Interfacial wave theory for dendritic structure of a growing needle crystal. II. Wave-emission mechanism at the turning point

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In this paper, we investigate the global instability of a needle crystal growth. It is shown that in the solidification system there exists a simple turning point, which plays a central role for the formation of self-sustaining dendritic structure of a growing needle crystal. A most remarkable signal reflection and wave-emission mechanism at the turning point is explored.

I. INTRODUCTION

In the preceding paper[1] (paper I), we have demonstrated the defects of the local normal-mode solutions and pointed out that these local normal modes cannot provide a coherent structure of global solution in the whole region ($0 \le \tilde{\xi} < \infty$), due to the absence of uniform validity near the tip $\tilde{\xi}=0$ and near the critical point $\tilde{\xi}_c$. This work is concerned with the global solutions of the problem. It will be seen that the dendritic growth problem can be formulated as a linear eigenvalue problem. The perturbed states of dendrite can be described by a series of uniform valid global modes and corresponding eigenvalues $\{\sigma_n\}$. In order to establish such a global-mode theory, we impose the following boundary conditions: (i) the smooth tip condition,

$$\text{as } \tilde{\xi} \to 0, \quad \tilde{h}(0) < \infty, \quad \tilde{h}'(0)=0 \;, \tag{1.1}$$

(ii) the Orr-Sommerfeld radiation condition at far field,

$$\tilde{\xi} \to \infty, \quad \tilde{h}(\tilde{\xi}) \text{ is an outgoing wave} \;. \tag{1.2}$$

Those two boundary conditions are easily justified from the observations. The form of the outgoing wave in (1.2) is to be specified. In this paper, we shall show that the critical point $\tilde{\xi}_c$ is actually a turning point of the system. In order to obtain the global solution uniformly valid in the neighborhood of the critical point $\tilde{\xi}_c$, the following procedure is adopted. In the "outer region" far away from the critical point $\tilde{\xi}_c$, the multiple-variable-expansion method is applicable. One can use the previous results derived in paper (I) as an "outer solution" whereas in the vicinity of the critical point $\tilde{\xi}_c$, one must employ a different length scale to simplify the system and then obtain an "inner solution." Finally, one must match the outer solution and inner solution in the intermediate region to construct a composite solution. The whole procedure is very complicated. We shall restrict ourselves in the lowest approximation. The most important result we obtain here is that, the critical point $\tilde{\xi}_c$ is a simple turning point; this turning point generates a signal reflection and wave-emission process which plays a central role for the formation of the dendritic structure in a needle crystal growth. This paper is arranged as follows. In Sec. II we shall derive the general governing equation for the interface perturbation. In Sec. III we shall show the uniformly valid solution in the vicinity of the turning point; the wave-emission mechanism will be described. Finally, in Secs. IV and V we shall give the outer solution and the matching condition.

II. GOVERNING EQUATION OF INTERFACE PERTURBATION

In order to derive the governing equation of the interface perturbation in the vicinity of the critical point $\tilde{\xi}_c$, we start with the system equations (4.2)–(4.7) of paper I. [Hereafter equations from paper I will be labeled with I, e.g., Eq. (I-4.2).] From Eq. (I-4.2), in general, it is derived

$$\left[\frac{\partial^2}{\partial\tilde{\xi}_+^2}+\frac{\partial^2}{\partial\tilde{\eta}_+^2}\right]\tilde{T}=\left[i\frac{\partial}{\partial\tilde{\xi}_+}+\frac{\partial}{\partial\tilde{\eta}_+}\right]\left[i\frac{\partial}{\partial\tilde{\xi}_+}-\frac{\partial}{\partial\tilde{\eta}_+}\right]\tilde{T} =\epsilon P\{T\} \;, \tag{2.1}$$

where the operator $P\{\tilde{T}\}$ represents all of the terms inside the brackets on the RHS of (I-4.2). Therefore we get

$$\begin{aligned}\left[i\frac{\partial}{\partial\tilde{\xi}_+}-\frac{\partial}{\partial\tilde{\eta}_+}\right]\tilde{T}&=\epsilon\left[i\frac{\partial}{\partial\tilde{\xi}_+}+\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{\tilde{T}\} \;,\\ \left[i\frac{\partial}{\partial\tilde{\xi}_+}+\frac{\partial}{\partial\tilde{\eta}_+}\right]\tilde{T}_s&=\epsilon\left[i\frac{\partial}{\partial\tilde{\xi}_+}-\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{\tilde{T}_s\}\end{aligned} \tag{2.2}$$

or

$$\begin{aligned}\frac{\partial}{\partial\tilde{\eta}_+}(\tilde{T}-\tilde{T}_s)=&\,i\frac{\partial}{\partial\tilde{\xi}_+}(\tilde{T}+\tilde{T}_s)\\ &-\epsilon\left[i\frac{\partial}{\partial\tilde{\xi}_+}+\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{\tilde{T}\}\\ &-\epsilon\left[i\frac{\partial}{\partial\tilde{\xi}_+}-\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{\tilde{T}_s\} \;.\end{aligned} \tag{2.3}$$

