Thermal noise from pure-state quantum correlations

A. Mann and M. Revzen
Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel
(Received 15 February 1989)

Given that two initially independent systems A and $\tilde{A}$ are in a pure state, we study the type of interaction between them that will lead an observer, limited to measuring system A only, to conclude that it is in a thermal bath. We show (under rather mild conditions) that there is always an interaction that does it, and study it for two particularly simple systems.

I. INTRODUCTION

Among the curious implications of quantum mechanics is the generation of thermal noise from a pure quantum state. This is realized in diverse disciplines: (i) Parametric interactions involving the radiation field. There, "if one has access to only one mode of a two-mode squeezed vacuum state—the photon statistics of this mode is indistinguishable from that of thermal distribution."[1] (ii) Decay of black holes. Here "observers whose observation of particle modes is limited by 'horizon' see a 'hot' thermal vacuum."[2] There is a considerable amount of literature on these two subjects. In addition, as was already noted,[3] the so-called thermo-field-dynamics[4] (TFD) formulation of a field theory that includes thermal phenomena is based on a closely related approach: here physical quantities are evaluated in pure states. But these states are in an expanded space that includes additional unobserved "tilde" fields. A large body of literature is based on this approach.

II. FORMULATION OF THE PROBLEM AND A GENERAL SOLUTION

We consider the following problem. Given two systems A and $\tilde{A}$, each with its respective Hamiltonian, H and $\tilde{H}$ [note that both A and/or $\tilde{A}$ could be a many-body system, one particle or mode(s), etc.], let the combined system at $t=t_0$ be with both systems in their respective ground states, i.e.,

$$|t_0\rangle=|0\rangle|\tilde{0}\rangle \ . \tag{1}$$

The two systems are coupled at $t\geq t_0$; hence at $t>t_0$ the state of the combined system is given by ($\hbar=1$)

$$|t\rangle=T\exp\left[-i\int_{t_0}^{t}H_{t'}dt'|t_0\rangle\right] . \tag{2}$$

Here T is the time ordering operator ("later" operators are moved to the left) and

$$H_t=H+\tilde{H}+H_I(t) \ . \tag{3}$$

The unknown interaction term H_I is, in general, time dependent. Our problem is: what should be the form of $H_I(t)$ so that for every operator F that involves only the coordinates of the first system (A) we shall have at all times $t>t_0$ (Ref. 5)

$$\bar{F}\equiv\langle t|F|t\rangle=\mathrm{Tr}(e^{-\beta H}F)/\mathrm{Tr}(e^{-\beta H}). \tag{4}$$

Here Tr involves the states pertaining to A only and β is a c-number parameter. Once we get such H_I we will show how known cases wherein thermal noise was derived from a pure state fit in as special cases.

We assume that a correspondence is set up (and indeed that it can be set up) which is one to one between the eigenstates of system A, $\{|n\rangle\}$, and those of system $\tilde{A}$, $\{|\tilde{n}\rangle\}$:

$$|n\rangle\leftrightarrow|\tilde{n}\rangle \ , \tag{5a}$$

with

$$|0\rangle\leftrightarrow|\tilde{0}\rangle \ , \tag{5b}$$

i.e., we stipulate correspondence between the respective ground states. The correspondence between $|n\rangle$ and $|\tilde{n}\rangle$ for $(n,\tilde{n})\neq(0,\tilde{0})$ is arbitrary at the moment. We remark that our results do not require such a severe (i.e., 1:1 correspondence) restriction—we comment on this in Appendix A—but it is easier to follow our argument with the 1:1 correspondence. We take the states $|n\rangle$ and $|\tilde{n}\rangle$ to be the eigenfunctions of the respective Hamiltonians, H and $\tilde{H}$. We now use a method of TFD (Ref. 4) and define

$$|0(\beta,\phi)\rangle=\frac{1}{Z^{1/2}}\sum_n e^{-\beta E_n/2+i\phi(n,\tilde{n})}|n,\tilde{n}\rangle \ , \tag{6}$$

where

$$|n,\tilde{n}\rangle=|n\rangle\otimes|\tilde{n}\rangle \ ,$$

$$H|n,\tilde{n}\rangle=E_n|n,\tilde{n}\rangle;\tilde{H}|n,\tilde{n}\rangle=E_{\tilde{n}}|n,\tilde{n}\rangle \ .$$

