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Dynamics of processes with a trilinear boson Hamiltonian

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We consider a Hamiltonian trilinear in boson operators, which describes several physical processes, such as the interaction of N two-level atoms with a single-mode resonant radiation field, parametric amplification, Raman and Brillouin scattering, and frequency conversion. The dynamics of systems obeying this trilinear Hamiltonian is analyzed and the equations of motion are solved exactly. So the time development of the boson operators is described by means of Laplace transforms of suitable functionals. The integration method is based on iteration techniques and makes use of the Manley-Rowe relations. Finally, the results are applied to describe the time evolution of the boson operators when one of the three modes is initially in a high state of excitation.

I. INTRODUCTION

Several processes of quantum optics are investigated by introducing Hamiltonians multilinear in boson operators. For example, parametric amplification, frequency conversion, Raman and Brillouin scattering, and the interaction of two-level atoms with a single-mode resonant radiation field are described by a trilinear Hamiltonian of the form[1–17]

$$\hat{H}=\hbar\sum_{i=1}^{3}\omega_i\hat{a}_i^\dagger\hat{a}_i+\hbar\Delta(\hat{a}_1^\dagger\hat{a}_2\hat{a}_3+\hat{a}_1\hat{a}_2^\dagger\hat{a}_3^\dagger)\ , \qquad (1)$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are boson operators of the ith mode with angular frequency ω_i and the frequencies obey the energy conservation law

$$\omega_1=\omega_2+\omega_3\ . \qquad (2)$$

Modes 1, 2, and 3 of the Hamiltonian (1) are identified as the pump, signal, and idler modes, respectively, in the parametric amplification and as the idler, pump, and signal modes, respectively, in the frequency conversion. The same Hamiltonian (1) describes the Raman and Brillouin scattering if modes 1, 2, and 3 represent input, vibrational, and Stokes modes, respectively, for a Stokes process, and anti-Stokes, input, and vibrational modes, respectively, for an anti-Stokes process. Finally, the Hamiltonian (1) is mathematically identical to the well-known Hamiltonian for the interaction of N identical two-level atoms with a single-mode resonant radiation field at frequency ω_3, which is read in Dicke's notations as

$$\hat{H}=\hbar\omega_3(\hat{J}_z+\hat{a}_3^\dagger\hat{a}_3)+\hbar\Delta(\hat{a}_3\hat{J}_++\hat{a}_3^\dagger\hat{J}_-)\ ,$$

where $\hat{J}_\pm,\hat{J}_z$ are the collective angular momentum operators for the atoms and $\hat{a}_3,\hat{a}_3^\dagger$ are the boson operators for the radiation-field mode. In fact, this Hamiltonian can be put in the form of the Hamiltonian (1) by using the Schwinger's representation of angular momentum operators in terms of boson operators:

$$\hat{J}_+=\hat{a}_1^\dagger\hat{a}_2\ ,$$

$$\hat{J}_-=\hat{a}_1\hat{a}_2^\dagger\ ,$$

and

$$\hat{J}_z=\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)\ .$$

So one identifies modes 1 and 2 of the Hamiltonian (1) as the upper and lower atomic state, respectively, and mode 3 as the radiation field with $\omega_1=-\omega_3=\frac{1}{2}\omega_2$, which is a particular case of the general condition (2).[13]

In order to study all these processes it is important to know the dynamics of the systems obeying the Hamiltonian (1). Actually it is impossible to obtain the exact quantum solution to this problem. In fact, although the eigenfunctions of the trilinear Hamiltonian have been exactly determined analytically, the results are quite complicated for studying the time evolution of particular systems.[2] Consequently, a number of authors have analyzed this problem using different kinds of approximations. In most of these analyses one makes the parametric approximation[3,10,14] in which the depletion of one of the modes is assumed to be slight through the evolution of the system, or the quasiclassical approximation[5,13] in which, at least, one of the three modes is assumed to be strongly populated so that some operators describing the process may be decoupled. Other authors have attempted to overcome calculation difficulties by using short-time solutions or numerical solutions.[4,9] But nearly all of these treatments give results that are not valid for long times.

The aim of this paper is to give an exact solution of the problem. Therefore we will describe a new method to study the dynamics of processes obeying the trilinear boson Hamiltonians. In order to investigate the time development of the boson operators we apply the mathematical techniques that have recently been used for studying other nonlinear processes.[18]

First, we write the equations of motion in a form for which the boson operators appear separated. Then, we look for a solution of these equations by applying iteration methods. But solutions expressed as a power series of time can be written only if a recursive operational relation among the terms of the power series is found and, at the same time, the expansion factor $[(n!)^{-1}]$ for the generic nth term of this series is taken into account. On making use of the Manley-Rowe relations and of some integral operators we are able to overcome these difficulties and to obtain formal solutions of the motion equations. Finally, we condense the resultant power series in integrals of analytical functionals. So, the final expressions appear in the shape of a Laplace transform and of a subsequent inverse Laplace transform of suitable operator functionals.

From these expressions one can directly analyze the characteristic properties of the processes described by the trilinear Hamiltonian without applying the usual calculation techniques. The utility of the present approach is illustrated by studying the time evolution of the boson operators in some simple cases. Hence, we assume that initially one of the three modes is strong. The present calculations require that the simplifying condition is verified only at time $t=0$, whereas in all the previous approximate methods every simplifying condition must be satisfied through the whole evolution of the system.

Section II is devoted to write the exact solutions of the equations of motion for the boson operators. In Sec. III these solutions are employed to study the time evolution of the boson operators when one of the three modes is in a quasiclassical state. The paper concludes with the Appendix where a particular functional necessary for our analysis is studied.

II. EQUATIONS OF MOTION AND THEIR SOLUTIONS

We will study the dynamics of the trilinear boson Hamiltonians, so we consider the model Hamiltonian (1) in which three modes of the field interact with each other. The boson operators of these modes, labeled by the subscripts 1, 2, and 3, obey the commutation rules

$$[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij} .$$

The coupling constant Δ is taken to be real and the energy conserving condition (2) is assumed to be satisfied. This Hamiltonian may be written as

$$\hat{H}=\hat{H}_0+\hat{H}_I , \tag{3}$$

where

$$\hat{H}_0=\hbar\sum_{i=1}^{3}\omega_i\hat{a}_i^\dagger\hat{a}_i \tag{4a}$$

and

$$\hat{H}_I=\hbar\Delta(\hat{a}_1^\dagger\hat{a}_2\hat{a}_3+\hat{a}_1\hat{a}_2^\dagger\hat{a}_3^\dagger) , \tag{4b}$$

with

$$[\hat{H}_0,\hat{H}_I]=0 . \tag{5}$$

The Heisenberg equation of motion for any operator $\hat{O}$, which does not depend on time explicitly, is given by

$$i\hbar\frac{d}{dt}\hat{O}(t)=[\hat{O}(t),\hat{H}] .$$

If we denote the photon-number operators of the three modes by

$$\hat{N}_i(t)=\hat{a}_i^\dagger(t)\hat{a}_i(t) , \tag{6}$$

from Eq. (1) we can readily verify that

$$\frac{d}{dt}[\hat{N}_1(t)+\hat{N}_2(t)]=0$$

and

$$\frac{d}{dt}[\hat{N}_1(t)+\hat{N}_3(t)]=0 .$$

Thus, we see that

$$\hat{W}_{12}=\hat{N}_1(t)+\hat{N}_2(t), \quad \hat{W}_{13}=\hat{N}_1(t)+\hat{N}_3(t) ,$$