By making use of the formulas (2.2) and (2.3) from the boundary conditions (I-4.5)–(I-4.7) it is derived that at $\tilde{\eta}_+=0$ and $\tilde{\eta}=\tilde{\eta}_0$

$$(\tilde{T}+\tilde{T}_s)=\tilde{\eta}_0\tilde{h}+\frac{2}{s}\left[\frac{\partial^2\tilde{h}}{\partial\tilde{\xi}_+^2}+\frac{\epsilon\tilde{\eta}_0^2}{\tilde{\xi}S^2}\frac{\partial\tilde{h}}{\partial\tilde{\xi}}\right] \tag{2.4}$$

and

$$i\frac{\partial}{\partial\tilde{\xi}_+}(\tilde{T}+\tilde{T}_s)+S\sigma\tilde{h}+\tilde{\xi}\frac{\partial\tilde{h}}{\partial\tilde{\xi}_+}+\epsilon(2+\delta^2\tilde{\eta}_0^2)\tilde{h}$$
$$=\epsilon\left[i\frac{\partial}{\partial\tilde{\xi}_+}+\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{T\}$$
$$+\epsilon\left[\frac{\partial}{\partial\tilde{\xi}_+}-\frac{\partial}{\partial\tilde{\eta}_+}\right]^{-1}P\{T\} . \tag{2.5}$$

Let $\epsilon\to 0$ from (2.4) and (2.5) as a leading approximation; we obtain the governing equation for general interface perturbations:

$$i\frac{2}{S}\frac{d^3\tilde{h}_0}{d\tilde{\xi}_+^3}+(\tilde{\xi}+i\tilde{\eta}_0)\frac{d\tilde{h}_0}{d\tilde{\xi}_+}-iS\sigma\tilde{h}_0=0 \tag{2.6}$$

or

$$i\frac{2\epsilon^3}{S}\frac{d^3\tilde{h}_0}{d\tilde{\xi}_1^3}+\epsilon(\tilde{\xi}+i\tilde{\eta}_0)\frac{d\tilde{h}_0}{d\tilde{\xi}_1}-iS\sigma\tilde{h}_0=0 , \tag{2.7}$$

where

$$\tilde{\xi}_1=\tilde{\xi}-\tilde{\xi}_c .$$

We introduce a variable transformation

$$\tilde{h}_0=\exp\left[\frac{i}{\epsilon}\int_{\tilde{\xi}_c}^{\tilde{\xi}}k_c(\tilde{\xi})d\tilde{\xi}\right]W(\tilde{\xi}_1) , \tag{2.8}$$

then the following identities hold:

$$\left[\frac{d}{d\tilde{\xi}_1}-\frac{ik_c}{\epsilon}\right]^n\tilde{h}_0=\exp\left[\frac{i}{\epsilon}\int_{\tilde{\xi}_c}^{\tilde{\xi}}k_c d\tilde{\xi}\right]\frac{d^nW}{d\tilde{\xi}_1^n}$$
$$(n=1,2,\ldots,) . \tag{2.9}$$

In terms of (2.8) and (2.9), Eq. (2.7) is transformed to

$$\frac{\epsilon^3}{3!}\left[\frac{\partial^3\sigma}{\partial k_0^3}\right]_{k_c}\frac{d^3W}{d\tilde{\xi}_1^3}+i\frac{\epsilon^2}{2!}\left[\frac{\partial^2\sigma}{\partial k_0^2}\right]_{k_c}\frac{d^2W}{d\tilde{\xi}_1^2}$$
$$-\left[\frac{\partial\sigma}{\partial k_0}\right]_{k_c}\frac{dW}{d\tilde{\xi}_1}+i[\sigma-\sigma(k_c,\tilde{\xi})]W=0 , \tag{2.10}$$

where the dispersion function $\sigma(k_0,\tilde{\xi})$ is defined as

$$\sigma(k_0,\tilde{\xi})=\frac{k_0}{S}\left[\tilde{\eta}_0-\frac{2k_0^2}{S}\right]-\frac{i\tilde{\xi}}{S}k_0 . \tag{2.11}$$

