Multiple-scattering expansion for $(e,2e)$ collisions in the presence of a laser field

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A multiple-scattering expansion is developed in this paper for the description of $(e,2e)$ collisions in the presence of a laser field. The dressed states of the incident electron, the scattered electron, the ejected electron, and the target atom due to the laser field are considered and then the scattering matrix is expanded as a series of multiple-scattering terms based on the time-dependent multiple-scattering theory, such as $S_{fi}=S_M^{(1)}+S_M^{(2)}+\cdots$. In the present approach, the effects of $S_M^{(1)}$ are almost equivalent to those of the first three orders of the Born expansion. Some aspects concerning the practical calculations of $S_M^{(1)}$ are also discussed at the end.

I. INTRODUCTION

$(e,2e)$ collisions in atoms and molecules in the absence of the laser field have greatly been developed during the last decade.[1-8] Recently, $(e,2e)$ collisions in a laser field have initially been discussed by Cavaliere *et al.*[9] and Joachain *et al.*[10] These authors' results show that some dramatic changes in the triply differential cross sections may occur because of the dressing effects of the target atom states due to the laser field.

The purpose of the present paper is to develop a theoretical method which possesses a wider valid region in both laser fields and energies of electron for the description of $(e,2e)$ collisions. The dressed states of the target atom are treated by an improved perturbation theory of multiphoton transitions. The scattering matrix is expanded as a series of multiple-scattering terms based on the time-dependent multiple-scattering theory, such as

$$S_{fi}=S_M^{(1)}+S_M^{(2)}+\cdots .$$

It is proved that the effects of $S_M^{(1)}$ are almost equivalent to those of the first three orders of the Born expansion. Finally, some aspects concerning the practical calculations of $S_M^{(1)}$ are also discussed.

II. THEORY

As a illustration, we only consider $(e,2e)$ collisions of atomic hydrogen in the present paper. In the present approach, the laser field is classically described and the effects of nuclear mass and spin and the spin-dependent electron-electron and electron-nucleus interactions are ignored for simplicity.

A. Hamiltonian of the system and scattering matrix

The total Hamiltonian of the e-H system in a laser field can be written as

$$H=H_0+V , \tag{2.1}$$

where

$$H_0=H_e+H_a . \tag{2.2}$$

H_e and H_a denote, respectively, the Hamiltonians of the incident (scattered) electron and of the target atom in the presence of the laser field and V is the e-H interaction. In detail, we have in atomic units and Coulomb gauge,

$$H_e=-\tfrac{1}{2}\nabla_1^2+\frac{1}{c}\mathbf{A}\cdot\mathbf{p}_1 , \tag{2.3}$$

$$H_a=-\tfrac{1}{2}\nabla_2^2+\frac{1}{c}\mathbf{A}\cdot\mathbf{p}_2+V_1(\mathbf{r}_2) , \tag{2.4}$$

$$V=V_1(\mathbf{r}_1)+V_2(\mathbf{r}) , \tag{2.5}$$

where

$$\mathbf{p}_1=-i\nabla_1 , \tag{2.6}$$

$$\mathbf{p}_2=-i\nabla_2 , \tag{2.7}$$

$$V_1(\mathbf{r})=-\frac{1}{r} , \tag{2.8}$$

$$V_2(\mathbf{r})=-\frac{1}{r} , \tag{2.9}$$

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2 . \tag{2.10}$$

In the above, $\mathbf{A}$ is the vector potential of the laser field and the terms with $\mathbf{A}^2$ have been ignored.

We assume that the laser field is a monochromatic plane wave, linearly polarized and the dipole approximation is valid. Thus the vector potential may be represented as

$$\mathbf{A}=\hat{\boldsymbol{\epsilon}}A_0\cos(\omega t) \tag{2.11}$$

or the electric vector is

$$\mathcal{E}=\hat{\boldsymbol{\epsilon}}\mathcal{E}_0\sin(\omega t) , \tag{2.12}$$

where $\hat{\boldsymbol{\epsilon}}$ is a unit polarization vector of the laser field and

$$\mathcal{E}_0=A_0\omega/c . \tag{2.13}$$

The initial and final states of the system, $\Psi_i^{(0)}$ and $\Psi_f^{(0)}$, are determined by

$$i\frac{\partial}{\partial t}\Psi_n^{(0)}=H_0\Psi_n^{(0)} \quad (n=i,f,\ldots) . \tag{2.14}$$

The complete wave function describing the collision process is the solution of the following Schrödinger equation:

$$i\frac{\partial}{\partial t}\Psi=H\Psi \ . \tag{2.15}$$

The scattering matrix from the initial state $\Psi_i^{(0)}$ to the final state $\Psi_f^{(0)}$ is yielded by the time-dependent scattering theory,

$$S_{fi}=-i\int_{-\infty}^{+\infty}dt\langle\Psi^{(-)}|V|\Psi_i^{(0)}\rangle \ , \tag{2.16}$$

where $\Psi^{(-)}$ is the solution of Eq. (2.15) which possesses the asymptotic conditions of a plane wave and ingoing waves.

B. Laser-dressed state

Given the properties of H_0, the initial state $\Psi_i^{(0)}$ may be separated into

$$\Psi_i^{(0)}(\mathbf{r}_1,\mathbf{r}_2,t)=\chi_{\mathbf{q}_i}(\mathbf{r}_1,t)\Phi_i(\mathbf{r}_2,t) \ , \tag{2.17}$$

where $\chi_{\mathbf{q}_i}$ is the Volkov solution of the incident electron in the laser field and satisfies the following equation:

$$i\frac{\partial}{\partial t}\chi_{\mathbf{q}_i}=H_e\chi_{\mathbf{q}_i} \ . \tag{2.18}$$

In detail, we have

$$\chi_{\mathbf{q}_i}(\mathbf{r}_i,t)=e^{-i\varepsilon_i t}e^{i[\mathbf{q}_i\cdot\mathbf{r}_1-\boldsymbol{\alpha}_0\cdot\mathbf{q}_i\sin(\omega t)]} \ , \tag{2.19}$$

where

$$\varepsilon_i=\frac{q_i^2}{2} \ , \tag{2.20}$$

$$\boldsymbol{\alpha}_0=\hat{\boldsymbol{\epsilon}}\frac{A_0}{c\omega}=\hat{\boldsymbol{\epsilon}}\frac{\sqrt{I}}{\omega^2} \ . \tag{2.21}$$

I is the intensity of the laser field (in a.u.).