and, consequently,

$$\hat{W}_{23}=-\hat{W}_{32}=\hat{N}_2(t)-\hat{N}_3(t)$$

are constants of motion in the trilinear processes.[19] These expressions, which are known in literature as Manley-Rowe relations, require that during the evolution of the system for every photon that is annihilated from mode 1 one photon each is created in modes 2 and 3. In the following we need to consider some other constants of motion that are linear combinations of the Manley-Rowe invariants. These constants are

$$\hat{W}_{23}^{(1)}=\hat{W}_{12}+\hat{W}_{13}+1$$
$$\equiv 2\hat{N}_1(t)+\hat{N}_2(t)+\hat{N}_3(t)+1 , \tag{7a}$$

$$\hat{W}_{13}^{(2)}=\hat{W}_{12}+\hat{W}_{23}\equiv\hat{N}_1(t)+2\hat{N}_2(t)-\hat{N}_3(t) , \tag{7b}$$

and

$$\hat{W}_{12}^{(3)}=\hat{W}_{13}+\hat{W}_{32}\equiv\hat{N}_1(t)-\hat{N}_2(t)+2\hat{N}_3(t) . \tag{7c}$$

Now, we derive the equations which describe the time evolution of the boson operators $\hat{a}_i$ in the trilinear processes. We will write these equations in a form in which we can apply a mathematical method previously used for studying some other nonlinear processes.[18] By using the interaction picture we see that the operators

$$\hat{\mathbf{a}}_i(t)=\exp(i\omega_i t)\hat{a}_i(t) \tag{8a}$$

and

$$\hat{\mathbf{a}}_i^\dagger(t)=\exp(-i\omega_i t)\hat{a}_i^\dagger(t) \tag{8b}$$

obey the equation of motion

$$i\hbar\frac{d}{dt}\hat{\mathbf{a}}_i(t)=[\hat{\mathbf{a}}_i(t),\hat{H}_I] .$$

So we have

$$\frac{d}{dt}[\hat{\mathbf{a}}_2(t)\hat{\mathbf{a}}_3(t)]=-i\Delta\hat{\mathbf{a}}_1(t)[\hat{N}_2(t)+\hat{N}_3(t)+1] \tag{9a}$$

and

$$\frac{d}{dt}[\hat{\mathbf{a}}_1(t)\hat{\mathbf{a}}_3^\dagger(t)]=-i\Delta\hat{\mathbf{a}}_2(t)[\hat{N}_3(t)-\hat{N}_1(t)] . \tag{9b}$$

If we introduce the constant of motion (7a) into Eq. (9a), for the annihilation operator $\hat{\mathbf{a}}_1$ we find the following equations of motion:

$$\frac{d^2}{dt^2}\hat{\mathbf{a}}_1(t)=\Delta^2\hat{\mathbf{a}}_1(t)\hat{\mathcal{G}}^{(1)}(t) \tag{10a}$$

and

$$\frac{d}{dt}\hat{\mathbf{a}}_1(t)=-i\Delta\hat{\mathbf{a}}_2(t)\hat{\mathbf{a}}_3(t)\ , \tag{10b}$$

where

$$\hat{\mathcal{G}}^{(1)}(t)=2\hat{N}_1(t)-\hat{W}^{(1)}_{23}\ . \tag{10c}$$

Analogously, for the operator $\hat{\mathbf{a}}_2$ we can write the equations of motion

$$\frac{d^2}{dt^2}\hat{\mathbf{a}}_2(t)=\Delta^2\hat{\mathbf{a}}_2(t)\hat{\mathcal{G}}^{2}(t) \tag{11a}$$

and

$$\frac{d}{dt}\hat{\mathbf{a}}_2(t)=-i\Delta\hat{\mathbf{a}}_1(t)\hat{\mathbf{a}}^{\dagger}_3(t)\ , \tag{11b}$$

where

$$\hat{\mathcal{G}}^{(2)}(t)=\hat{W}^{(2)}_{13}-2\hat{N}_2(t)\ . \tag{11c}$$

Here $\hat{W}^{(2)}_{13}$ is the constant of motion defined in Eq. (7b). The integrals of the equations of motion must satisfy the initial conditions

$$\hat{\mathbf{a}}_i(t=0)=\hat{a}_i$$

and

$$\hat{\mathbf{a}}^{\dagger}_i(t=0)=\hat{a}^{\dagger}_i\ .$$

For the particular symmetry of the Hamiltonian $\hat{H}_I$ the equations of motion of the operator $\hat{\mathbf{a}}_3$ can be directly obtained from the equations of motion of the operator $\hat{\mathbf{a}}_2$. To this end we must exchange the subscripts 2 and 3 in Eqs. (11). Finally, we note that

$$[\hat{\mathbf{a}}_i(t),\hat{\mathcal{G}}^{(i)}(t)]=0\ . \tag{12}$$

In the following study of trilinear processes we will use some integral operators. Therefore we consider the operator

$$\hat{I}(t;t_1)=\int_0^t dt_1\ , \tag{13}$$

for which we put

$$\hat{I}^{\,n}(t;t_n)=\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\ . \tag{14}$$

Clearly, we have

$$\hat{I}^{\,n}(t;t_n)(t_n)^k=k![(k+n)!]^{-1}t^{k+n} \tag{15a}$$

and, in particular,

$$\hat{I}^{\,n}(t)g\equiv\hat{I}^{\,n}(t;t_n)g=(n!)^{-1}gt^n\ , \tag{15b}$$

provided the function g does not depend on the variables t_i. Then, we introduce the operators $\hat{\mathcal{J}}^{(+)}$ and $\hat{\mathcal{J}}^{(-)}$ defined by the following relations:

$$\hat{\mathcal{J}}^{(+)}(\eta;\zeta)\zeta^n=n!\eta^n \tag{16a}$$

and

$$\hat{\mathcal{J}}^{(-)}(\zeta;\eta)\eta^n=(n!)^{-1}\zeta^n\ . \tag{16b}$$

These operators can be expressed in explicit form by using integral transforms. If

$$\hat{\mathcal{L}}(\eta;\zeta)f(\zeta)=\int_0^\infty d\zeta\exp(-\eta\zeta)f(\zeta)\equiv\varphi(\eta)$$

is the Laplace transform of the function $f(\zeta)$ and

$$\hat{\mathcal{L}}^{-1}(\zeta;\eta)\varphi(\eta)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}d\eta\exp(\eta\zeta)\varphi(\eta)\equiv f(\zeta)$$

is the inverse Laplace transform of the function $\varphi(\eta)$, we have

$$\hat{\mathcal{L}}(\eta;\zeta)\zeta^n=n!\eta^{-n-1}$$

and

$$\hat{\mathcal{L}}^{-1}(\zeta;\eta)\eta^{-n-1}=(n!)^{-1}\zeta^n\ .$$

Consequently, for the operators (16) we can write

$$\hat{\mathcal{J}}^{(+)}(\eta;\zeta)=\hat{\mathcal{L}}(\eta^{-1};\zeta)\eta^{-1} \tag{17a}$$

and

$$\hat{\mathcal{J}}^{(-)}(\zeta;\eta)=\hat{\mathcal{L}}^{-1}(\zeta;\eta^{-1})\eta\ . \tag{17b}$$