So far we have made no restriction on the reference wave-number function $k_c(\tilde{\xi})$ yet. Now, we suppose (i) $k_c(\tilde{\xi})$ is determined by the equality

$$\left[\frac{\partial\sigma}{\partial k_0}\right]_{k_c}=0 \tag{2.12}$$

so that

$$k_c=\left[\frac{S}{6}(\tilde{\eta}_0-i\tilde{\xi})\right]^{1/2} \quad [\mathrm{Re}(k_c)>0] , \tag{2.13a}$$

$$\sigma(k_c,\tilde{\xi})=\sigma_c(\tilde{\xi})=\left[\frac{2}{27}\right]^{1/2}\frac{1}{S^{1/2}}(\tilde{\eta}_0-i\tilde{\xi})^{3/2} . \tag{2.13b}$$

(ii) The eigenvalue σ satisfies the pattern-formation condition, and

$$\sigma=\sigma(k_c(\tilde{\xi}_c),\tilde{\xi}_c)=\sigma_c(\tilde{\xi}_c) . \tag{2.14}$$

With the formulas (2.12)–(2.14), Eq. (2.10) is reduced to

$$\epsilon^3\Omega_3(\tilde{\xi})\frac{d^3W}{d\tilde{\xi}_1^3}+i\epsilon^2\Omega_2(\tilde{\xi})\frac{d^2W}{d\tilde{\xi}_1^2}+i\Omega_0(\tilde{\xi})W=0 , \tag{2.15}$$

where

$$\Omega_0(\tilde{\xi})=\sigma(k_c(\tilde{\xi}_c),\tilde{\xi}_c)-\sigma(k_c(\tilde{\xi}),\tilde{\xi})$$
$$=\sigma-\sigma(k_c(\tilde{\xi}),\tilde{\xi}) ,$$
$$\Omega_2(\tilde{\xi})=\frac{1}{2!}\left[\frac{\partial^2\sigma}{\partial k_0^2}\right]_{k_c} , \tag{2.16}$$
$$\Omega_2(\tilde{\xi})=\frac{1}{3!}\left[\frac{\partial^3\sigma}{\partial k_0^3}\right]_{k_c} .$$

Here notes should be made that (1) since $\Omega_0(\tilde{\xi}_c)=0$, the critical point $\tilde{\xi}_c$ is actually a simple turning point of Eq. (2.15); (2) the perturbation equation (2.15) or (2.7) is a leading-order approximate equation valid over the whole region $(0<\tilde{\xi}<\infty)$. In the region far away from the turning point $\tilde{\xi}_c$, one can consider the normal-mode solutions

$$\tilde{h}_0\sim\exp\left[\frac{i}{\epsilon}\int k_0(\tilde{\xi}_1)d\tilde{\xi}_1\right] . \tag{2.17}$$

By substituting the solutions (2.17) into (2.7), one can regain the dispersion relation formula (4.22) in paper I.

III. INNER EXPANSION OF SOLUTION IN THE VICINITY OF TURNING POINT $\tilde{\xi}_c$

In the vicinity of the turning point $\tilde{\xi}_c$, Eq. (2.15) can be simplified further. We make the Taylor expansions in the neighborhood of $\tilde{\xi}_c$:

$$\Omega_0(\tilde{\xi})=\Omega_0'(\tilde{\xi}_c)\tilde{\xi}_1+\cdots ,$$
$$\Omega_2(\tilde{\xi})=\Omega_2(\tilde{\xi}_c)+O(\tilde{\xi}_1) , \tag{3.1}$$
$$\Omega_3(\tilde{\xi})=\Omega_3(\tilde{\xi}_c)+O(\tilde{\xi}_1) .$$

By introducing an inner variable

$$\xi_*=\frac{\tilde{\xi}_1}{\beta(\epsilon)} , \tag{3.2a}$$

where

$$\beta(\epsilon)\to 0 \quad \text{as } \epsilon\to 0 , \tag{3.2b}$$

Eq. (2.15) is transformed to

$$\frac{\epsilon^3}{\beta^3}\Omega_3(\tilde{\xi}_c)\frac{d^3W}{d\xi_*^3}+i\frac{\epsilon^2}{\beta^2}\Omega_2(\tilde{\xi}_c)\frac{d^2W}{d\xi_*^2}+i\Omega_0'(\tilde{\xi}_c)\beta\xi_*W+\cdots=0\ . \quad (3.3)$$