Φ_i in (2.17) is the initial dressed state of the H atom determined by

$$i\frac{\partial}{\partial t}\Phi_i=H_a\Phi_i \ . \tag{2.22}$$

Clearly it is quite difficult to find the exact solutions of (2.22). According to Joachain *et al.*'s approach,[10] the approximate expression of Φ_i is obtained by the traditional first-order time-dependent perturbation theory. In particular, the dressed ground-state wave function of atomic hydrogen is expressed as

$$\Phi_0(\mathbf{r}_2,t)=e^{-iE_0t}e^{-i\mathbf{a}\cdot\mathbf{r}_2}\left[\phi_0(\mathbf{r}_2)+\frac{i}{2}\sum_n\left[\frac{e^{i\omega t}}{E_n-E_0+\omega}-\frac{e^{-i\omega t}}{E_n-E_0-\omega}\right]M_{n0}\phi_n(\mathbf{r}_2)\right] \ , \tag{2.23}$$

where

$$M_{n0}=\langle\phi_n|\mathcal{E}_0\cdot\mathbf{r}_2|\phi_0\rangle \ , \tag{2.24}$$

$$\mathbf{a}=\frac{1}{\omega}\mathcal{E}_0\cos(\omega t)\hat{\boldsymbol{\epsilon}} \ . \tag{2.25}$$

In the above, ϕ_n is a target state with energy E_n in the absence of the laser field; ω and $\mathcal{E}_0$ are, respectively, the frequency and the electric amplitude of the laser field.

Let us now analyze the properties of (2.23). Because of the limitation of the dipole selection rule, only p states can be mixed into the dressed state if the initial state ϕ_0 is an s state. It is true for a low-intensity laser. Otherwise, for a high-intensity laser, multiphoton exchange becomes quite important, thus more other states (s, d states and so on) will also enter into the dressed state. Therefore it seems that the traditional first-order time-dependent perturbation theory is not enough to describe the dressing of the target atom due to a strong laser field.

In the following, we use an improved time-dependent perturbation theory of multiphoton transitions[11] to get a new expression of Φ_i. The main points of this approach are as follows.

We first apply the Kramers-Henneberger transformation to (2.22) and obtain the Schrödinger equation in the Kramers picture,

$$i\frac{\partial}{\partial t}\Phi_i^{(K)}=H_a^{(K)}\Phi_i^{(K)} \ , \tag{2.26}$$

where

$$\Phi_i^{(K)}(\mathbf{r}_2,t)=e^{i\boldsymbol{\alpha}\cdot\mathbf{p}_2}\Phi_i(\mathbf{r}_2,t) \ , \tag{2.27}$$

$$H_a^{(K)}=H_a^0+\Delta V \ . \tag{2.28}$$

In the above, H_a^0 is the Hamiltonian of the H atom without lasers and

$$\Delta V=V_1(\mathbf{r}_2+\boldsymbol{\alpha})-V_1(\mathbf{r}_2) \ , \tag{2.29}$$

$$\boldsymbol{\alpha}=\boldsymbol{\alpha}_0\sin(\omega t) \ . \tag{2.30}$$

In the present case, ΔV is a periodic function of t. We Fourier analyze it,

$$\Delta V=\sum_{l=-\infty}^{+\infty}v^{(l)}(\mathbf{r}_2)e^{-il\omega t} \ . \tag{2.31}$$

The Fourier components can be written as[12]

$$v^{(l)}=\frac{i^l}{\pi}\int_{-1}^{+1}ds\{[V_1(\mathbf{r}_2+\boldsymbol{\alpha}_0 s)-V_1(\mathbf{r}_2)] \times T_l(s)(1-s^2)^{-1/2}\} \ , \tag{2.32}$$

where $T_l(s)$ are Chebyshev polynomials. We also have

$$v^{(l)}=\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega}\Delta V e^{il\omega t}dt \ . \tag{2.33}$$

In the traditional perturbation theory of laser-atom interactions, one considers $H_1=\mathcal{E}\cdot\mathbf{r}_2$ as a perturbation.

However, in some cases, we prefer to consider ΔV as a perturbation.[11] Thus, by the standard perturbation method, we obtain

$$\Phi_i=\Phi_i^{(1)}+\Phi_i^{(2)}+\cdots , \tag{2.34}$$

where

$$\Phi_i^{(1)}=e^{-iE_it}e^{-i\alpha\cdot p_2}\phi_i(\mathbf{r}_2) , \tag{2.35}$$

$$\Phi_i^{(2)}=\sum_{n,l} C_{nl}^{(i)}e^{-i\alpha\cdot p_2}\phi_n(\mathbf{r}_2)e^{-il\omega t}e^{-iE_it} , \tag{2.36}$$

where

$$C_{nl}^{(i)}=\frac{v_{ni}^{(l)}}{E_i-E_n+l\omega} , \tag{2.37}$$

$$v_{ni}^{(l)}=\langle \phi_n|v^{(l)}|\phi_i\rangle . \tag{2.38}$$

E and ϕ are, respectively, the energies and the wave functions of atomic hydrogen in the absence of the laser field. The summation over n extends to all the discrete and continuum states of the H atom, except $n=i$.

It is of interest to analyze the properties of (2.34)–(2.38) and compare them with (2.23)–(2.25). On the one hand, we can prove that the present dressed wave functions are essentially in agreement with those obtained by the traditional first-order time-dependent perturbation theory under the weak-field limit. For the weak laser, we may make three approximations as follows.