Now, in order to study the time evolution of the boson operators $\hat{a}_i$ in the trilinear processes, we begin by solving the equations of motion (11) for the operator $\hat{\mathbf{a}}_2$. For the integration we will use the mathematical method, previously applied to other problems, which is very convenient for studying nonlinear equations. First we give the solution of the equation of motion as a power series in t and then we study the analytical function to which the series converges. In order to express the solution of the equation as a power series we must find a recursive operational relation among the terms of the series and, at the same time, take into account the expansion factor $[(n!)^{-1}]$ for the generic nth term of the series. For this purpose we use the integral operators previously introduced, since they allow us to deal with these difficulties in different phases of the calculations. A formal solution of Eqs. (11) can be easily obtained by iteration techniques. With straightforward argumentations we see that the solution is given by the following expression:

$$\hat{\mathbf{a}}_2(t)=\hat{A}_2(t)+\hat{I}^{\,2}(t)\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^{\,2}(t)\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^{\,2}(t)\hat{G}^{(2)}\{\cdots\}\}\}\ , \tag{18}$$

where the functional $\hat{A}_2(t)$ and $\hat{G}^{(2)}(\hat{A}_2)$ are written as

$$\hat{A}_2(t)=\hat{a}_2-i\Delta\hat{I}(t)\hat{a}_1\hat{a}^{\dagger}_3 \tag{19}$$

and

$$\hat{G}^{(2)}(\hat{A}_2)=\Delta^2\hat{A}_2(\hat{W}_{13}^{(2)}-2\hat{A}_2^\dagger\hat{A}_2)\ , \tag{20}$$

respectively, and the operator $\hat{I}(t)$ has been defined in Eq. (13). It is trivial to verify that Eq. (18) is solution of the equations of motion (11), since the derivatives of Eq. (18) give

$$\frac{d}{dt}\hat{\mathbf{a}}_2(t)=-i\Delta\hat{a}_1\hat{a}_3^\dagger+\hat{I}(t)\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^2(t)\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^2(t)\hat{G}^{(2)}\{\cdots\}\}\}$$

and

$$\frac{d^2}{dt^2}\hat{\mathbf{a}}_2(t)=\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^2(t)\hat{G}^{(2)}\{\hat{A}_2(t)+\hat{I}^2(t)\hat{G}^{(2)}\{\cdots\}\}\}\ .$$

From Eq. (12) we see that the functional $\hat{G}^{(2)}(\hat{A}_2)$ can also be written in the form

$$\hat{G}^{(2)}(\hat{A}_2)=\Delta^2(\hat{W}_{13}^{(2)}-2\hat{A}_2^\dagger\hat{A}_2)\hat{A}_2\ .$$

Moreover, since

$$[\hat{A}_2,\hat{W}_{13}^{(2)}]=2\hat{A}_2\ ,$$

the operators $(\hat{A}_2^\dagger\hat{A}_2)$ and $\hat{W}_{13}^{(2)}$ commute,

$$[\hat{A}_2^\dagger\hat{A}_2,\hat{W}_{13}^{(2)}]=0\ . \tag{21}$$

Now we must find the analytical function to which the series (18) converges. For this purpose we consider the simpler series

$$\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})=\hat{\mathcal{A}}+\hat{I}(\tau)\hat{G}\{\hat{\mathcal{A}}+\hat{I}(\tau)\hat{G}\{\mathcal{A}+\hat{I}(\tau)\hat{G}\{\cdots\}\}\}, \tag{22}$$

where $\mathcal{A}$ is an annihilation boson operator. Then, we define, in a similar way to Eqs. (20) and (21), the functional $\hat{G}(\hat{\mathcal{A}})$ as

$$\hat{G}(\hat{\mathcal{A}})=\Delta^2\hat{\mathcal{A}}(\hat{W}-2\hat{\mathcal{A}}^\dagger\mathcal{A}) \tag{23}$$

with the operator $\hat{W}$ that obeys the commutation rule

$$[\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}},\hat{W}]=0\ . \tag{24}$$

In the Appendix we see that, on making use of the condition (24), the auxiliary functional $\hat{F}_2^{(0)}$ can be written as

$$\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})=\hat{\mathcal{A}}(\tfrac{1}{2}\hat{W})^{1/2}\exp(\Delta^2\tau\hat{W})\times\{\tfrac{1}{2}\hat{W}-\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}}[1-\exp(2\Delta^2\tau\hat{W})]\}^{-1/2}\ . \tag{25}$$

For our purposes it useful to express the τ dependence of the functional $\hat{F}_2^{(0)}$ by means of the operator $\hat{I}(\tau)$. So, with the help of the operator $\hat{\mathcal{J}}^{(+)}$, we write the following identity:

$$\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})=\exp[\hat{I}(\tau)\hat{D}(\eta)]\hat{\mathcal{J}}^{(+)}(\eta;\tau')\times\hat{F}_2^{(0)}(\tau=\tau';\hat{\mathcal{A}})|_0\ , \tag{26}$$

where

$$\hat{D}(\eta)=\frac{d}{d\eta}\ .$$

The $|_0$ notation means that the functional on the right-hand side of Eq. (26) must be evaluated for $\eta=0$. Then we compare the operators $\hat{F}_2^{(0)}$ and $\hat{\mathbf{a}}_2$ which have been defined by Eqs. (22) and (18), respectively. We see that $\hat{\mathbf{a}}_2$ can easily be obtained from the expression of $\hat{F}_2^{(0)}$. To this end it is sufficient, in Eq. (22), to replace (Δ^2) with $(\Delta^2\hat{I})$, $\hat{\mathcal{A}}$ with $\hat{A}_2(t)$, $\hat{W}$ with $\hat{W}_{13}^{(2)}$, and to assume $\tau=t$. Since we will use Eqs. (25) and (26) to express $\hat{F}_2^{(0)}$, we can take into account the above substitutions by replacing (Δ^2) with $(\Delta^2\eta)$, $\hat{W}$ with $\hat{W}_{13}^{(2)}$, and $\hat{\mathcal{A}}$ with $(\hat{a}_2-i\Delta\eta\hat{a}_1\hat{a}_3^\dagger)$ in Eq. (25). So, for $\hat{\mathbf{a}}_2$, we obtain the following expression:

$$\hat{\mathbf{a}}_2(t)=\exp[\hat{I}(t)\hat{D}(\eta)]\hat{\mathcal{J}}^{(+)}(\eta;\tau')\hat{F}_2(\eta;\tau')\ , \tag{27}$$

where we have put

$$\hat{F}_2(\eta;\tau)=\hat{A}_2(\eta)(\tfrac{1}{2}\hat{W}_{13}^{(2)})^{1/2}\exp(\Delta^2\eta\tau\hat{W}_{13}^{(2)})\times\{\tfrac{1}{2}\hat{W}_{13}^{(2)}-\hat{A}_2^\dagger(\eta)\hat{A}_2(\eta)\times[1-\exp(2\Delta^2\eta\tau\hat{W}_{13}^{(2)})]\}^{-1/2} \tag{28}$$

with

$$\hat{A}_2(\eta)=\hat{a}_2-i\Delta\eta\hat{a}_1\hat{a}_3^\dagger\ ,\quad \hat{A}_2^\dagger(\eta)=\hat{a}_2^\dagger+i\Delta\eta\hat{a}_1^\dagger\hat{a}_3\ . \tag{29}$$

Now, on making use of the operator $\hat{\mathcal{J}}^{(-)}$, we free Eq. (27) from the operator $\hat{I}(t)$. So we see from the definitions (15b) and (16b) that $\hat{\mathbf{a}}_2$ can be expressed in the compact form

$$\hat{\mathbf{a}}_2(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{F}_2(\eta;\tau)\ . \tag{30}$$

When this result is introduced in Eq. (8a) we find that

$$\hat{a}_2(t)=\exp(-i\omega_2 t)\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{F}_2(\eta;\tau)\ . \tag{31}$$

Thus we have obtained the desired expression that describes the time evolution of the operator $\hat{a}_2$ in the trilinear processes.