It is evident that one must set

$$\beta(\epsilon)=\epsilon^{2/3}\ . \quad (3.4)$$

We make the following inner expansion:[2]

$$W(\xi_*,\epsilon)=q_0(\epsilon)W_0(\xi_*)+q_1(\epsilon)W_1(\xi_*)+\cdots\ . \quad (3.5)$$

As a leading-order approximation from (3.3) we derive

$$\frac{d^2W_0}{d\xi_*^2}+A^2\xi_*W_0=0\ , \quad (3.6)$$

where

$$A=\left[\frac{\Omega_0'(\tilde{\xi}_c)}{\Omega_2(\tilde{\xi}_c)}\right]=\left[\frac{-2\partial\sigma_c/\partial\tilde{\xi}}{\partial^2\sigma/\partial k_c^2}\right]_{\tilde{\xi}_c}=\left[\frac{S}{6}\left[1-\frac{\tilde{\xi}(\tilde{\xi}-i\tilde{\eta}_0)}{3S^2}\right]\right]^{1/2}_{\tilde{\xi}=\tilde{\xi}_c}$$

$$-\frac{\pi}{2}<\arg(A)<0\ . \quad (3.7)$$

We are only interested in the global-mode solutions that satisfy the radiation condition at the far field (1.2). Thus the following boundary condition for $W_0(\xi_*)$ is posed:

$$\mathrm{Re}(\xi_*)\to\infty,\quad W_0(\xi_*)\sim\frac{1}{\sqrt{k_*}}\exp\left[i\int_0^{\xi_*}k_*d\tilde{\xi}\right],\quad \mathrm{Re}(k_*)>0\ . \quad (3.8)$$

The general solutions of the Airy equation (3.6) are

$$W_0(\tilde{\xi}^*)=\xi_*^{1/2}[CH^{(1)}_{1/3}(\tfrac{2}{3}A\xi_*^{3/2})+DH^{(2)}_{1/3}(\tfrac{2}{3}A\xi_*^{3/2})]\ , \quad (3.9)$$

where $C;D$ are the arbitrary constants, $H^{(1)}_{1/3}(Z)$, $H^{(2)}_{1/3}(Z)$ are the Hankel functions (see Ref. 3). The boundary condition (3.8) determines $D=0$. So that we have the solution

$$W_0(\xi_*)=C\xi_*^{1/2}H^{(1)}_{1/3}(\tfrac{2}{3}A\xi_*^{3/2})\ . \quad (3.10)$$

We denote

$$\zeta=\tfrac{2}{3}A|\xi_*|^{3/2} \quad (3.11)$$

and define a wave-number function k_* on the complex ξ_* plane as

$$k_*=A\xi_*^{1/2}\ . \quad (3.12)$$

The branch cut line is chosen on the lower half ξ_* plane (see Fig. 1). We suppose that the pattern-formation condition (6.7) in paper I is satisfied; the eigenvalue σ is inside the domain (Σ_1) on the complex σ plane and the corresponding critical point $\tilde{\xi}_c$ is below the real axis of $\tilde{\xi}$. Thus the branch cut line chosen above will not cross the real axis of $\tilde{\xi}$; that is necessary to ensure the continuity of the solution for all real variable $\tilde{\xi}$. In doing so,

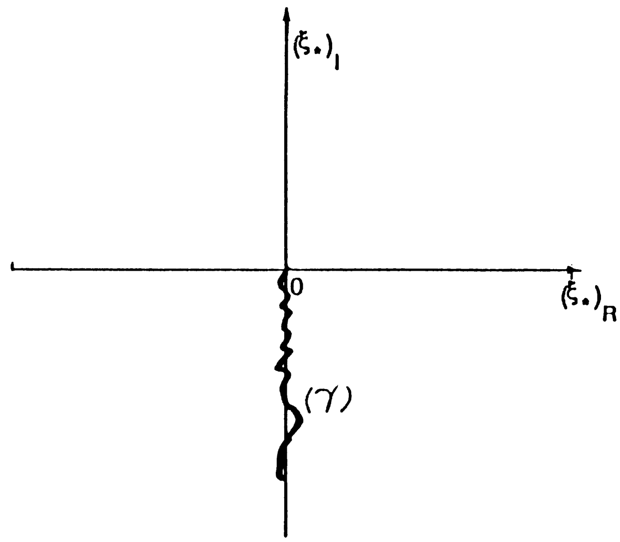

FIG. 1. Branch cut line used in formula (3.13).

