Counting statistics of partial coherent light with Lorentzian spectrum

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Analytic aspects of the photon-counting distributions in quantum optics are investigated for the case where the underlying field includes a mixture of harmonic signal and thermal noise. A Lorentzian spectral profile for the thermal light is used to find simple iterative relations for the factorial cumulant moments. Explicit analytic expressions for the moments are then given to eighth order. Approximate interpolation formulas, which display proper limiting behavior at small and large correlation length, are also investigated.

I. INTRODUCTION

Recently there has been renewed interest in the quantum stochastic nature of the multiplicity distribution in production processes.[1,2] The formulation and phenomenology is substantially based on earlier studies in the field of quantum optics.[3–6] The desire for a better understanding of the intrinsic nature of the multiplicity distribution in quantum stochastic processes encourages a further investigation of the photon multiplicity distribution.[7–9]

Statistical properties of a harmonic signal mixing with noise have been investigated since the 1960s.[3,4] Glauber and Lachs first used coherent-state representations and the associated density matrix to obtain a simple formulation of the photon probability distribution P_n.[3,4] This problem was later considered by Bedard,[10] Jakeman and Pike,[11] Dielatis,[12] and Roussecau.[13] Particularly, Jaiswal and Mehta included more explicitly the spectrum profile of the thermal field.[14,15] The nature of photon correlations were studied in greater detail. What has been available in the literature that follows the approach of Jaiswal and Mehta are either studies of the low-order cumulant moments, or numerical analyses of the P_n distribution. Few results are expressed in analytic form.[10] It is desirable to extend the study of Jaiswal and Mehta in order to further test the underlying nature of these correlations in quantum statistics.

In this paper we report on a study of the analytic expression of the factorial cumulants for the case of a thermal light with Lorentzian profile. This is achieved through a set of iterative relations with respect to the order of the strength of the chaotic spectrum profile. These iterative relations can then be used for evaluations of the multiplicity distribution P_n, and for a base in testing approximate expressions of the distribution. Iteration formulations presented here can be easily generated for the case where the harmonic signal and the mean frequency of the thermal light do not coincide. The structure of the iterative relationships can also be generalized for spectrum profiles of other kinds.

In Sec. II we shall briefly summarize the known formulation for later discussion.[14] In Sec. III, we shall present the details of the iterative relationships for the factorial cumulants. Explicit calculation of the moments are then carried out to eighth order. Limiting behaviors of the moments are finally used in Sec. IV to discuss the nature of the photon probability distributions as well as to construct interpolating formulas of the moment generating function.

II. BASIC FORMULATION OF THE DISTRIBUTION

A. Multiplicity distribution

In the formulation of Jaiswal and Mehta (JM), the probability of n photons being registered in an interval (t_1, t_2) is given by[14]

$$P_n = \int_0^\infty \frac{(\alpha W)^n}{n!} e^{-\alpha W} P(W)\, dW \ . \tag{1}$$

Here α represents the efficiency of the field of radiation to hadronization, and is taken to be 1. W is the integrated intensity over the interval in t:

$$W = \int_{t_1}^{t_2} I(t)\, dt \ . \tag{2}$$

$P(W)$ is the probability density of the random variable W. From here on, we shall study the plane stationary quasimonochromatic incident wave described by a single complex scalar random variable $V(t)$, with the intensity $I(t)$ given by

$$I(t) = |V(t)|^2 \ . \tag{3}$$

$P(W)$ now stands for all the stochastic characteristics of the radiating field, i.e.,

$$P_n = \langle (W^n/n!) e^{-W} \rangle_V \tag{4}$$

through

$$V(t) = V_c(t) + V_T(t) \ , \tag{5}$$

with V_c a harmonic signal, and V_T a thermal field. While $V_c(t)$ is represented by a constant magnitude V_0 and a random phase ϕ,

$$V_c(t)=V_0 e^{-i(\omega_c t+\phi)} , \quad (6)$$

the thermal field $V_T(t)$ is represented by a complex Gaussian random process with an autocorrelation function $\Gamma(t)$,