The solution of Eqs. (10) which describe the time evolution of the operator $\hat{\mathbf{a}}_1$ can be easily obtained from Eq. (30). To this purpose we must suitably exchange the subscripts and replace the parameter (Δ^2) by $(-\Delta^2)$ in Eqs. (28)–(30). So, for the operator $\hat{\mathbf{a}}_1$ we find the expression

$$\hat{\mathbf{a}}_1(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{F}_1(\eta;\tau)\ , \tag{32}$$

where it is

$$\hat{F}_1(\eta;\tau)=\hat{A}_1(\eta)(\tfrac{1}{2}\hat{W}_{23}^{(1)})^{1/2}\exp(-\Delta^2\eta\tau\hat{W}_{23}^{(1)})\{\tfrac{1}{2}\hat{W}_{23}^{(1)}-\hat{A}_1^\dagger(\eta)\hat{A}_1(\eta)[1-\exp(-2\Delta^2\eta\tau\hat{W}_{23}^{(1)})]\}^{-1/2}\ , \tag{33}$$

with

$$\hat{A}_1(\eta)=\hat{a}_1-i\Delta\eta\hat{a}_2\hat{a}_3,\quad \hat{A}_1^\dagger(\eta)=\hat{a}_1^\dagger+i\Delta\eta\hat{a}_2^\dagger\hat{a}_3^\dagger\ . \tag{34}$$

Consequently, the expression that describes the time evolution of the operator $\hat{a}_1$ is given by

$$\hat{a}_1(t)=\exp(-i\omega_1 t)\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{F}_1(\eta;\tau)\ . \tag{35}$$

The time evolution operator $\hat{a}_3$ also can be obtained from Eq. (31). We must exchange the subscripts 2 and 3 in Eq. (28), as we have previously noted.

For the sake of completeness we rewrite Eqs. (31) and (35) by expressing the integral operators $\hat{\mathcal{J}}^{(-)}$ and $\hat{\mathcal{J}}^{(+)}$ in explicit form. If we use the Laplace transform and the inverse Laplace transform, from Eqs. (17) we see that the operators $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\hat{a}_1(t)=\exp(-i\omega_1 t)\hat{\mathcal{L}}^{-1}(t;\chi^{-1})\hat{\mathcal{L}}(\chi^{-1};\tau)\times\hat{F}_1(\eta=\chi^{-1};\tau) \tag{36a}$$

and

$$\hat{a}_2(t)=\exp(-i\omega_2 t)\hat{\mathcal{L}}^{-1}(t;\chi^{-1})\hat{\mathcal{L}}(\chi^{-1};\tau)\times\hat{F}_2(\eta=\chi^{-1};\tau)\ . \tag{36b}$$

We point out that the integral expressions (36) contain a Laplace transform and an inverse Laplace transform only. Therefore these expressions can be used to study the properties of trilinear processes or to calculate handy approximate values of the quantities which describe other particular multilinear boson systems.

III. APPLICATIONS

We will show the utility of the present approach to the analysis of the trilinear processes by applying the results to some simple cases. In these calculations we assume that, at time $t=0$, the three boson modes are described by a coherent state

$$|\alpha\rangle\equiv|\alpha_1,\alpha_2,\alpha_3\rangle$$

with

$$\hat{a}_i|\alpha\rangle=\alpha_i|\alpha\rangle\ .$$

Moreover, we assume that the amplitude α_j of one of the modes is such that $|\alpha_j|\gg 1$. Experimentally, this condition is verified when the mode j represents the pump. In fact, the pump is always in a high state of excitation and it is generally large compared with the other signals of the process. We point out that, in the present calculation, all simplifying conditions must be satisfied only at time $t=0$, as we use the exact expressions for the time evolution of the boson operators.

In the following we will indicate the mean value of an operator $\hat{O}$ as

$$\langle\hat{O}\rangle_\alpha^{(j)}\equiv\langle\alpha|\hat{O}|\alpha\rangle^{(j)}\ ,$$

where the label j indicates the pump mode.

First, we will analyze the time evolution of the boson operators when the pump mode is represented by mode 2, so it is $|\alpha_2|\gg 1$. In this case we can identify the process described by the trilinear Hamiltonian with the frequency conversion where modes 1 and 3 are the idler and signal, respectively. In order to simplify the following calculations the amplitudes α_1 and α_3 are assumed such that

$$|\alpha_2|\gg|\alpha_1|,\quad |\alpha_2|\gg|\alpha_3|\ . \tag{37}$$

We begin by considering the mean value of the boson operator $\hat{a}_1$,

$$\langle\hat{a}_1(t)\rangle_\alpha^{(2)}=\exp(-i\omega_1 t)\langle\mathbf{\hat{a}}_1(t)\rangle_\alpha^{(2)}\ . \tag{38}$$

From Eq. (32) we see that

$$\langle\mathbf{\hat{a}}_1(t)\rangle_\alpha^{(2)}=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle\hat{F}_1(\eta;\tau)\rangle_\alpha^{(2)}\ . \tag{39}$$

When we expand the functional $\hat{F}_1$ as a power series, from Eqs. (21) and (33) we obtain

$$\hat{F}_1(\eta;\tau)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}c_{nl}\hat{A}_1(\eta)[\hat{A}_1^\dagger(\eta)\hat{A}_1(\eta)]^n\times(\hat{W}_{23}^{(1)})^{-n}\times\exp[-(2l+1)\hat{W}_{23}^{(1)}\Delta^2\eta\tau] \tag{40}$$

with

$$c_{nl}=(-1)^l(2n)![2^n n!(n-l)!l!]^{-1}\ . \tag{41}$$

In order to give a handy form to the mean value $\langle\hat{F}_1\rangle_\alpha^{(2)}$ we introduce some approximations. Since the conditions (37) allow us to neglect the terms of superior order with respect to $|\alpha_1|\,|\alpha_2|^{-1}$ and $|\alpha_3|\,|\alpha_2|^{-1}$, we see that the mean value $\langle\hat{F}_1\rangle_\alpha^{(2)}$ can be written as

$$\langle\hat{F}_1(\eta;\tau)\rangle_\alpha^{(2)}=\mathcal{F}_1^{(2)}(\eta;\tau;\gamma=1)\ , \tag{42}$$

where

$$\mathcal{F}_1^{(2)}(\eta;\tau;\gamma)=\sum_{n,l}c_{nl}[(\alpha_1-i\alpha_2\alpha_3\Delta\eta)-n(\alpha_1^*\alpha_2\alpha_3-\alpha_1\alpha_2^*\alpha_3^*)\alpha_2\alpha_3(|\alpha_2|^2|\alpha_3|^2)^{-1}]\times(|\alpha_3|^2\Delta^2\eta^2)^n\exp[-(2l+1)|\alpha_2|^2\gamma\Delta^2\eta\tau]\ . \tag{43}$$