1. Approximation 1

Only one-photon exchange is important. It means that we may only keep the terms with $l=\pm 1$ in (2.36), thus

$$\Phi_0=e^{-iE_0t}e^{-i\alpha\cdot p_2}\left[\phi_0(\mathbf{r}_2)-\sum_n\left[\frac{e^{i\omega t}}{E_n-E_0+\omega}v_{n0}^{(-1)}-\frac{e^{-i\omega t}}{E_n-E_0-\omega}v_{n0}^{(+1)}\right]\phi_n(\mathbf{r}_2)\right] . \tag{2.39}$$

2. Approximation 2

ΔV can be expanded as a Taylor series up to the first two terms,

$$\Delta V\simeq(\boldsymbol{\alpha}\cdot\nabla_2)V_1(\mathbf{r}_2)=\sin(\omega t)(\boldsymbol{\alpha}_0\cdot\nabla_2)V_1(\mathbf{r}_2) . \tag{2.40}$$

Substituting (2.40) into (2.33) and performing the integration for t, we have

$$v^{(\pm1)}=\pm\frac{i}{2}(\boldsymbol{\alpha}_0\cdot\nabla_2)V_1(\mathbf{r}_2) = \mp\tfrac{1}{2}(\boldsymbol{\alpha}_0\cdot\mathbf{p}_2)V_1(\mathbf{r}_2) . \tag{2.41}$$

Thus

$$v_{n0}^{(\pm1)}=\mp\tfrac{1}{2}\sum_m \boldsymbol{\alpha}_0\cdot\langle\phi_n|\mathbf{p}_2|\phi_m\rangle\langle\phi_m|V_1(\mathbf{r}_2)|\phi_0\rangle . \tag{2.42}$$

Since $\langle\phi_0|V_1|\phi_0\rangle$ is the maximum among all the $\langle\phi_m|V_1|\phi_0\rangle$, we obtain, approximately,

$$v_{n0}^{(\pm1)}\approx\mp\tfrac{1}{2}\boldsymbol{\alpha}_0\cdot\langle\phi_n|\mathbf{p}_2|\phi_0\rangle\langle\phi_0|V_1|\phi_0\rangle . \tag{2.43}$$

Using the Virial theorem and the relation of matrix elements of $\mathbf{r}_2$ and $\mathbf{p}_2$, (2.43) becomes

$$v_{n0}^{(\pm1)}\approx\pm i\eta\langle\phi_n|\mathcal{E}_0\cdot\mathbf{r}_2|\phi_0\rangle , \tag{2.44}$$

where

$$\eta=\left|\frac{(E_n-E_0)E_0}{\omega^2}\right| . \tag{2.45}$$

For the H atom, $E_0-E_n\sim E_0$, and $\omega\sim E_0$ for single-photon exchange, hence, in magnitude,

$$\eta\sim 1 . \tag{2.46}$$

3. Approximation 3

We have

$$e^{-i\alpha\cdot p_2}\approx 1 . \tag{2.47}$$

We sum up (2.44) and (2.47) and obtain

$$\Phi_0=e^{-iE_0t}\left[\phi_0(\mathbf{r}_2)+i\eta\sum_n\left[\frac{e^{i\omega t}}{E_n-E_0+\omega}-\frac{e^{-i\omega t}}{E_n-E_0-\omega}\right]M_{n0}\phi_n(\mathbf{r}_2)\right] . \tag{2.48}$$

Obviously, (2.48) is essentially in agreement with (2.23) because the factor $\exp(-i\mathbf{a}\cdot\mathbf{r}_2)$ in (2.23) is no contribution to the S-matrix elements.

On the other hand, for the strong laser, both dressed wave functions expressed respectively by (2.23) and (2.34) in the above are quite different from each other. We would like to emphasize three points as follows.

Firstly, either $\Phi_i^{(1)}$ expressed by (2.35) or $\Phi_i^{(2)}$ expressed by (2.36) already contains all the nonperturbation intermediate state (not only p states) through one- and multiple-photon exchange.

Secondly, the first and the second terms of (2.34) expressed respectively by (2.35) and (2.36) describe respectively multiphoton transitions with or without multiphoton intermediate resonances.

Thirdly, for the absence of multiphoton intermediate resonances, the second term is smaller than the first one in (2.34). This fact leads to the simplicity of practical calculations. Otherwise, we must carefully consider the terms with intermediate resonances.

From the above analysis, it seems that the present dressed wave functions are more general than those obtained by the traditional first-order time-dependent perturbation theory.

Similarly, we can get the final state of the system $\Psi_f^{(0)}$ as follows:

$$\Psi_f^{(0)}=\chi_{\mathbf{q}_A}(\mathbf{r}_1,t)\Phi_f(\mathbf{r}_2,t) , \quad (2.49)$$

where $\chi_{\mathbf{q}_A}$ is the dressed state of the scattered electron with momentum $\mathbf{q}_A$ due to the laser field and is determined by (2.18). We have, in detail,

$$\chi_{\mathbf{q}_A}(\mathbf{r}_1,t)=e^{-i\varepsilon_A t}e^{i[\mathbf{q}_A\cdot\mathbf{r}_1-\alpha_0\cdot\mathbf{q}_A\sin(\omega t)]} , \quad (2.50)$$

where

$$\varepsilon_A=q_A^2/2 . \quad (2.51)$$

Φ_f in (2.49) is the final dressed state of the target atom and is determined by (2.22). Here we must note that the final state of the target atom in $(e,2e)$ collisions of the H atom is a continuum state, that is a Coulomb scattering wave function. Similar to Φ_i, we obtain

$$\Phi_f=\Phi_f^{(1)}+\Phi_f^{(2)}+\cdots , \quad (2.52)$$

where

$$\Phi_f^{(1)}=e^{-i\varepsilon_B t}e^{-i\alpha\cdot\mathbf{p}_2}\phi_{\mathbf{q}_B}(\mathbf{r}_2) , \quad (2.53)$$

$$\Phi_f^{(2)}=e^{-i\varepsilon_B t}\sum_{n,l}C_{nl}^{(f)}e^{-i\alpha\cdot\mathbf{p}_2}\phi_n(\mathbf{r}_2)e^{-il\omega t} , \quad (2.54)$$

and

$$\varepsilon_B=q_B^2/2 , \quad (2.55)$$

$$\phi_{\mathbf{q}_B}(\mathbf{r}_2)=e^{\pi/2q_B}\Gamma\left[1+\frac{i}{q_B}\right]e^{i\mathbf{q}_B\cdot\mathbf{r}_2}\times F\left[-\frac{i}{q_B},1,-i(q_B r_2-\mathbf{q}_B\cdot\mathbf{r}_2)\right] . \quad (2.56)$$

In the above, the notations are the same with the case of Φ_i.