$$k_*=\begin{cases}A\xi_*^{1/2} & [\text{as } \mathrm{Re}(\xi_*)>0]\\ +iA|\xi_*|^{1/2} & [\text{as } \mathrm{Re}(\xi_*)<0]\end{cases} \quad (3.13)$$

and since $-\frac{\pi}{2}<\arg(A)<0$, we have

$$\mathrm{Re}(k_*)>0\ \ [\text{as } \mathrm{Re}(\xi_*)>0]\ ,\qquad \mathrm{Re}(k_*)<0\ \ [\text{as } \mathrm{Re}(\xi_*)<0]\ . \quad (3.14)$$

In terms of (3.12) and (3.13) the variable ζ is expressed as

$$\zeta=\tfrac{2}{3}A|\xi_*|^{3/2}=\begin{cases}\int_0^{\xi_*}k_*d\tilde{\xi} & [\mathrm{Re}(\xi_*)>0]\\ -i\int_0^{\xi_*}k_*d\tilde{\xi} & [\mathrm{Re}(\xi_*)<0]\ .\end{cases} \quad (3.15)$$

Hence, as $\mathrm{Re}(\xi_*)\to\infty$, the asymptotic form of the solution (3.10) can be written as

$$W_0(\xi_*)\sim W_T^{(+)}(\xi_*)=\frac{C_3}{\sqrt{k_*}}\exp\left[+i\int_0^{\xi_*}k_*d\tilde{\xi}\right]\quad[\mathrm{Re}(\xi_*)\to+\infty]\ , \quad (3.16)$$

where

$$C_3=C\sqrt{3/\pi}e^{(-5\pi/12)i}\ . \quad (3.17)$$

$W_T^{(+)}(\xi_*)$ represents an outgoing short sidebranching wave with an increasing amplitude. On the other hand, as $\mathrm{Re}(\xi_*)<0$, the solution (3.10) can be written as

$$W_0(\xi_*) = -2Ce^{(\pi/3)i}|\xi_*|^{1/2} \times [I_{1/3}(\zeta) + (e^{\pi i/6}/\pi)K_{1/3}(\zeta)] \; . \quad (3.18)$$

As $\mathrm{Re}(\xi_*) \to -\infty$, it has the following asymptotic expansion form:

$$W_0(\xi_*) \sim C_1 W_0^{(+)}(\xi_*) + C_2 W_0^{-}(\xi_*) \quad [\text{as } \mathrm{Re}(\xi_*) \to -\infty] \; , \quad (3.19)$$

where $[\mathrm{Re}(\xi_*) < 0]$

$$W_0^{(+)}(\xi_*) = \frac{1}{\sqrt{k_*}} \exp\left[+i \int_0^{\xi_*} k_* d\xi_* \right] \quad \text{(outgoing wave)} \quad (3.20)$$

$$W_0^{(-)}(\xi_*) = \frac{1}{\sqrt{k_*}} \exp\left[-i \int_0^{\xi_*} k_* d\xi_* \right] \quad \text{(incoming wave)}$$

and

$$C_2 = -\sqrt{3/\pi} C e^{\pi i/12} \; , \qquad C_1 = -\sqrt{3/\pi} C e^{\pi i/4} \; . \quad (3.21)$$

The results (3.16) and (3.19) show that in the vicinity of the turning point $\bar{\xi}_c$, there exists the interaction of the three waves (see Fig. 2). The incident outgoing wave $W_0^{(+)}$ is reflected from the turning point and becomes an incoming wave $W_0^{(-)}$, the transmitted outgoing wave $W_T^{(+)}$ has an increasing amplitude, which generates the sidebranching structure on the interface of the needle crystal. The reflection ratio of wave is

$$R_c = \frac{C_2}{C_1} = e^{-\pi i/6} \; . \quad (3.22)$$

The transmission ratio of wave is

$$T_c = \frac{C_3}{C_1} = -e^{\pi i/3} \; . \quad (3.23)$$

Return back to the variable $\tilde{h}_0$; we obtain

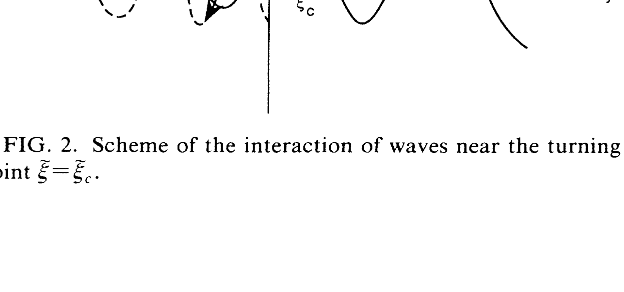

FIG. 2. Scheme of the interaction of waves near the turning point $\bar{\xi} = \bar{\xi}_c$.