The utility of the parameter γ in Eq. (43) will appear clear in the following. Now it is useful to introduce a property of the operator $\hat{\mathcal{J}}^{(-)}$. If $f(\eta)$ is an arbitrary function, from the definition (16b) we have

$$\hat{\mathcal{J}}^{(-)}(t;\eta)[\eta^{-v}f(\eta)]=\hat{D}^{v}(t)[\hat{\mathcal{J}}^{(-)}(t;\eta)f(\eta)] .$$

Consequently, from Eq. (39) we find that the mean value of the operator $\hat{\mathbf{a}}_1$ is given by

$$\langle \hat{\mathbf{a}}_1(t)\rangle_\alpha^{(2)}=[1+i(\Delta\alpha_2\alpha_3)^{-1}\alpha_1\hat{D}(t)-i(\Delta^2|\alpha_2|^2)^{-1}(\alpha_1^*\alpha_2\alpha_3-\alpha_1\alpha_2^*\alpha_3^*)\hat{D}(|\alpha_3|^2)\hat{D}(t)]\mathcal{F}_{1,1}^{(2)}(t;\alpha_i;\gamma=1) , \tag{44}$$

where the function $\mathcal{F}_{1,1}^{(2)}$ is defined as

$$\mathcal{F}_{1,1}^{(2)}(t;\alpha_i;\gamma)=-i\alpha_2\alpha_3\hat{\mathcal{J}}^{(-)}(t;\eta)\sum_{n,l}c_{nl}|\alpha_3|^{2n}(\Delta\eta)^{2n+1}\sum_{h=0}^{\infty}[-(2l+1)|\alpha_2|^2\gamma\Delta^2\eta^2]^h . \tag{45}$$

By a straightforward calculation we see that the function $\mathcal{F}_{1,1}^{(2)}$ can be expressed as

$$\mathcal{F}_{1,1}^{(2)}(t;\alpha_i;\gamma)=-i\alpha_2\alpha_3\sum_{n,l}c_{nl}|\alpha_3|^{2n}\{(-1)^n[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}]^{-2n-1}\sin[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]$$
$$+[(2n-1)!]^{-1}(2l+1)^{-1}|\alpha_2|^{-1}\gamma^{-1}(\Delta t)^{2n-1}\} .$$

Then, after a little algebra, we obtain the more compact expression

$$\mathcal{F}_{1,1}^{(2)}(t;\alpha_i;\gamma)=-i\alpha_2\alpha_3\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}[(2l+1)^{1/2}|\alpha_2|]^{-1}$$
$$\times\hat{D}^{l}(\theta)\{[1+2\theta(2l+1)^{-1}|\alpha_3|^2|\alpha_2|^{-2}\gamma^{-1}]^{-1/2}\gamma^{-1/2}$$
$$\times\sin[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,1}^{(2)}(t;|\alpha_3|;\theta)\}|_1 , \tag{46}$$

where θ is a real parameter, the function $Y_{l,1}^{(2)}$ is defined as

$$Y_{l,1}^{(2)}(t;|\alpha_3|;\theta)=(2\theta)^{1/2}|\alpha_3|[(2l+1)^{1/2}|\alpha_2|]^{-1}B_1[(2\theta)^{1/2}|\alpha_3|\Delta t] , \tag{47}$$

and B_h is the modified Bessel function

$$B_h(y)=\sum_{u=0}^{\infty}[u!(u+h)!]^{-1}(\tfrac{1}{2}y)^{h+2u} .$$

Here and in the following by the $|_1$ notation we mean that the function on the right-hand side of the equations must be evaluated for $\theta=1$. When we introduce the result given by Eq. (46) into Eq. (44), from Eq. (38) we see that the mean value of the operator $\hat{a}_1$ can be written as

$$\langle \hat{a}_1(t)\rangle_\alpha^{(2)}=\exp(-i\omega_1 t)[\mathcal{F}_{1,1}^{(2)}(t;\alpha_i;\gamma=1)+\mathcal{F}_{1,2}^{(2)}(t;\alpha_i;\gamma=1)+\mathcal{F}_{1,3}^{(2)}(t;\alpha_i;\gamma=1)] , \tag{48}$$

where we have put

$$\mathcal{F}_{1,2}^{(2)}(t;\alpha_i;\gamma)=\alpha_1\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}\hat{D}^{l}(\theta)\{[1+2\theta(2l+1)^{-1}|\alpha_3|^2|\alpha_2|^{-2}\gamma^{-1}]^{-1/2}$$
$$\times\cos[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,2}^{(2)}(t;|\alpha_3|;\theta)\}|_1 \tag{49a}$$

and

$$\mathcal{F}_{1,3}^{(2)}(t;\alpha_i;\gamma)=\alpha_2\alpha_3(\alpha_1\alpha_2^*\alpha_3^*-\alpha_1^*\alpha_2\alpha_3)|\alpha_2|^{-2}$$
$$\times\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}\hat{D}^{l}(\theta)\{-\theta[(2l+1)|\alpha_2|^2\gamma]^{-1}[1+2\theta(2l+1)^{-1}|\alpha_3|^2|\alpha_2|^{-2}\gamma^{-1}]^{-3/2}$$
$$\times\cos[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,3}^{(2)}(t;|\alpha_3|;\theta)\}|_1 \tag{49b}$$

with

$$Y_{l,2}^{(2)}(t;|\alpha_3|;\theta)=[(2l+1)|\alpha_2|^2]^{-1}\theta|\alpha_3|^2\{B_0[(2\theta)^{1/2}|\alpha_3|\Delta t]+B_2[(2\theta)^{1/2}|\alpha_3|\Delta t]\} \tag{50a}$$

and

$$Y_{l,3}^{(2)}(t;|\alpha_3|;\theta)=[(2l+1)|\alpha_2|^2]^{-1}\theta\{(2\theta)^{1/2}|\alpha_3|\Delta t B_1[(2\theta)^{1/2}|\alpha_3|\Delta t]+B_0[(2\theta)^{1/2}|\alpha_3|\Delta t]\} . \tag{50b}$$

Now, we study the mean value of the operator $\hat{a}_3$,

$$\langle \hat{a}_3(t)\rangle_\alpha^{(2)}=\exp(-i\omega_3 t)\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle \hat{F}_3(\eta;\tau)\rangle_\alpha^{(2)} ,$$

where, from Eq. (28), we have

$$\hat{F}_3(\eta;\tau)=\sum_{n,l} c_{nl}\,\hat{A}_3(\eta)[\hat{A}_3^{\dagger}(\eta)\hat{A}_3(\eta)]^n[\hat{W}_{12}^{(3)}]^{-n}\exp[(2l+1)\hat{W}_{12}^{(3)}\Delta^2\eta\tau] .$$

On the present conditions we can approximate the mean value $\langle \hat{F}_3\rangle_\alpha^{(2)}$ by writing

$$\langle \hat{F}_3(\eta;\tau)\rangle_\alpha^{(2)}=\mathcal{F}_3^{(2)}(\eta;\tau;\gamma=1) \tag{51}$$

with

$$\mathcal{F}_3^{(2)}(\eta;\tau;\gamma)=\sum_{n,l} c_{nl}[(\alpha_3-i\alpha_1\alpha_2^*\Delta\eta)-n(\alpha_1\alpha_2^*\alpha_3^*-\alpha_1^*\alpha_2\alpha_3)\alpha_1\alpha_2^*(|\alpha_2|^2|\alpha_1|^2)^{-1}]$$
$$\times(|\alpha_1|^2\Delta^2\eta^2)^n\exp[-(2l+1)|\alpha_2|^2\gamma\Delta^2\eta\tau] . \tag{52}$$