C. Time-dependent multiple-scattering theory

In this section, the time-independent multiple-scattering theory proposed by Dewangan[13] is extended to the time-dependent case. According to Dewangan, let us define two distorted waves by

$$\langle\xi_1^{(-)}|=\langle\Psi_f^{(0)}|+\langle\xi_1^{(-)}|V_1G_0 , \quad (2.57)$$

$$\langle\xi_2^{(-)}|=\langle\Psi_f^{(0)}|+\langle\xi_2^{(-)}|V_2G_0 , \quad (2.58)$$

where

$$G_0=\left[i\frac{\partial}{\partial t}-H_0+i\epsilon\right]^{-1} . \quad (2.59)$$

Thus[14]

$$\langle\Psi^{(-)}|=\langle\Psi_f^{(0)}|+\langle\Psi^{(-)}|VG_0 \quad (2.60)$$

$$=\langle\xi_1^{(-)}|+\langle\Psi^{(-)}|V_2G_1 \quad (2.61)$$

$$=\langle\xi_2^{(-)}|+\langle\Psi^{(-)}|V_1G_2 , \quad (2.62)$$

where

$$G_1=\left[i\frac{\partial}{\partial t}-H_0-V_1+i\epsilon\right]^{-1} , \quad (2.63)$$

$$G_2=\left[i\frac{\partial}{\partial t}-H_0-V_2+i\epsilon\right]^{-1} . \quad (2.64)$$

The equivalent equations of $\xi_1^{(-)}$ and $\xi_2^{(-)}$ are in the following

$$i\frac{\partial}{\partial t}\xi_1^{(-)}=(H_e+V_1+H_a)\xi^{(-)} , \quad (2.65)$$

$$i\frac{\partial}{\partial t}\xi_2^{(-)}=(H_e+V_2+H_a)\xi_2^{(-)} , \quad (2.66)$$

with the corresponding asymptotic conditions.

Since (H_e+V_1) and H_a are respectively functions of $\mathbf{r}_1$ and $\mathbf{r}_2$, $\xi_1^{(-)}$ can be separated into

$$\xi_1^{(-)}=\eta_1^{(-)}(\mathbf{r}_1,t)\Phi_f(\mathbf{r}_2,t) , \quad (2.67)$$

where Φ_f has been defined by (2.52) and $\eta_1^{(-)}$ is determined by

$$i\frac{\partial}{\partial t}\eta_1^{(-)}=[H_e+V_1(\mathbf{r}_1)]\eta_1^{(-)} , \quad (2.68)$$

with the corresponding asymptotic conditions.

For similar reasons, we also have

$$\xi_2^{(-)}=\eta_2^{(-)}(\mathbf{r},t)\Phi_f(\mathbf{r}_2,t) , \quad (2.69)$$

where $\eta_2^{(-)}$ is determined by

$$i\frac{\partial}{\partial t}\eta_2^{(-)}=\left[-\tfrac{1}{2}\nabla^2-\frac{i}{c}\mathbf{A}\cdot\nabla+V_2(\mathbf{r})\right]\eta_2^{(-)} , \quad (2.70)$$

with the corresponding asymptotic conditions similar to $\xi_2^{(-)}$.

From (2.58) and (2.62), we have

$$\langle\Psi_f^{(0)}|=\langle\xi_2^{(-)}|-\langle\xi_2^{(-)}|V_2G_0 =\langle\xi_2^{(-)}|-\langle\Psi^{(-)}|V_2G_0+\langle\Psi^{(-)}|V_1G_2V_2G_0 . \quad (2.71)$$

Since $G_2V_2G_0=G_2-G_0$, (2.71) becomes

$$\langle\Psi_f^{(0)}|=\langle\xi_2^{(-)}|-\langle\Psi^{(-)}|VG_0+\langle\Psi^{(-)}|V_1G_2 . \quad (2.72)$$

It is clear from (2.50) that we always have

$$\chi_{\mathbf{q}_A}^*\chi_{\mathbf{q}_A}=\chi_{\mathbf{q}_A}^{-1}\chi_{\mathbf{q}_A}=1 . \quad (2.73)$$

Hence $\langle\xi_1^{(-)}|$ can be written as

$$\langle\xi_1^{(-)}|=\langle\eta_1^{(-)}\chi_{\mathbf{q}_A}^{-1}\chi_{\mathbf{q}_A}\Phi_f| =\langle\chi_{\mathbf{q}_A}^{-1}\eta_1^{(-)}\Psi_f^{(0)}| . \quad (2.74)$$

Substituting (2.74) and (2.72) into (2.61), and using the iteration procedure, we obtain

$$\langle\Psi^{(-)}|=\langle\Psi_{M1}^{(-)}|+\langle\Psi_{M2}^{(-)}|+\cdots , \quad (2.75)$$

where $\langle \Psi_{M1}^{(-)}|$ and $\langle \psi_{M2}^{(-)}|$ are defined as

$$\langle \Psi_{M1}^{(-)}| = \langle \chi_{\mathbf{q}_A}^{-1} \eta_1^{(-)} \xi_2^{(-)}| , \quad (2.76)$$

$$\langle \Psi_{M2}^{(-)}| = \langle \xi_1^{(-)}| V_2 G_1 + \langle \chi_{\mathbf{q}_A}^{-1} \eta_1^{(-)}| \langle \xi_2^{(-)}| V_1 G_2 - \langle \chi_{\mathbf{q}_A}^{-1} \eta_1^{(-)}| \langle \Psi_f^{(0)}| V G_0 . \quad (2.77)$$

D. Multiple-scattering expansion of the scattering matrix

Substituting (2.76) and (2.77) in (2.16), we obtain the multiple-scattering expansion of the scattering matrix, such as

$$S_{fi} = S_M^{(1)} + S_M^{(2)} + \cdots , \quad (2.78)$$

where

$$S_M^{(1)} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_{M1}^{(-)}| V | \Psi_i^{(0)} \rangle , \quad (2.79)$$

$$S_M^{(2)} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_{M2}| V | \Psi_i^{(0)} \rangle . \quad (2.80)$$