$$\tilde{h}_0 = \exp\left[\frac{i}{\epsilon} \int_{\bar{\xi}_c}^{\bar{\xi}} k_c d\bar{\xi} \right] W_0(\xi_*) = C_1 e\left[\frac{i}{\epsilon} \int_{\bar{\xi}_c}^{\bar{\xi}} k_c d\bar{\xi} \right] \times [W_0^{(+)}(\xi_*) + e^{-\pi i/6} W_0^{(-)}(\xi_*)] \quad [\mathrm{Re}(\xi_*) < 0] \; , \quad (3.24)$$

and

$$\tilde{h}_0 = e^{\pi i/3} C_1 \exp\left[\frac{i}{\epsilon} \int_{\bar{\xi}_c}^{\bar{\xi}} k_c d\bar{\xi} \right] W_T^{(+)}(\xi_*) \quad [\mathrm{Re}(\xi_*) > 0] \; . \quad (3.25)$$

The formulas (3.22) and (3.23) also exhibit an important wave-emission mechanism at the turning point ξ_c. By stimulation of incident wave $W_0^{(+)}$ from the tip, the turning point emits a sidebranching wave towards the backside of the dendrite.

It is seen that if the signal $W_0^{(+)}$ is continuously supplied, the sidebranching wave $W_T(+)$ will be continuously emitted from the turning point. The dendrite structure then will be maintained. Thus, significant questions are naturally raised: Is it possible to obtain a continuous supply of the signal $W_0^{(+)}$? How will this supply be maintained? These questions will be answered elsewhere.

Here, we only want to point out the solidification system is an inconceivable dynamical system; this system not only possesses a wave emitter at the turning point but also possesses a signal reflector at the tip. The leading edge of the dendrite tip produces a signal feedback process. Thus the head of the dendrite is like a well-constructed sidebranching wave-emission device. It is certainly possible for this system to possess self-sustaining modes.

IV. OUTER EXPANSION OF SOLUTION

Now we consider the outer solution in the region $0 < \bar{\xi} < \bar{\xi}_c$ and far away from $\bar{\xi}_c$. In this region the multiple-variable expansions are applicable. Thus the previous results in paper I Sec. IV can be used as the outer solution. We list these results as follows:

$$\tilde{h} = \left[\hat{D}_0^{(1)} \exp\left[\frac{i}{\epsilon} \int_0^{\bar{\xi}} k_0^{(1)} d\bar{\xi} \right] + \hat{D}_0^{(3)} \exp\left[\frac{i}{\epsilon} \int_0^{\bar{\xi}} k_0^{(3)} d\bar{\xi} \right] \right] e^{(\sigma \bar{t}/\epsilon)} + \cdots \quad (0 < \bar{\xi} < \bar{\xi}_c) \; , \quad (4.1)$$

where the wave-number functions $k_0^{(1)}(\bar{\xi})$ and $k_0^{(3)}(\bar{\xi})$ are determined by the formula (6.3) in paper I, while the coefficients $\hat{D}_0^{(1)}, \hat{D}_0^{(3)}$, are constants to be determined by the matching condition with the inner solution.

Similarly, as $\bar{\xi} \gg \bar{\xi}_c$ we have

$$\tilde{h} = \hat{D}_{T0}^{(1)} e^{(\sigma \bar{t}/\epsilon)} \exp\left[\frac{i}{\epsilon} \int_0^{\bar{\xi}} k_0^{(1)} d\bar{\xi} \right] + \cdots \; , \quad (4.2)$$

where the amplitude $\hat{D}_{T0}^{(1)}$ is also a constant.

V. MATCHING

In the intermediate regions, the solution (4.1) should match with the solution (3.24) and the solution (4.2) should match with the solution (3.25). To study the matching condition, let us introduce an intermediate variable

$$\tilde{\xi}_I=\frac{\tilde{\xi}-\tilde{\xi}_c}{\mu(\epsilon)}\ , \tag{5.1}$$

where $\mu(\epsilon)$ satisfies the conditions:

$$\lim_{\epsilon\to 0}\mu(\epsilon)=0,\quad \lim_{\epsilon\to 0}\frac{\mu(\epsilon)}{\epsilon^{2/3}}=\infty\ . \tag{5.2}$$

Hence

$$\tilde{\xi}_1=\tilde{\xi}-\tilde{\xi}_c=\mu(\epsilon)\tilde{\xi}_I,\quad \xi_*=\frac{\mu(\epsilon)}{\epsilon^{2/3}}\tilde{\xi}_I\ . \tag{5.3}$$