$$\Gamma(t')=\langle V_T(t)^* V_T(t+t')\rangle_V . \quad (7)$$

Let us expand the wave field V_T and V_c in an orthogonal base $\{\phi_m\}$,

$$V_T(t)=\sum_m C_m \phi_m(t) , \quad (8)$$

$$V_c(t)=\sum_m C_m^{(0)}\phi_m(t) , \quad (9)$$

where $\{\phi_m\}$ are the orthonormal eigenfunctions of the integral equation

$$\int_0^T \Gamma(t-t')\phi_m(t')dt'=\lambda_m\phi_m(t) . \quad (10)$$

The set of coefficients $\{C_m\}$ are uncorrelated,

$$\langle C_m^* C_n\rangle=\lambda_m\delta_{m,n} , \quad (11)$$

and possess the complex Gaussian distribution,

$$P(\{C_m\})=\prod_m \frac{1}{\pi\lambda_m}\exp\left[-\frac{|C_m|^2}{\lambda_m}\right] , \quad (12)$$

so that

$$P_n=\prod_m \int d\,\mathrm{Re}(C_m)d\,\mathrm{Im}(C_m)P(\{C_m\})\frac{W^n}{n!}e^{-W} . \quad (13)$$

With

$$W=V_0^2 T+\sum_m (|C_m|^2+V_0 e^{-i\phi}f_m C_m^* + V_0 e^{i\phi}f_m^* C_m) , \quad (14)$$

$$=\sum_m |C_m+C_m^{(0)}|^2 , \quad (15)$$

after performing the integration over $\{C_m\}$ (Refs. 6 and 14),

$$P_n=\sum_{\{n_m\}}\prod_m P_{n_m}^{\mathrm{GL}}(N_m,S_m)\delta\left[\sum n_m-n\right] , \quad (16)$$

with

$$N_m=1/(1+\lambda_m) ,$$

$$S_m=C_m\lambda_m/(1+\lambda_m) , \quad (17)$$

where P_n^{GL} is the Glauber-Lachs distribution (GL) in quantum optics,

$$P_n^{\mathrm{GL}}(N,S)=e^{-S/(1+N)}\frac{(N)^n}{(1+N)^{n+1}}L_n^0\left[-\frac{S/N}{1+N}\right] . \quad (18)$$

Here L_n^0 is the generalized Laguerre polynomial of order 0. Expressing P_n alternatively in terms of the generating function $G(s)$, we get

$$G(s)=\sum_0^\infty (1-s)^n P_n$$
$$=\exp(-s\langle n_c\rangle)$$
$$\times\prod_m \frac{1}{1+s\lambda_m}\exp\left[\frac{\langle n_c\rangle|f_m|^2 s^2\lambda_m}{(t_2-t_1)(1+s\lambda_m)}\right] , \quad (19)$$

with

$$\langle n_c\rangle=(t_2-t_1)V_0^2 .$$

Thus the JM formulation can be considered as a specific convolution of an infinite number of GL distributions.

B. Factorial cumulant moments

Because of the difficulties in manipulating overlapping integrals of the orthornormal functions, very few analytical results are known for P_n.[10] To avoid these difficulties, it is traditional to start with the factorial-cumulant generating function $H(s)$ as given by the relation

$$H(s)=\ln G(1-s)$$
$$=\sum_{k=1}^\infty (-s)^k\mu_k/k! . \quad (20)$$

Substituting λ_m of Eq. (10) into the above equation, and integrating over the parameter space of the C_m we obtain

$$\mu_k=(k-1)!\int_{t_i}^{t_f}\Gamma^{(k)}(t,t)dt+k!\langle n_c\rangle\frac{1}{t_f-t_i}\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_f}dt_2 e^{i\omega_c(t_1-t_2)}\Gamma^{(k-1)}(t_1,t_2) , \quad (21)$$