It is of interest to analyze the properties of (2.79) and compare it with the Born expansion. From (2.60), we obtain directly

$$S_{fi} = S_B^{(1)} + S_B^{(2)} + S_B^{(3)} + \cdots , \quad (2.81)$$

where

$$S_B^{(1)} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_f^{(0)}| V | \Psi_i^{(0)} \rangle , \quad (2.82)$$

$$S_B^{(2)} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_f^{(0)}| V G_0 V | \Psi_i^{(0)} \rangle , \quad (2.83)$$

$$S_B^{(3)} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_f^{(0)}| V G_0 V G_0 V | \Psi_i^{(0)} \rangle = S_B^{(3)a} + S_B^{(3)b} , \quad (2.84)$$

and

$$S_B^{(3)a} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_f^{(0)}| V (G_0 V_1 G_0 V_1 + G_0 V_2 G_0 V_2) | \Psi_i^{(0)} \rangle , \quad (2.85)$$

$$S_B^{(3)b} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_f^{(0)}| V (G_0 V_1 G_0 V_2 + G_0 V_2 G_0 V_1) | \Psi_i^{(0)} \rangle . \quad (2.86)$$

On the other hand, $S_M^{(1)}$ can be transformed as

$$S_M^{(1)} = -i \int_{-\infty}^{+\infty} dt \langle (\Psi_f^{(0)})^{-1} \xi_1^{(-)} \xi_2^{(-)}| V | \Psi_i^{(0)} \rangle . \quad (2.87)$$

Using (2.57) and (2.58), we show easily that

$$S_M^{(1)} = S_B^{(1)} + S_B^{(2)} + (S_B^{(3)a} + \tilde{S}_B^{(3)b}) + \cdots , \quad (2.88)$$

where $S_B^{(1)}$, $S_B^{(2)}$, and $S_B^{(3)a}$ have been defined by (2.82), (2.83), and (2.85). The expression of $\tilde{S}_B^{(3)b}$ is

$$\tilde{S}_B^{(3)b} = -i \int_{-\infty}^{+\infty} dt \langle \Psi_3 | V | \Psi_i^{(0)} \rangle , \quad (2.89)$$

where

$$\langle \Psi_3 | = [\langle (\Psi_f^{(0)})^{-1} |](\langle \Psi_f^{(0)} | V_1 G_0)(\langle \Psi_f^{(0)} | V_2 G_0) . \quad (2.90)$$

Therefore it is concluded that the contributions of $S_M^{(1)}$ are almost equivalent to the total one of the first three orders of the Born expansion.

For practical calculations, $\langle \Psi_{M1}^{(-)}|$ can also be deformed as

$$\langle \Psi_{M1}^{(-)}| = \langle \chi_{\mathbf{q}_A} \Phi_f \eta_1^{(-)} \eta_2^{(-)}| . \quad (2.91)$$

E. Approximate solutions of the distorted-wave equation

The distorted wave $\eta_1^{(-)}$ is determined by (2.68). It is difficult to find its exact expression. Let us put

$$\eta_1^{(-)} = \chi_{\mathbf{q}_A}(\mathbf{r}_1, t) f_1(\mathbf{r}_1, t) , \quad (2.92)$$

where $\chi_{\mathbf{q}_A}$ has been defined by (2.50) and f_1 satisfies the following equation with corresponding asymptotic conditions:

$$i \frac{\partial}{\partial t} f_1 = \left[-\tfrac{1}{2}\nabla_1^2 + V_1(\mathbf{r}_1) - i \left[\mathbf{q}_A + \frac{1}{c}\mathbf{A} \right] \cdot \nabla_1 \right] f_1 . \quad (2.93)$$

Usually, as shown in Table I,

$$|\mathbf{q}_A| \gg \left| \frac{1}{c}\mathbf{A} \right| . \quad (2.94)$$

Hence (2.93) is reduced to[15]

$$i \frac{\partial}{\partial t} f_1 = [-\tfrac{1}{2}\nabla_1^2 - i \mathbf{q}_A \cdot \nabla_1 + V_1(\mathbf{r}_1)] f_1 . \quad (2.95)$$

We can exactly solve (2.95) and obtain[16]

$$\eta_1^{(-)} = \chi_{\mathbf{q}_A}(\mathbf{r}_1, t) e^{\pi/2q_A} \Gamma \left[1 + \frac{i}{q_A} \right] \times F \left[-\frac{i}{q_A}, 1, -i(q_A r_1 + \mathbf{q}_A \cdot \mathbf{r}_1) \right] . \quad (2.96)$$

The distorted wave $\eta_2^{(-)}$ is determined by (2.70). Using similar treatment, we have

$$\eta_2^{(-)} = \chi_{\mathbf{q}_A}(\mathbf{r}_1, t) e^{-\pi/2q_A} \Gamma \left[1 - \frac{1}{q_A} \right] \times F \left[\frac{i}{q_A}, 1, -i(q_A r + \mathbf{q}_A \cdot \mathbf{r}) \right] . \quad (2.97)$$

Substituting (2.96) and (2.97) into (2.91), we obtain

$$\Psi_{M1}^{(-)} = \Phi_f(\mathbf{r}_2, t) \chi_{\mathbf{q}_A}(\mathbf{r}_1, t) \frac{(\pi/q_A)}{\sinh(\pi/q_A)} F \left[-\frac{i}{q_A}, 1, -i(q_A r_1 + \mathbf{q}_A \cdot \mathbf{r}_1) \right] F \left[\frac{i}{q_A}, 1, -i(q_A r + \mathbf{q}_A \cdot \mathbf{r}) \right] . \quad (2.98)$$

It is clear from (2.98) that the approximate complete wave function of the system, $\Psi_{M1}^{(-)}$, contains the space correla-

TABLE I. Comparisons of q_i with $(1/c)A_0$ in eV/c.

Scattered electron ε_i (eV)	q_i (eV/c)	I (W/cm^2)	Wavelength of Laser (Å) 10 600	3080	2430
10	3200	1.3×10^{9}	5.57×10^{-2}	4.7×10^{-3}	2.9×10^{-3}
40	6400	1.3×10^{11}	5.57×10^{-1}	4.7×10^{-2}	2.9×10^{-2}
100	10 000	1.3×10^{13}	5.57	4.7×10^{-1}	2.9×10^{-1}
200	14 000	1.3×10^{15}	5.57×10^{1}	4.7	2.9

tions between the scattered and the ejected electrons. This may be one of the advantages of the present expression. Let us now examine some special cases. In the absence of the laser field, $\Psi_{M1}^{(-)}$ becomes

$$\Psi_{M1}^{(-)}=[e^{-i\varepsilon_B t}\phi_{\mathbf{q}_B}(\mathbf{r}_2)][e^{-i\varepsilon_A t}\phi_{\mathbf{q}_A}(\mathbf{r}_1)]\left[e^{-\pi/2q_A}\Gamma\left[1-\frac{i}{q_A}\right]F\left[\frac{i}{q_A},1,-i(q_A r+\mathbf{q}_A\cdot\mathbf{r})\right]\right], \tag{2.99}$$

where $\phi_{\mathbf{q}_A}(\mathbf{r}_1)$ and $\phi_{\mathbf{q}_B}(\mathbf{r}_2)$ denote, respectively, the Coulomb scattering wave functions of the scattered and the ejected electrons by the nucleus of the atomic hydrogen alone. The large square brackets describe the correlations between the scattered and the ejected electrons in the coordinate space.