In Eq. (6) the sum is over the corresponding pairs of states, Eq. (5); ϕ is an arbitrary real function of its argument and the normalizer

$$Z=\sum_n e^{-\beta E_n}=\mathrm{Tr}(e^{-\beta H}) \ . \tag{7}$$

It is now evident[4] that, for F involving the coordinates of system A only, we have

$$\langle 0(\beta,\phi)|F|0(\beta,\phi)\rangle=1/Z[\mathrm{Tr}(e^{-\beta H}F)] \ . \tag{8}$$

 ©1989 The American Physical Society

This stems from the orthonormality of the states of the $\tilde{A}$ system ($|\tilde{n}\rangle$). We will associate below the state $|t\rangle$, Eq. (2), with $|0(\beta,\phi)\rangle$, Eq. (6).

An alternative way to write the right-hand side (rhs) of Eq. (6) is

$$|0(\beta,\phi)\rangle = Z^{-1/2}(1+A_\beta|J\rangle\langle\tilde{0}0|)|0\tilde{0}\rangle \ , \qquad (9)$$

with

$$|J(\phi)\rangle = \frac{1}{A_\beta}\sum_{n\neq 0} e^{-\beta E_n/2+i\phi(n,\tilde{n})}|n,\tilde{n}\rangle \ , \qquad (10)$$

$$A_\beta = \sqrt{Z-1} \ . \qquad (11)$$

We now return to the time evolution, Eq. (2) (we choose $t_0=0$ for simplicity)

$$|t\rangle = T\exp\left[-i\int_0^t (H+\tilde{H}+H_I)dt'\right]|t_0\rangle \ , \qquad (12)$$

which can be written as

$$|t\rangle = e^{-iH_0 t}T\exp\left[-i\int_0^t \bar{H}_I(t')dt'\right]|t_0\rangle \ , \qquad (13)$$

with

$$\bar{H}_I(t) = e^{iH_0 t}H_I(t)e^{-iH_0 t} \ . \qquad (14)$$

Furthermore, the temperature (i.e., β) is independent of the phases $\phi(n,\tilde{n})$ of the state $|n,\tilde{n}\rangle$. Hence, the state

$$e^{iH_0 t}Z^{-1/2}[1+A_\beta|J(\phi)\rangle\langle\tilde{0}0|]|0\tilde{0}\rangle$$
$$= Z^{-1/2}[1+A_\beta|J(\tilde{\phi})\rangle\langle\tilde{0}0|]|0\tilde{0}\rangle = |0(\beta,\phi')\rangle \ , \qquad (15)$$

with $\phi'(n,\tilde{n}) = \phi(n,\tilde{n}) + (E_n+E_{\tilde{n}})t$ is equivalent to $|0(\beta,\phi)\rangle$ as far as thermal properties of system A are concerned. Our task, is, then, to find H_I, such that

$$e^{iH_0 t}|t\rangle \equiv T\exp\left[-i\int_0^t \bar{H}_I(t')dt'\right]|t_0\rangle$$

could be written as $|0(\beta,\phi')\rangle$, Eq. (15). In some cases this can be done directly (by reading off the solution), and we illustrate this in Sec. III.

For general systems A and $\tilde{A}$, finding such an H_I is more involved. Thus we look for an H_I ensuring, for arbitrary H and $\tilde{H}$, that an observer limited to measuring system A only will be led to conclude that this system is in a thermal bath. To this end we return to Eq. (9) [or (6)]; it can be written in another, equivalent form

$$|0(\beta,\phi)\rangle = Z^{-1/2}\exp(A_\beta|J\rangle\langle\tilde{0}0|)|0\tilde{0}\rangle \ . \qquad (16)$$

