modeled using the distribution function of Eq. (5).[24] We have also developed a (special) relativistic theory of kinematic expansion for this purpose which will be presented elsewhere.[24]

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