From Eqs. (44) and (52) we directly infer that the mean value of the operator $\hat{\mathbf{a}}_3$ can be expressed as

$$\langle \hat{\mathbf{a}}_3(t)\rangle_\alpha^{(2)}=[1+i(\Delta\alpha_1\alpha_2^*)^{-1}\alpha_3\hat{D}(t)-i(\Delta^2|\alpha_2|^2)^{-1}(\alpha_1\alpha_2^*\alpha_3^*-\alpha_1^*\alpha_2\alpha_3)\hat{D}(|\alpha_1|^2)\hat{D}(t)]\mathcal{F}_{3,1}^{(2)}(t;\alpha_i;\gamma=1) , \tag{53}$$

where the function $\mathcal{F}_{3,1}^{(2)}$ is given by

$$\mathcal{F}_{3,1}^{(2)}(t;\alpha_i;\gamma)=-i\alpha_1\alpha_2^*\hat{\mathcal{J}}^{(-)}(t;\eta)\sum_{n,l} c_{nl}|\alpha_1|^{2n}(\Delta\eta)^{2n+1}\sum_{h=0}^{\infty}[-(2l+1)|\alpha_2|^2\gamma\Delta^2\eta^2]^h . \tag{54}$$

From this result we conclude that the mean value of the operator $\hat{a}_3$ can be written as

$$\langle \hat{a}_3(t)\rangle_\alpha^{(2)}=\exp(-i\omega_3 t)[\mathcal{F}_{3,1}^{(2)}(t;\alpha_i;\gamma=1)+\mathcal{F}_{3,2}^{(2)}(t;\alpha_i;\gamma=1)+\mathcal{F}_{3,3}^{(2)}(t;\alpha_i;\gamma=1)] . \tag{55}$$

Hence, on the analogy of Eqs. (46) and (49), we have

$$\mathcal{F}_{3,1}^{(2)}(t;\alpha_i;\gamma)=-i\alpha_2\alpha_3^*\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}[(2l+1)^{1/2}|\alpha_2|]^{-1}$$
$$\times\hat{D}^{\,l}(\theta)\{[1+2\theta(2l+1)^{-1}|\alpha_1|^2|\alpha_2|^{-2}\gamma^{-1}]^{-1/2}\gamma^{-1/2}$$
$$\times\sin[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,1}^{(2)}(t;|\alpha_1|;\theta)\}|_1 , \tag{56a}$$

$$\mathcal{F}_{3,2}^{(2)}(t;\alpha_i;\gamma)=\alpha_3\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}\hat{D}^{\,l}(\theta)\{[1+2\theta(2l+1)^{-1}|\alpha_1|^2|\alpha_2|^{-2}\gamma^{-1}]^{-1/2}$$
$$\times\cos[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,2}^{(2)}(t;|\alpha_1|;\theta)\}|_1 , \tag{56b}$$

and

$$\mathcal{F}_{3,3}^{(2)}(t;\alpha_i;\gamma)=\alpha_1\alpha_2^*(\alpha_1^*\alpha_2\alpha_3-\alpha_1\alpha_2^*\alpha_3^*)|\alpha_2|^{-2}$$
$$\times\sum_{l=0}^{\infty}(-1)^l(l!)^{-1}\hat{D}^{\,l}(\theta)\{-\theta[(2l+1)|\alpha_2|^2\gamma]^{-1}[1+2\theta(2l+1)^{-1}|\alpha_1|^2|\alpha_2|^{-2}\gamma^{-1}]^{-3/2}$$
$$\times\cos[(2l+1)^{1/2}|\alpha_2|\gamma^{1/2}\Delta t]+\gamma^{-1}Y_{l,3}^{(2)}(t;|\alpha_1|;\theta)\}|_1 . \tag{56c}$$

Finally, we study the mean value of the operator $\hat{a}_2$. For the sake of simplicity we consider only times t for which we have

$$|\alpha_2|\gg|\alpha_1||\alpha_3|\Delta t .$$

So, from Eq. (28) we can write that

$$\langle \hat{F}_2(\eta;\tau)\rangle_\alpha^{(2)}=\alpha_2-i\alpha_1\alpha_3^*\Delta\eta+i\alpha_2(\alpha_1^*\alpha_2\alpha_3-\alpha_1\alpha_2^*\alpha_3^*)(2|\alpha_2|^2)^{-1}\Delta\eta[\exp(-4|\alpha_2|^2\Delta^2\eta\tau)-1] .$$

Since

$$\langle \hat{\mathbf{a}}_2(t)\rangle_\alpha^{(2)}=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle \hat{F}_2(\eta;\tau)\rangle_\alpha^{(2)} , \tag{57}$$

after a little algebra we obtain that

$$\langle \hat{a}_2(t)\rangle_\alpha^{(2)}=\exp(-i\omega_2 t)\{\alpha_2-i\alpha_1\alpha_3^*\Delta t+i\alpha_2(\alpha_1^*\alpha_2\alpha_3-\alpha_1\alpha_2^*\alpha_3^*)(2|\alpha_2|^2)^{-1}[(2|\alpha_2|)^{-1}\sin(2|\alpha_2|\Delta t)-\Delta t]\} . \tag{58}$$

Thus we have obtained the expressions that describe the time evolution of the three boson operators when the pump mode is mode 2. We point out that the resulting mean values are presented as a superposition of circular and modified Bessel functions. The solutions, therefore, are not periodic as the Bessel functions cause a drift in the time evolution.

Then, we will study the time evolution of the boson operators when the pump mode is represented by mode 1. In this case the process described by the trilinear Hamiltonian can be identified with the parametric amplification where modes 2 and 3 are the idler and signal modes. So, we assume the amplitudes α_i such as

$$|\alpha_1| \gg 1 \ ,$$

$$|\alpha_1| \gg |\alpha_2| \ ,$$

and

$$|\alpha_1| \gg |\alpha_3| \ .$$

We start by considering the mean values of the operators $\hat{a}_2$ and $\hat{a}_3$. On the present conditions we can write the mean values $\langle \hat{F}_2 \rangle_\alpha^{(1)}$ and $\langle \hat{F}_3 \rangle_\alpha^{(1)}$ as

$$\langle \hat{F}_2(\eta;\tau) \rangle_\alpha^{(1)} = \sum_{n,l} c_{nl} [(\alpha_2 - i\alpha_1\alpha_3^*\Delta\eta) - n(\alpha_1\alpha_2^*\alpha_3^* - \alpha_1^*\alpha_2\alpha_3)\alpha_1\alpha_3^*(|\alpha_1|^2|\alpha_3|^2)^{-1}] \times (|\alpha_3|^2\Delta^2\eta^2)^n \exp[(2l+1)|\alpha_1|^2\Delta^2\eta\tau] \tag{59a}$$

and

$$\langle \hat{F}_3(\eta;\tau) \rangle_\alpha^{(1)} = \sum_{n,l} c_{nl} [(\alpha_3 - i\alpha_1\alpha_2^*\Delta\eta) - n(\alpha_1\alpha_2^*\alpha_3^* - \alpha_1^*\alpha_2\alpha_3)\alpha_1\alpha_2^*(|\alpha_1|^2|\alpha_2|^2)^{-1}] \times (|\alpha_2|^2\Delta^2\eta^2)^n \exp[(2l+1)|\alpha_1|^2\Delta^2\eta\tau] \ . \tag{59b}$$