We show in Appendix B that another equivalent form is

$$|0(\beta,\phi)\rangle = \exp[B(J\rangle\langle 0| - |0\rangle\langle J|)]|0\tilde{0}\rangle \ , \qquad (17)$$

with

$$Z^{-1/2} = \cos B \ . \qquad (18)$$

We now equate (choosing $|t_0\rangle = |0\tilde{0}\rangle$)

$$\exp\{B[|J(\phi)\rangle\langle 0| - |0\rangle\langle J(\phi)|]\}|0\tilde{0}\rangle$$
$$= T\exp\left[-i\int_0^t \bar{H}_I(t')dt'\right]|0\tilde{0}\rangle \ , \qquad (19)$$

and regard this as our equation for H_I. An obvious solution is

$$H_I = ik(t)e^{-H_0 t}(|J\rangle\langle\tilde{0}0| - |0\tilde{0}\rangle\langle J|)e^{iH_0 t} \ , \qquad (20)$$

with $k(t)$ a c-number function whose time integral gives

$$B(t) = \int_0^t k(t')dt' \ .$$

Equation (20) is an answer to the posed question, i.e., given H and $\tilde{H}$, and given that at some initial time t_0 system A and $\tilde{A}$ were in their ground state—H_I, as deduced from Eq. (20), will result in system A being in, apparently, a thermal bath at the time t. This is so, provided the observation is limited to system A only.

III. REALIZATIONS OF THERMAL NOISE FROM PURE QUANTUM CORRELATIONS

A particularly convenient example is the free radiation field. We consider the simple case of a two-mode radiation field, i.e.,

$$H = \hbar\omega a^\dagger a \ , \qquad (21a)$$

$$\tilde{H} = \hbar\tilde{\omega}\tilde{a}^\dagger\tilde{a} \ , \qquad (21b)$$

$$[a,a^\dagger] = [\tilde{a},\tilde{a}^\dagger] = 1 \ , \qquad (22)$$

and all other commutators vanish. The natural 1:1 correspondence in this case is between the number eigenstates of H and $\tilde{H}$, i.e., $\tilde{n}=n$.

We thus return to Eqs. (9) and (10) and obtain for this case [choosing for simplicity $\phi(n,n)=n\phi$ and $\hbar=1$]

$$Z = (1-e^{-\beta\omega})^{-1} \ , \qquad (23)$$

$$|J\rangle = \frac{1}{\sqrt{Z-1}}\sum_{n\neq 0} e^{-\beta n\omega/2+in\phi}\frac{(a^\dagger\tilde{a}^\dagger)^n}{n!}|0\tilde{0}\rangle$$
$$= \frac{1}{\sqrt{Z-1}}[\exp(e^{-\beta\omega/2+i\phi}a^\dagger\tilde{a}^\dagger)-1]|0\tilde{0}\rangle \ , \qquad (24)$$

and thus

$$|0(\beta,\phi)\rangle = Z^{-1/2}\exp(e^{-\beta\omega/2+i\phi}a^\dagger\tilde{a}^\dagger)|0\tilde{0}\rangle \ . \qquad (25)$$

It is convenient for the evaluation of H_I to bring the exponent on the rhs of Eq. (25) to an anti-Hermitian form. Because the operators[1]

$$K_+ = a^\dagger\tilde{a}^\dagger \ ,$$

$$K_- = \tilde{a}a \ ,$$

$$K_0 = \tfrac{1}{2}(a^\dagger a + \tilde{a}\tilde{a}^\dagger)$$

close the su (1,1) algebra, we have[1]

$$\text{sech}|\gamma|\exp\left[\left[\frac{\gamma}{|\gamma|}\tanh|\gamma|\right]a^\dagger\tilde{a}^\dagger\right]|0\tilde{0}\rangle$$
$$= \exp[(\gamma a^\dagger\tilde{a}^\dagger - \gamma^*\tilde{a}a)]|0\tilde{0}\rangle \ .$$

Taking $\gamma=|\gamma|e^{i\phi}$ with

$$\tanh|\gamma|=e^{-\beta\omega/2} ,$$
$$\operatorname{sech}|\gamma|=(1-e^{-\beta\omega})^{1/2}=Z^{-1/2} ,$$

we get

$$|0(\beta,\phi)\rangle=\exp[(\gamma a^{\dagger}\tilde{a}^{\dagger}-\gamma^{*}\tilde{a}a)]|0\tilde{0}\rangle . \tag{26}$$