If we neglect the interaction between the scattered and the ejected electrons, that is, put $V_2=0$, the space correlations will be absent, and (2.99) becomes

$$\Psi_{M1}^{(-)}=[e^{-i\varepsilon_B t}\phi_{\mathbf{q}_B}(\mathbf{r}_2)][e^{-i\varepsilon_A t}\phi_{\mathbf{q}_A}(\mathbf{r}_1)] . \tag{2.100}$$

This is just a usual form of the distorted-wave Born approximation.

If we further ignore the interaction of electron-nucleus, (2.100) is simplified as the plane-wave Born approximation,

$$\Psi_{M1}^{(-)}=(e^{-i\varepsilon_B t}e^{i\mathbf{q}_B\cdot\mathbf{r}_2})(e^{-i\varepsilon_A t}e^{i\mathbf{q}_A\cdot\mathbf{r}_1}) . \tag{2.101}$$

F. Collision cross section

Since momentum operators appear in (2.35), (2.36), (2.53), and (2.54), a convenient treatment is as follows. Let us rewrite,

$$\Phi_i^{(1)}=e^{-iE_i t}\int d\mathbf{q}_1 e^{-i\alpha_0\cdot\mathbf{q}_1\sin(\omega t)}\tilde{\phi}_i(\mathbf{q}_1)|\mathbf{q}_1\rangle , \tag{2.102}$$

$$\Phi_i^{(2)}=\sum_{n_1,l_1} C_{n_1,l_1}^{(i)} e^{-i(E_i+l_1\omega)t}\int d\mathbf{q}_1 e^{-i\alpha_0\cdot\mathbf{q}_1\sin(\omega t)}\tilde{\phi}_{n_1}(\mathbf{q}_1)|\mathbf{q}_1\rangle , \tag{2.103}$$

$$\Phi_f^{(1)}=e^{-i\varepsilon_B t}\int d\mathbf{q}_2 e^{-i\alpha_0\cdot\mathbf{q}_2\sin(\omega t)}\tilde{\phi}_{\mathbf{q}_B}(\mathbf{q}_2)|\mathbf{q}_2\rangle , \tag{2.104}$$

$$\Phi_f^{(2)}=\sum_{n_2,l_2} C_{n_2 l_2}^{(f)} e^{-i(\varepsilon_B+l_2\omega)t}\int d\mathbf{q}_2 e^{-i\alpha_0\cdot\mathbf{q}_2\sin(\omega t)}\tilde{\phi}_{n_2}(\mathbf{q}_2)|\mathbf{q}_2\rangle , \tag{2.105}$$

where

$$|\mathbf{q}\rangle=(2\pi)^{-3/2}e^{i\mathbf{q}\cdot\mathbf{r}_2} . \tag{2.106}$$

$\tilde{\phi}_i(\mathbf{q})$, $\tilde{\phi}_n(\mathbf{q})$, and $\tilde{\phi}_{\mathbf{q}_B}(\mathbf{q})$ are, respectively, the momentum representations of $\phi_i(\mathbf{r}_2)$, $\phi_n(\mathbf{r}_2)$, and $\phi_{\mathbf{q}_B}(\mathbf{r}_2)$.

Let us now rewrite,

$$\Psi_{M1}^{(-)}=\Phi_f(\mathbf{r}_2,t)\chi_{\mathbf{q}_A}(\mathbf{r}_1,t)f(\mathbf{r}_1,\mathbf{r}_2) , \tag{2.107}$$

where

$$f(\mathbf{r}_1,\mathbf{r}_2)=\frac{(\pi/q_A)}{\sinh(\pi/q_A)}F\left[-\frac{i}{q_A},1,-i(q_A r_1+\mathbf{q}_A\cdot\mathbf{r}_1)\right]F\left[\frac{i}{q_A},1,-i(q_A r+\mathbf{q}_A\cdot\mathbf{r})\right] . \tag{2.108}$$

Substituting (2.107) into (2.16), we have

$$S_{fi}=S_1+S_2+S_3+S_4 , \tag{2.109}$$

where

$$S_1=-i\int_{-\infty}^{+\infty}dt\langle\chi_{\mathbf{q}_A}\Phi_f^{(1)}f(\mathbf{r}_1,\mathbf{r}_2)|V|\chi_{\mathbf{q}_i}\Phi_i^{(1)}\rangle\ , \tag{2.110}$$

$$S_2=-i\int_{-\infty}^{+\infty}dt\langle\chi_{\mathbf{q}_A}\Phi_f^{(2)}f(\mathbf{r}_1,\mathbf{r}_2)|V|\chi_{\mathbf{q}_i}\Phi_i^{(1)}\rangle\ , \tag{2.111}$$

$$S_3=-i\int_{-\infty}^{+\infty}dt\langle\chi_{\mathbf{q}_A}\Phi_f^{(1)}f(\mathbf{r}_1,\mathbf{r}_2)|V|\chi_{\mathbf{q}_i}\Phi_i^{(2)}\rangle\ , \tag{2.112}$$

$$S_4=-i\int_{-\infty}^{+\infty}dt\langle\chi_{\mathbf{q}_A}\Phi_f^{(2)}f(\mathbf{r}_1,r_2)|V|\chi_{\mathbf{q}_i}\Phi_i^{(2)}\rangle\ . \tag{2.113}$$