If we compare these expressions with Eq. (52), we immediately note that the mean values $\langle \hat{F}_2 \rangle_\alpha^{(1)}$ and $\langle \hat{F}_3 \rangle_\alpha^{(1)}$ are very similar to the mean value $\langle \hat{F}_3 \rangle_\alpha^{(2)}$. Consequently, $\langle \hat{F}_2 \rangle_\alpha^{(1)}$ and $\langle \hat{F}_3 \rangle_\alpha^{(1)}$ can be obtained by $\langle \hat{F}_3 \rangle_\alpha^{(2)}$, provided we exchange the indices suitably and we put $\gamma = -1$ in the expressions which give $\langle \hat{F}_3 \rangle_\alpha^{(2)}$. So we see that the mean values of the boson operators $\hat{a}_2$ and $\hat{a}_3$ can be written in the known form

$$\langle \hat{a}_2(t) \rangle_\alpha^{(1)} = \exp(-i\omega_2 t)[\mathcal{F}_{2,1}^{(1)}(t;\alpha_i) + \mathcal{F}_{2,2}^{(1)}(t;\alpha_i) + \mathcal{F}_{2,3}^{(1)}(t;\alpha_i)] \tag{60a}$$

and

$$\langle \hat{a}_3(t) \rangle_\alpha^{(1)} = \exp(-i\omega_3 t)[\mathcal{F}_{3,1}^{(1)}(t;\alpha_i) + \mathcal{F}_{3,2}^{(1)}(t;\alpha_i) + \mathcal{F}_{3,3}^{(1)}(t;\alpha_i)] \ . \tag{60b}$$

The different functions of Eqs. (60) are defined through the following expressions:

$$\mathcal{F}_{u,1}^{(1)}(t;\alpha_i) = i\alpha_1\alpha_v^* \sum_{l=0}^{\infty} (-1)^l (l!)^{-1} [(2l+1)^{1/2}|\alpha_1|]^{-1} \hat{D}^{\,l}(\theta) \times \{[1-2\theta(2l+1)^{-1}|\alpha_v|^2|\alpha_1|^{-2}]^{-1/2} \sinh[(2l+1)^{1/2}|\alpha_1|\Delta t] - Y_{l,1}^{(1)}(t;|\alpha_v|;\theta)\}|_1 \ , \tag{61a}$$

$$\mathcal{F}_{u,2}^{(1)}(t;\alpha_i) = \alpha_u \sum_{l=0}^{\infty} (-1)^l (l!)^{-1} \hat{D}^{\,l}(\theta) \times \{[1-2\theta(2l+1)^{-1}|\alpha_v|^2|\alpha_1|^{-2}]^{-1/2} \cosh[(2l+1)^{1/2}|\alpha_1|\Delta t] - Y_{l,2}^{(1)}(t;|\alpha_v|;\theta)\}|_1 \ , \tag{61b}$$

and

$$\mathcal{F}_{u,3}^{(1)}(t;\alpha_i) = \alpha_1\alpha_v^*(\alpha_1^*\alpha_2\alpha_3 - \alpha_1\alpha_2^*\alpha_3^*)|\alpha_1|^{-2} \times \sum_{l=0}^{\infty} (-1)^l (l!)^{-1} \hat{D}^{\,l}(\theta) \{ \theta[(2l+1)|\alpha_1|^2]^{-1} [1-2\theta(2l+1)^{-1}|\alpha_v|^2|\alpha_1|^{-2}]^{-3/2} \times \cosh[(2l+1)^{1/2}|\alpha_1|\Delta t] - Y_{l,3}^{(1)}(t;|\alpha_v|,\theta)\}|_1 \ . \tag{61c}$$

In fact, the functions $\mathcal{F}_{2,1}^{(1)}$, $\mathcal{F}_{2,2}^{(1)}$, and $\mathcal{F}_{2,3}^{(1)}$ are obtained from Eqs. (61) by letting $u=2$ and $v=3$ and the functions $\mathcal{F}_{3,1}^{(1)}$, $\mathcal{F}_{3,2}^{(1)}$, and $\mathcal{F}_{3,3}^{(1)}$ by letting $u=3$ and $v=2$.

In order to study the mean value $\langle \hat{a}_1 \rangle_\alpha^{(1)}$ we write the mean value of the operator $\hat{F}_1$. If we consider times t for which it is

$$|\alpha_1| \gg |\alpha_2|\,|\alpha_3|\Delta t \ ,$$

from Eq. (33) we find

$$\langle \hat{F}_1(\eta;\tau) \rangle_\alpha^{(1)} = \alpha_1 - i\alpha_2\alpha_3\Delta\eta + i\alpha_1(\alpha_1\alpha_2^*\alpha_3^* - \alpha_1^*\alpha_2\alpha_3)(2|\alpha_1|^2)^{-1}\Delta^2\eta \exp(4|\alpha_1|^2\Delta\eta\tau) \ .$$

Consequently, for the operator $\hat{a}_1$ we can write the mean value

$$\langle \hat{a}_1(t)\rangle_\alpha^{(1)}=\exp(-i\omega_1 t)[\alpha_1 - i\alpha_2\alpha_3\Delta t + i\alpha_1(\alpha_1\alpha_2^*\alpha_3^* - \alpha_1^*\alpha_2\alpha_3)(4|\alpha_1|^3)^{-1}\sinh(2|\alpha_1|\Delta t)] \ . \tag{62}$$

To conclude, we have obtained the desired expressions that give the time evolution of the boson operators when the pump mode is represented by mode 1. The resulting expressions are presented as a superposition of hyperbolic and modified Bessel functions. The Bessel functions help the solutions to get the correct convergence at long times.

It may be of some interest to deduce from the previous results the well-known expressions that describe the boson operators for the frequency conversion and parametric amplification in the quasiclassical approximation.[20] If we assume $|\alpha_3|\,|\alpha_2|^{-1}\rightarrow 0$ and $|\alpha_1|\,|\alpha_2|^{-1}\rightarrow 0$, from Eqs. (48), (55), and (58) we find the following mean values for the boson operators in the frequency conversion:

$$\langle \hat{a}_1(t)\rangle_{\alpha,0}^{(2)}=\exp(-i\omega_1 t)\times[\alpha_1\cos(|\alpha_2|\Delta t) - i\alpha_3\alpha_2|\alpha_2|^{-1}\sin(|\alpha_2|\Delta t)] \ , \tag{63a}$$

$$\langle \hat{a}_3(t)\rangle_{\alpha,0}^{(2)}=\exp(-i\omega_3 t)\times[\alpha_3\cos(|\alpha_2|\Delta t) - i\alpha_1\alpha_2^*|\alpha_2|^{-1}\sin(|\alpha_2|\Delta t)] \ , \tag{63b}$$

and

$$\langle \hat{a}_2(t)\rangle_{\alpha,0}^{(2)}=\exp(-i\omega_2 t)\alpha_2 \ . \tag{63c}$$