For any fixed $\tilde{\xi}_I=O(1)$, we have

$$\lim_{\epsilon\to 0}\tilde{\xi}_1=0,\quad \lim_{\epsilon\to 0}\xi_*=-\infty\ . \tag{5.4}$$

In terms of the intermediate variable $\tilde{\xi}_I$, the matching condition of the outer solution (4.1) and inner solution (3.24) yields

$$\hat{D}_0^{(1)}\exp\left[\frac{i}{\epsilon}\int_0^{\mu\tilde{\xi}_I}(k_0^{(1)}-k_c)d\tilde{\xi}_1+\frac{i\phi_1}{\epsilon}\right]+\hat{D}_0^{(3)}\exp\left[\frac{i}{\epsilon}\int_0^{\mu\tilde{\xi}_I}(k_0^{(3)}-k_c)d\tilde{\xi}_1+\frac{i\phi_2}{\epsilon}\right]+\cdots$$
$$=q_0(\epsilon)\frac{1}{\sqrt{k_*}}\left[C_1\exp\left[i\int_0^{\xi_*}k_*d\xi_*\right]+C_2\exp\left[i\int_0^{\xi_*}k_*d\xi_*\right]\right]+\cdots\quad(\xi_*<0;\tilde{\xi}<\tilde{\xi}_c)\ , \tag{5.5a}$$

where

$$\phi_1=\int_0^{\tilde{\xi}_c}k_0^{(1)}d\tilde{\xi},\quad \phi_2=\int_0^{\tilde{\xi}_c}k_0^{(3)}d\tilde{\xi}\ . \tag{5.5b}$$

Notice that we have

$$k_*=A\xi_*^{1/2}=A\frac{\mu^{1/2}(\epsilon)}{\epsilon^{1/3}}\tilde{\xi}_I^{1/2}\ , \tag{5.6}$$

as

$$k_0=k_c,\quad \tilde{\xi}=\tilde{\xi}_c,\quad \sigma=\sigma(k_c\tilde{\xi}_c)\ . \tag{5.7}$$

Thus, as $\tilde{\xi}\to\tilde{\xi}_c$ it follows that

$$0=\left[\frac{\partial\sigma}{\partial k_0}\right]_{k_c\tilde{\xi}_c}(k_0-k_c)+\frac{1}{2!}\left[\frac{\partial^2\sigma}{\partial k_0^2}\right]_{k_c\tilde{\xi}_c}(k_0-k_c)^2+\left[\frac{\partial\sigma}{\partial\tilde{\xi}}\right]_{k_c\tilde{\xi}_c}(\tilde{\xi}-\tilde{\xi}_c)+\cdots\ , \tag{5.8}$$

$$(k_0^{(1,3)}-k_c)=\pm A\tilde{\xi}_1^{1/2}+\cdots\ . \tag{5.9}$$

In terms of the formulas (5.6)–(5.9), from the matching condition (5.5) one can derive

$$q_0(\epsilon)=\epsilon^{-1/6} \tag{5.10}$$

and

$$\frac{\hat{D}_0^{(3)}e^{(i\phi_3/\epsilon)}}{\hat{D}_0^{(1)}e^{(i\phi_1/\epsilon)}}=\frac{C_2}{C_1}=e^{-\pi i/6}\ . \tag{5.11}$$

Finally, we obtain the outer solutions in the region $(0,\tilde{\xi}_c)$:

$$\tilde{h}=C_1e^{\sigma\tilde{t}_+}\exp\left[\frac{i}{\epsilon}\int_0^{\tilde{\xi}}k_0^{(1)}(\tilde{\xi})d\tilde{\xi}\right]+e^{i\chi}\exp\left[\frac{i}{\epsilon}\int_0^{\tilde{\xi}}k_0^{(3)}(\tilde{\xi})d\tilde{\xi}\right]+\cdots\ , \tag{5.12}$$

where

$$\chi=(2\pi-\tfrac{1}{6})\pi+\frac{\phi_1-\phi_3}{\epsilon}\quad(n=0,\pm1,\pm2,\ldots,) \tag{5.13}$$

or

$$\tilde{h}=C_1\exp\left[\frac{\sigma_R\tilde{t}}{\epsilon}+\frac{i}{\epsilon}\int_0^{\tilde{\xi}}k_c(\tilde{\xi})d\tilde{\xi}\right]\times\left[e^{i\chi}\exp\left[\frac{i}{\epsilon}\int_0^{\tilde{\xi}}\Delta k_0^{(3)}d\tilde{\xi}-\frac{i\omega}{\epsilon}\tilde{t}\right]+\exp\left[\frac{i}{\epsilon}\int_0^{\tilde{\xi}}\Delta k_0^{(1)}d\tilde{\xi}-\frac{i\omega}{\epsilon}\tilde{t}\right]\right]+\cdots\ , \tag{5.14}$$

where the reference wave-number function $k_c(\tilde{\xi})$ is determined by the condition (2.12) or the formula (2.13a), while

$$\Delta k_0^{(1)}=k_0^{(1)}-k_c\ ,\qquad \Delta k_0^{(3)}=k_0^{(3)}-k_c\ . \tag{5.15}$$