Using Eqs. (13)–(15) and recalling that $\overline{H}_I(t)$ has only trivial time dependence,

$$\overline{H}_I(t)=ik(t)H_I' \tag{27}$$

(where H_I' has no explicit time dependence), we have then [rhs of (19)]

$$e^{iH_0t}|t\rangle=\exp[-iB(t)H_I']|0\tilde{0}\rangle , \tag{28}$$

with

$$B(t)=\int_0^t k(t')dt' . \tag{29}$$

Comparing with (26) leads to

$$H_I=ik(t)(a^{\dagger}\tilde{a}^{\dagger}e^{-i(\omega+\tilde{\omega})t}-\text{H.c.}) , \tag{30}$$

i.e., we recovered the parametric interaction that was used by Yurke and Potasek as the interaction that brings about an effective temperature for the mode A. The effective temperature of the mode is

$$T=\hbar\omega/[2k_B\ln(\coth|B|)] ,$$

where k_B is Boltzmann's constant. Note that in this case the temperature T is time dependent in general.

Another example that we wish to consider is a two-mode fermion system ($\hbar=1$)

$$H=\omega C^{\dagger}C ,$$
$$\tilde{H}=\tilde{\omega}\tilde{C}^{\dagger}\tilde{C} ,$$

with

$$[C,C^{\dagger}]_+=[\tilde{C},\tilde{C}^{\dagger}]_+=1 ,$$

where all other anticommutators vanish. In this case it is also advantageous to go back to Eq. (9), and hence we get directly

$$|0(\beta)\rangle=1/Z^{1/2}\exp(e^{-\beta\omega/2}C^{\dagger}\tilde{C}^{\dagger})|0\tilde{0}\rangle,\quad Z=1+e^{-\beta\omega} .$$

Here, too, in order to solve for H_I it is useful to write the exponent in an antihermitian form. We show in Appendix B that the result is

$$|0(\beta)\rangle=\exp[\nu(C^{\dagger}\tilde{C}^{\dagger}-\tilde{C}C)]|0\tilde{0}\rangle ,$$

with

$$\sin\nu=e^{-\beta\omega/2}/(1+e^{-\beta\omega})^{1/2} .$$

Comparing this is to our previous study we see that H_I in this case is of the same form as Eq. (30):

$$H_I=ik(t)(C^{\dagger}\tilde{C}^{\dagger}e^{-i(\omega+\tilde{\omega})t}-\tilde{C}Ce^{i(\omega+\tilde{\omega})t}) . \tag{31}$$

The formula for the temperature is, however,

$$T=\hbar\omega/2k_B\ln\cot|\nu| ,$$

with ν given by

$$\nu=\int_0^t k(t')dt'$$

A particular interaction leading to time-independent temperature is ($t\ge t_0$)

$$H_I=i\delta(t-t_0)\nu_0(\tilde{a}a-a^{\dagger}\tilde{a}^{\dagger}) .$$

This represents an instantaneous interaction at the initial time and leads to

$$|t\rangle=\exp\left[\nu_0\left[a\tilde{a}e^{i(\omega+\tilde{\omega})(t-t_0)}-a^{\dagger}\tilde{a}^{\dagger}e^{-i(\omega+\tilde{\omega})(t-t_0)}\right]\right]|t_0\rangle .$$

Thus, in this case of an instantaneous interaction the resultant temperature is time independent.

We end this section with the remark that the interactions H_I deduced in this section [Eqs. (30) and (31)] are much simpler than implied by Eq. (20). Nonetheless, they are equivalent to the H_I of Eq. (20) as far as the time evolution of $|0\tilde{0}\rangle$ is concerned.

IV. CONCLUDING REMARKS

A procedure for constructing interaction between two general systems A and $\tilde{A}$ was outlined, an interaction such that if only A is accessible to our measurements then it will be in an apparent temperature bath. This holds even though the combined system of A and $\tilde{A}$ is in a pure quantum state, i.e., H_I induces the exact correlations between $\tilde{A}$ and A to have A in what appears to be a thermal bath. Two well-known generic examples for the realization of such a situation were mentioned: parametric interactions between two radiation modes[1] and the effect of states beyond the relativistic horizon on local measurements.[2] The formalism used was the one devised in thermo field dynamics.[4] It would be of interest to show explicitly that the Hawking radiation[2] emerges from an interaction of the type considered here.