Then, if we assume $|\alpha_2|\,|\alpha_1|^{-1}\rightarrow 0$ and $|\alpha_3|\,|\alpha_1|^{-1}\rightarrow 0$, we obtain, from Eqs. (61) and (62), that the mean values of the boson operators in the parametric amplification are given by

$$\langle \hat{a}_2(t)\rangle_{\alpha,0}^{(1)}=\exp(-i\omega_2 t)\times[\alpha_2\cosh(|\alpha_1|\Delta t) - i\alpha_3^*\alpha_1|\alpha_1|^{-1}\sinh(|\alpha_1|\Delta t)] \ , \tag{64a}$$

$$\langle \hat{a}_3(t)\rangle_{\alpha,0}^{(1)}=\exp(-i\omega_3 t)\times[\alpha_3\cosh(|\alpha_1|\Delta t) - i\alpha_2^*\alpha_1|\alpha_1|^{-1}\sinh(|\alpha_1|\Delta t)] \ , \tag{64b}$$

and

$$\langle \hat{a}_1(t)\rangle_{\alpha,0}^{(1)}=\exp(-i\omega_1 t)\alpha_1 \ . \tag{64c}$$

Thus, we have found the expressions that describe the mean values of the boson operators for the frequency conversion and parametric amplification in the quasiclassical approximation. Finally, we note that Eqs. (63) and (64) can be directly obtained from Eq. (18) provided the functionals $\hat{G}^{(2)}$ and $\hat{G}^{(1)}$ are expressed as

$$\hat{G}^{(2)}(\hat{A}_2)=\Delta^2|\alpha_2|^2\hat{A}_2$$

and

$$\hat{G}^{(1)}(\hat{A}_1)=-\Delta^2|\alpha_1|^2\hat{A}_1 \ .$$

In a subsequent paper we will analyze the statistical properties of the boson operators after the trilinear processes by using the above method, which turns out to be very useful to study the time evolution of the operators in these processes.

IV. CONCLUSIONS

We have studied the time evolution of the boson operators in the processes described by a trilinear boson Hamiltonian. The equations of motion have been solved by using iteration methods and the solutions have been presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals of the boson operators. The solutions written in this form allow to analyze the properties of the processes obeying the trilinear Hamiltonian and to write handy approximate values of quantities describing these processes. In fact, the inverse Laplace transform, as well as the Laplace transform, can be evaluated by using the convolution law or one of the many approximate methods of calculation reported in literature. So, the operators obeying the trilinear Hamiltonian can be expressed in forms which facilitate the study of particular processes without applying the usual approximation techniques.

In order to verify the feasibility of the present method we have analyzed the time evolution of the boson operators when one of the modes is initially in a quasiclassical state and for this particular case we have written the resulting functionals in analytical form.

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APPENDIX: AUXILIARY FUNCTIONAL

In this appendix we study the functional $\hat{F}_2^{(0)}$ defined in Eq. (22) as

$$\hat{F}_2^{(0)}(\tau;\mathcal{A})=\hat{\mathcal{A}}+\hat{I}(\tau)\hat{G}\{\hat{\mathcal{A}}+\hat{I}(\tau)\hat{G}\times\{\hat{\mathcal{A}}+\hat{I}(\tau)\hat{G}\{\cdots\}\}\} \ , \tag{A1}$$

where $\mathcal{A}$ is an annihilation boson operator, the integral operator $\hat{I}(\tau)$ is given by Eq. (13) and the functional $\hat{G}$ is expressed as

$$\hat{G}(\hat{\mathcal{A}})=\Delta^2\hat{\mathcal{A}}(\hat{W}-2\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}}) \ .$$

Moreover, for the operators $(\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}})$ and $\hat{W}$ we assume that

$$[\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}},\hat{W}]=0 \ . \tag{A2}$$

We note that the series (A1) is the solution of the equation

$$\frac{d}{d\tau}\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})=\hat{G}[\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})] \tag{A3}$$

with the initial condition

$$\hat{F}_2^{(0)}(\tau=0;\hat{\mathcal{A}})=\hat{\mathcal{A}} .$$

Since we will use a method previously applied for studying similar problems,[18] we consider the function

$$F(\tau;\xi)=\xi+\hat{I}(\tau)G\{\xi+\hat{I}(\tau)G\{\xi+\hat{I}(\tau)G\{\cdots\}\}\} , \tag{A4}$$

where ξ and τ are independent c-number variables and the function G is expressed as

$$G(\xi)=\Delta^2\xi(w-2|\xi|^2) .$$

We assume that w is a real parameter, as it is Δ^2. We begin by writing the function (A4) in a more useful form. Since this function obeys the equation

$$\frac{d}{d\tau}F(\tau;\xi)=G[F(\tau;\xi)]$$

with the initial condition $F(\tau=0;\xi)=\xi$, we find that

$$F(\tau;\xi)=\exp[\tau G(\xi)\hat{D}(\xi)]\xi , \tag{A5}$$

where

$$\hat{D}(\xi)=\frac{d}{d\xi} .$$

In order to evaluate the function (A5) we start by assuming that ξ is real. In this case we have

$$F^{(R)}(\tau;\xi)=\exp[\tau G^{(R)}(\xi)\hat{D}(\xi)]\xi$$

with

$$G^{(R)}(\xi)=\Delta^2\xi(w-2\xi^2) .$$

If we put

$$\hat{D}(\mu)=G^{(R)}(\xi)\hat{D}(\xi) ,$$

the variable μ can be written as function of ξ in the form

$$\mu=(2\Delta^2 w)^{-1}\ln[\xi^2(\tfrac{1}{2}w-\xi^2)^{-1}] .$$

We, therefore, have

$$\xi=(\tfrac{1}{2}w)^{1/2}\exp(\Delta^2 w\mu)[1+\exp(2\Delta^2 w\mu)]^{-1/2} \tag{A6}$$

and, when we express $F^{(R)}(\tau;\xi)$ in terms of μ, we find

$$F^{(R)}(\tau;\xi)=\exp[\tau\hat{D}(\mu)]\{(\tfrac{1}{2}w)^{1/2}\exp(\Delta^2 w\mu)\times[1+\exp(2\Delta^2 w\mu)]^{-1/2}\} .$$

On performing the derivatives we obtain, from Eq. (A6), that

$$F^{(R)}(\tau;\xi)=\xi(\tfrac{1}{2}w)^{1/2}\exp(\Delta^2 w\tau)\times\{\tfrac{1}{2}w-\xi^2[1-\exp(2\Delta^2 w\tau)]\}^{-1/2} . \tag{A7}$$

Since the parameters Δ^2 and w which define the functions $G^{(R)}$ and G are real, we can obtain the expression of the function F from Eq. (A7) directly. So we find that

$$F(\tau;\xi)=\xi(\tfrac{1}{2}w)^{1/2}\exp(\Delta^2 w\tau)\times\{\tfrac{1}{2}w-|\xi|^2[1-\exp(2\Delta^2 w\tau)]\}^{-1/2} . \tag{A8}$$

If we compare Eqs. (A1) and (A4) we see that the functional $\hat{F}_2^{(0)}$ can be easily obtained from the expression of the function F. To this end it is sufficient, in Eq. (A8), to replace ξ, $|\xi|^2$, and w by the operators $\hat{\mathcal{A}}$, $(\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}})$, and $\hat{W}$, respectively. This operation does not present ordering problems as the commutation rule (A2) must be satisfied. So, we can write that

$$\hat{F}_2^{(0)}(\tau;\hat{\mathcal{A}})=\hat{\mathcal{A}}(\tfrac{1}{2}\hat{W})^{1/2}\exp(\Delta^2\tau\hat{W})\times\{\tfrac{1}{2}\hat{W}-\hat{\mathcal{A}}^\dagger\hat{\mathcal{A}}[1-\exp(2\Delta^2\tau\hat{W})]\}^{-1/2} .$$

This is the desired expression of the analytical function to which the series (A1) converges. This result can be easily verified by using the equation of motion (A3).

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