The solution (5.14) has a very interesting physical implication. First of all, we notice that for the planar interface case, the condition (2.12) yields the most dangerous constant wave number k_{cp} as shown in Fig. 3. In the dendrite growth case, the reference wave number $k_c(\tilde{\xi})$ plays a similar role as k_{cp} in the 2D case. The solution (5.14) shows that a global solution of interface perturbation consists of two factors: (i) the morphological instability factor

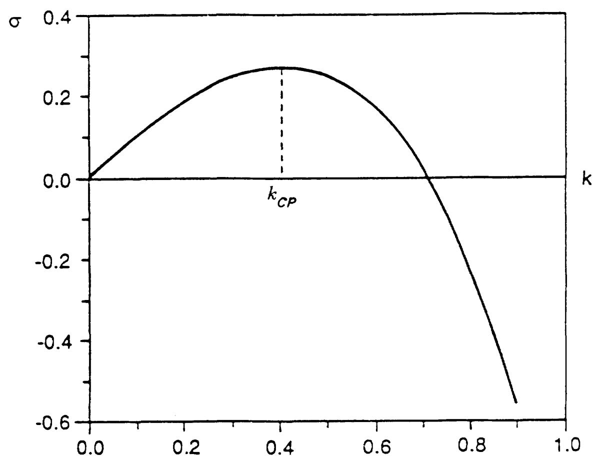

FIG. 3. Dispersion relation curve for the planar interface case.

$$C_1\exp\left[\frac{\sigma_R\tilde t}{\epsilon}+\frac{1}{\epsilon}\int_0^{\tilde\xi}k_c(\tilde\xi)d\tilde\xi\right] \tag{5.16}$$

which implies that the interface of a dendrite grows (or decays) like a "planar" interface with the "preferred" reference wave number $k_c(\tilde\xi)$ at every local position $\tilde\xi$; and (ii) the traveling wave factor

$$[\tilde h^{(+)}+e^{i\chi}\tilde h^{(-)}]\ , \tag{5.17}$$

where

$$\tilde h^{(+)}=\exp\left[\frac{i}{\epsilon}\left[\int_0^{\tilde\xi}\Delta k_0^{(1)}d\tilde\xi-\omega\tilde t\right]\right]\ ,$$
$$\tilde h^{(-)}=\exp\left[\frac{1}{\epsilon}\left[\int_0^{\tilde\xi}\Delta k_0^{(3)}d\tilde\xi-\omega\tilde t\right]\right]\ . \tag{5.18}$$

The outer solution in the region $(\tilde\xi_c,\infty)$ can also be written in the form

$$\tilde h=C_1e^{\pi i/3}\exp\left[\frac{\sigma_R\tilde t}{\epsilon}+\frac{1}{\epsilon}\int_0^{\tilde\xi}k_c d\tilde\xi\right]\tilde h^{(+)}(\tilde\xi)\ . \tag{5.19}$$

VI. SUMMARY

We have considered the global solution of the interface perturbations. It is identified that the global solution in a dendrite growth process contains two factors: the morphological instability factor and the traveling wave factor. The following conclusions are drawn.

(1) The existence of morphological instability and the interacting sidebranching waves are the cause of the coherent sidebranching structure of a growing dendrite.

(2) The presence of a simple turning point $\tilde\xi_c$ plays a crucial role to the global instability mechanism. It is discovered that in the turning-point region, there exists a remarkable signal reflection and wave-emission mechanism. By means of this mechanism at the turning point and a signal feedback at the leading point of the tip, a set of self-sustaining global modes will be possible in the dendritic growth system. Such a global-mode theory is planned to be presented in another paper.

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[1]J. J. Xu, preceding paper, Phys. Rev. A **40**, 1599 (1989).

[2]J. Kevorkian and J. D. Cole, in *Perturbation Methods in Applied Mathematics*, Vol. 34 of *Perturbation Methods in Applied Mathematics* (Springer-Verlag, New York, 1981).

[3]*Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, edited by M. Abramovitz and I. A. Stegun (Dover, New York, 1970).