After some manipulation, we obtain

$$S_1=-2\pi i\sum_N\delta(\varepsilon_i+E_i-\varepsilon_A-\varepsilon_B-N\omega)T_1(N)\ , \tag{2.114}$$

$$S_2=-2\pi i\sum_N\delta(\varepsilon_i+E_i-\varepsilon_A-\varepsilon_B-N\omega)T_2(N)\ , \tag{2.115}$$

$$S_3=-2\pi i\sum_N\delta(\varepsilon_i+E_i-\varepsilon_A-\varepsilon_B-N\omega)T_3(N)\ , \tag{2.116}$$

$$S_4=-2\pi i\sum_N\delta(\varepsilon_i+E_i-\varepsilon_A-\varepsilon_B-N\omega)T_4(N)\ , \tag{2.117}$$

where

$$T_1(N)=\int d\mathbf{q}_1 d\mathbf{q}_2[\tilde{\phi}^*_{\mathbf{q}_B}(\mathbf{q}_2)\tilde{\phi}_i(\mathbf{q}_1)g(\mathbf{q}_1,\mathbf{q}_2)J_N(\boldsymbol{\alpha}_0\cdot(\mathbf{q}_A-\mathbf{q}_i+\mathbf{q}_2-\mathbf{q}_1))]\ , \tag{2.118}$$

$$g(\mathbf{q}_1,\mathbf{q}_2)=(2\pi)^{-3}\langle e^{i(\mathbf{q}_2-\mathbf{q}_1)\cdot\mathbf{r}_2}f(\mathbf{r}_1,\mathbf{r}_2)|V|e^{i(\mathbf{q}_i-\mathbf{q}_A)\cdot\mathbf{r}_1}\rangle\ , \tag{2.119}$$

$$T_2(N)=\sum_{n_2,l_2}C^{(f)*}_{n_2l_2}\int d\mathbf{q}_1 d\mathbf{q}_2[\tilde{\phi}^*_{n_2}(\mathbf{q}_2)\tilde{\phi}_i(\mathbf{q}_1)g(\mathbf{q}_1,\mathbf{q}_2)J_{N-l_2}(\boldsymbol{\alpha}_0\cdot(\mathbf{q}_A-\mathbf{q}_i+\mathbf{q}_2-\mathbf{q}_1))]\ , \tag{2.120}$$

$$T_3(N)=\sum_{n_1,l_1}C^{(i)}_{n_1l_1}\int d\mathbf{q}_1 d\mathbf{q}_2[\tilde{\phi}^*_{\mathbf{q}_B}(\mathbf{q}_2)\tilde{\phi}_{n_1}(\mathbf{q}_1)g(\mathbf{q}_1,\mathbf{q}_2)J_{N+l_1}(\boldsymbol{\alpha}_0\cdot(\mathbf{q}_A-\mathbf{q}_i+\mathbf{q}_2-\mathbf{q}_1))]\ , \tag{2.121}$$

$$T_4(N)=\sum_{n_1,l_1}\sum_{n_2,l_2}C^{(i)}_{n_1l_1}C^{(f)*}_{n_2l_2}\int d\mathbf{q}_1 d\mathbf{q}_2[\tilde{\phi}^*_{n_2}(\mathbf{q}_2)\tilde{\phi}_{n_1}(\mathbf{q}_1)g(\mathbf{q}_1,\mathbf{q}_2)J_{N+l_1-l_2}(\boldsymbol{\alpha}_0\cdot(\mathbf{q}_A-\mathbf{q}_i+\mathbf{q}_2-\mathbf{q}_1))]\ . \tag{2.122}$$

In the above, $J_l(x)$ is a Bessel function of the first kind.

So that, we have

$$S_{fi}=-2\pi i\sum_N\delta(\varepsilon_i+E_i-\varepsilon_A-\varepsilon_B-N\omega)T(N)\ , \tag{2.123}$$

where

$$T(N)=\sum_{i=1}^{4}T_i(N)\ . \tag{2.124}$$

The individual triply differential cross sections, that is, the cross sections for $(e,2e)$ collisions of the H atom while N photons are absorbed ($N<0$) or emitted ($N<0$) are

$$\frac{d^3\sigma(N)}{d\Omega_A d\Omega_B d\varepsilon_B}=(2\pi)^{-5}\frac{q_Aq_B}{q_i}|T(N)|^2\ . \tag{2.125}$$

The total triply differential cross sections may be written as

$$\frac{d^3\sigma}{d\Omega_A d\Omega_B d\varepsilon_B}=\sum_N\frac{d^3\sigma(N)}{d\Omega_A d\Omega_B d\varepsilon_B}\ . \tag{2.126}$$

III. DISCUSSION

A general description for $(e,2e)$ collisions of the H atom in a laser field has been obtained by the combinations of the time-dependent multiple-scattering theory and the improved perturbation theory of multiphoton transitions.

The space correlations between the scattered and the ejected electrons have been taken into account in the present formalism. In particular, the contributions of the first-order multiple-scattering expansion are almost equivalent to the total one of the first three orders of the Born expansion. It means that the present method possesses high accuracy.

The present description may be applied to other precesses, for examples, electron scattering by the H atom in a laser field,

$$e^-+\mathrm{H}(1s)+N\omega\rightarrow e^-+\mathrm{H}(nl)$$

and laser-assisted charge exchange,

$$\mathrm{H}^++\mathrm{D}+N\omega\rightarrow\mathrm{H}+\mathrm{D}^+$$

and laser-assisted antihydrogen-atom formation in collisions of antiprotons with positronium,

$$\bar{p}+\mathrm{Ps}+N\omega\rightarrow\overline{\mathrm{H}}+e^-\ .$$

In the following, let us discuss, in principle, some aspects concerning the practical calculations of $S_M^{(1)}$.

Firstly, we must know the momentum representations of the wave functions of the H atom including discrete and continuum states. As is well known, for the bound states of the H atom, the momentum representations can analytically be expressed. As to the continuum states, the momentum representations can also be written as

$$\tilde{\phi}_{q_B}(\mathbf{q})=\sum_l R_l(q)Y_{l0}(\theta_q) , \tag{3.1}$$

where θ_q is the angle between $\mathbf{q}_B$ and $\mathbf{q}$, and $R_l(q)$ may be expressed by an analytical formula.[17]

Secondly, we must also know how to calculate $v_{ni}^{(l)}$. Generally, $v^{(l)}$ may be written as

$$v^{(l)}=\frac{1}{r_2}\delta_{l0}-\frac{1}{2\pi^2}\int dq\, d\Omega_q J_l(\boldsymbol{\alpha}_0\cdot\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}_2} . \tag{3.2}$$

Hence

$$v_{ni}^{(l)}=\left\langle\phi_n\left|\frac{1}{r_2}\right|\phi_i\right\rangle\delta_{l0}-\frac{1}{2\pi^2}\int dq\, d\Omega_q J_l(\boldsymbol{\alpha}_0\cdot\mathbf{q})\langle\phi_n|e^{i\mathbf{q}\cdot\mathbf{r}_2}|\phi_i\rangle . \tag{3.3}$$

Usually, $\langle\phi_n|e^{i\mathbf{q}\cdot\mathbf{r}_2}|\phi_i\rangle$ may be reduced to one-dimensional integrals.