APPENDIX A

In the text, Eq. (5), we considered the case where there is 1:1 correspondence between the eigenstates of systems A and $\tilde{A}$. In this case, whenever H_I (the interaction term between these systems) is such as to lead to a thermal bath for system A, then the converse is also true, i.e., in this case we may consider system A to be in an apparent thermal bath due to its interaction with A. (This requires a trivial change in the definition of $|J\rangle$; their respective temperatures need not be equal.) However, if we study the case of A only, then the number of eigenstates of $\tilde{A}$ may be different ("larger") than that of A. The argument given in the text still holds, provided we associate the two ground states $|0\rangle\leftrightarrow|\tilde{0}\rangle$ and we associate a distinct state $|\tilde{n}\rangle$ to each state $|n\rangle$. All other states of $\tilde{A}$ are then taken to be uncoupled to system A. With this provision all our results remain intact.

APPENDIX B: FACTORIZATION OF EXPONENTIAL

Consider the equality of the normalized states

$$1/Z^{1/2}[\exp(A_\beta|J\rangle\langle 0|)]|0\rangle$$
$$=\exp[B(|J\rangle\langle 0|-|0\rangle\langle J|)]|0\rangle \ . \quad (32)$$

Here

$$\langle J|0\rangle=0, \quad \langle 0|0\rangle=\langle J|J\rangle=1 \ , \quad (33)$$

and we define $E=|J\rangle\langle J|+|0\rangle\langle 0|$.

This is an equation for B in terms of $A_\beta[=(Z-1)^{1/2}]$ and $Z^{1/2}$. Define

$$R=|J\rangle\langle 0|-|0\rangle\langle J| \ ,$$
$$R^2=-(|J\rangle\langle J|+|0\rangle\langle 0|)=-E \ .$$

Expand both sides of Eq. (32) to get

$$1/Z^{1/2}(|0\rangle+A_\beta|J\rangle)=\cos B\,|0\rangle+\sin B\,|J\rangle \ .$$

The orthogonality relation, (33), implies

$$1/Z^{1/2}=\cos B \ ,$$
$$A_\beta/Z^{1/2}=\sin B \ .$$

The above is now illustrated for the case of free fermions. Thus consider the exponential operator

$$\exp[\nu(C^\dagger\tilde{C}^\dagger-\tilde{C}C)] \ . \quad (34)$$

Here ν is a real number and C, $\tilde{C}$, $C^\dagger$, and $\tilde{C}^\dagger$ obey the fermionic anticommutation rules. Defining

$$P=C^\dagger\tilde{C}^\dagger-\tilde{C}C \ , \quad (35)$$

we have

$$P^2=-(C^\dagger\tilde{C}^\dagger\tilde{C}C+\tilde{C}CC^\dagger\tilde{C}^\dagger) \ ; \quad (36)$$

it then follows that

$$P^{2n+1}=(-1)^n P, \quad n=0,1,2,\ldots$$
$$P^{2n}=(-1)^n(-P^2), \quad n=1,2,3,\ldots .$$

Hence

$$\exp(\nu P)=1+P^2+\sin\nu P+\cos\nu(-P^2) \ ,$$

and hence, with $C|0\tilde{0}\rangle=\tilde{C}|0\tilde{0}\rangle=0$, we have

$$\exp[\gamma(C^\dagger\tilde{C}^\dagger-\tilde{C}C)]|0\tilde{0}\rangle=(\sin\gamma)C^\dagger\tilde{C}^\dagger|0\tilde{0}\rangle+\cos\gamma|0\tilde{0}\rangle$$
$$=(\cos\gamma)\exp[(\tan\gamma)C^\dagger\tilde{C}^\dagger]|0\tilde{0}\rangle \ .$$

---

[1]B. Yurke and M. Potasek, Phys. Rev. A **36**, 3464 (1987).
[2]W. Israel, Phys. Lett. **57A**, 107 (1976).
[3]S. M. Barnett and P. L. Knight, Phys. Rev. A **38**, 1657 (1988).
[4]H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982); Y. Takahashi and H. Umezawa, Collect Phenom. **2**, 55 (1975).
[5]A related problem wherein the A system achieves thermal equilibrium at some later time is not considered here.