Thirdly, $g(\mathbf{q}_1,\mathbf{q}_2)$ in (2.119) only is a function of $\mathbf{q}=\mathbf{q}_1-\mathbf{q}_2$ and may be expressed by

$$g(\mathbf{q}_1,\mathbf{q}_2)=g(\mathbf{q})=I_a-I_b , \tag{3.4}$$

where

$$I_a=\frac{\pi\gamma}{(2\pi)^3\sinh(\pi\gamma)}\left[\int d\mathbf{r}_1[e^{i\mathbf{q}\cdot\mathbf{r}_1}e^{i\mathbf{Q}\cdot\mathbf{r}_1}F^*(-i\gamma,1,-i(q_Ar_1+\mathbf{q}_A\cdot\mathbf{r}_1))]\right]\left[\int d\mathbf{r}\left[\frac{1}{r}e^{-i\mathbf{q}\cdot\mathbf{r}}F^*(i\gamma,1,-i(q_Ar+\mathbf{q}_A\cdot\mathbf{r}))\right]\right] , \tag{3.5}$$

$$I_b=\frac{\pi\gamma}{(2\pi)^3\sinh(\pi\gamma)}\left[\int d\mathbf{r}_1\left[\frac{1}{r_1}e^{i\mathbf{q}\cdot\mathbf{r}_1}e^{i\mathbf{Q}\cdot\mathbf{r}_1}F^*(-i\gamma,1,-i(\mathbf{q}_Ar_1+\mathbf{q}_A\cdot\mathbf{r}_1))\right]\right]\left[\int d\mathbf{r}[e^{-i\mathbf{q}\cdot\mathbf{r}}F^*(i\gamma,1,-i(q_Ar+\mathbf{q}_A\cdot\mathbf{r}))]\right], \tag{3.6}$$

and

$$\gamma=1/q_A,\quad \mathbf{Q}=\mathbf{q}_i-\mathbf{q}_A . \tag{3.7}$$

Using some expansions, such as

$$e^{i\mathbf{q}\cdot\mathbf{r}}=4\pi\sum_{l,m}i^l j_l(q_r)Y_{lm}^*(\hat{\mathbf{q}})Y_{lm}(\hat{\mathbf{r}}) \tag{3.8}$$

$$F(i\gamma,1,-i(q_Ar+\mathbf{q}_A\cdot\mathbf{r}))=\sum_{l,m}\bar{R}_l(\gamma,r)Y_{lm}^*(\hat{\mathbf{q}}_A)Y_{lm}(\hat{\mathbf{r}}) , \tag{3.9}$$

$\bar{R}l(\gamma,r)$ can analytically be expressed as a function of r. Hence I_a and I_b are reduced to some one-dimensional integrals, such as

$$I_a\to\sum_{\substack{l_1,l_2,l_3,l_4\\ m_1,m_4}}\left[\int_0^\infty dr_1[r_1^2j_{l_1}(qr_1)j_{l_2}(Qr_1)\bar{R}_{l_3}^*(-\gamma,r_1)]\right]Y_{l_1m_1}^*(\hat{\mathbf{q}})Y_{l_4m_4}(\hat{\mathbf{q}})\left[\int_0^\infty dr[rj_{l_4}(qr)\bar{R}_{l_4}^*(\gamma,r)]\right] , \tag{3.10}$$

$$I_b\to\sum_{\substack{l_1,l_2,l_3,l_4\\ m_1,m_4}}\left[\int_0^\infty dr_1[r_1j_{l_1}(qr_1)j_{l_2}(Qr_1)\bar{R}_{l_3}^*(-\gamma,r_1)]\right]Y_{l_1m_1}^*(\hat{\mathbf{q}})Y_{l_4m_4}(\hat{\mathbf{q}})\left[\int_0^\infty dr[r^2j_{l_4}(qr)\bar{R}_{l_4}^*(\gamma,r)]\right] . \tag{3.11}$$

These one-dimensional integrals can efficiently be performed by the techniques of analytical continuation.[18,19]

Hence T-matrix elements are further reduced, such as

$$T_4(N)\to\int d\mathbf{q}\, g(\mathbf{q})J_{N'}(\boldsymbol{\alpha}_0\cdot(\mathbf{Q}+\mathbf{q}))\tilde{g}(\mathbf{q}) , \tag{3.12}$$

where

$$\tilde{g}(\mathbf{q})=\int d\mathbf{q}_1\tilde{\phi}_{n_1}(\mathbf{q}_1)\tilde{\phi}_{n_2}^*(\mathbf{q}_1-\mathbf{q}) . \tag{3.13}$$

Because $\tilde{\phi}_{n_1}(\mathbf{q})$ and $\tilde{\phi}_{n_2}(\mathbf{q})$ are, respectively, the Fourier transformations of $\phi_{n_1}(\mathbf{r})$ and $\phi_{n_2}(\mathbf{r})$, (3.13) is transformed as, based on the Fourier convolution theorem,

$$\tilde{g}(\mathbf{q})=\langle\phi_{n_2}(\mathbf{r})|e^{-i\mathbf{q}\cdot\mathbf{r}}|\phi_{n_1}(\mathbf{r})\rangle . \tag{3.14}$$

As the explanation in the above, (3.14) can be reduced to one-dimensional integrals, even for the case in which ϕ_{n_1} and ϕ_{n_2} are continuum states. Therefore the integrals appearing in the expressions of T_4 can be reduced to what we can perform nowadays although the specific computations are heavy. As to T_1, T_2 and T_3, they are much simpler than T_4.

Finally, the summation over n in (2.120, (2.121), and (2.122) can approximately be treated by the truncated summation or by the closure approximation.[20] As to the summation over l, they may also be carried out by the truncated summation.

Therefore we may practically calculate the triply differential cross sections for $(e,2e)$ collisions of the H atom according to the above formulas which possess quite high accuracy.

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