where the kth kernel of the equation $\Gamma^{(k)}(t_1,t_2)$ satisfies

$$\Gamma^{(1)}(t_1,t_2)=\Gamma(t_1-t_2) , \quad (22)$$

$$\Gamma^{(k)}(t_1,t_2)=\int_{t_i}^{t_f}\Gamma(t_1,t)\Gamma^{(k-1)}(t,t_2)dt, \quad k\ge 2 \quad (23)$$

and μ_k can be reexpressed as

$$\mu_k=(k-1)!\langle n_T\rangle^k B_k+k!\langle n_T\rangle^{k-1}\langle n_c\rangle\bar{B}_k , \quad (24)$$

where

$$B_k = \frac{1}{(t_f - t_i)^k} \int_{t_i}^{t_f} dt_1 \cdots \int_{t_i}^{t_f} dt_k \gamma(t_1 - t_2) \cdots \gamma(t_2 - t_1)\gamma(t_k - t_1) , \tag{25}$$

$$\overline{B}_k = \frac{1}{(t_f - t_i)^k} \int_{t_i}^{t_f} dt_1 \cdots \int_{t_i}^{t_f} dt_k e^{i(\omega_c - \omega T)} \gamma(t_1 - t_2) \cdots \gamma(t_{k-1} - t_k) ,$$

$$\langle n_T \rangle = \langle V_T^* V_T \rangle (t_f - t_i) , \tag{26}$$

$$\langle n_c \rangle = V_c^* V_c (t_f - t_i) .$$

We shall also assume that the $\Gamma(t)$ is given by

$$\Gamma(t) = \Gamma_0 e^{-i\omega_T t} \gamma(t) , \tag{27}$$

where ω_T is the mean frequency of the thermal component and $\gamma(t)$ is a slowly varying function of t. For the important special case of a Lorentzian autocorrelation function,

$$\gamma(t) = e^{-\tau|t|} , \tag{28}$$

and the above expression is greatly simplified. Explicit evaluation is given in the paper of Jaiswal and Mehta. For example, for the interval $\Delta_T = t_2 - t_1$

$$B_1 = \overline{B}_1 = 1 , \tag{29}$$

$$B_2 = (e^{-2\beta} + 2\beta - 1)/2\beta^2 , \tag{30}$$

$$\overline{B}_2 = 2(\beta^2 + \Omega^2)^{-2} [e^{-\beta}(\beta^2 - \Omega^2)\cos(\Omega) - 2\beta\Omega \sin(\Omega) - (\beta^2 - \Omega^2) + \beta(\beta^2 + \Omega^2)] , \tag{31}$$

where

$$\beta = \tau \Delta_T , \tag{32}$$

$$\Omega = (\omega_c - \Omega_T)\Delta_T . \tag{33}$$

Explicit expressions of the higher-order terms for B_k and $\overline{B}_k$ are rather involved, and are unknown for $k > 5$. In the next section, we shall extend the known expressions to B_8 and $\overline{B}_8$ for the case $\omega = 0$.

III. EVALUATION OF FACTORIAL CUMULANT MOMENTS

A. Evaluation of $\overline{B}_{k+1}$

In this section we shall discuss an iterative procedure for the evaluation of $\overline{B}_{k+1}$ for the specific case of the Lorentzian correlation function and $\omega_c = \omega_T$. We first set $(t_i, t_f) = (0, \Delta_T)$ and introduce the scaled variable

$$x_j = t_j / \Delta_T$$

so that the range of integration for all the x_j is between 0 and 1. $\overline{B}_{k+1}$ is then given by

$$\overline{B}_{k+1} = \int_0^1 dx_{k+1} \int_0^1 dx_k \gamma(x_{k+1}, x_k) \cdots \int_0^1 dx_1 \gamma(x_2, x_1) . \tag{34}$$

In order to evaluate $\gamma(x,y)$ explicitly, we notice for any function $f(x,y)$ with $0 < y < 1$:

$$\int_0^1 dx\, \gamma(y,x) f(x,y) = e^{-\beta y} \int_0^y e^{\beta x} f(x,y) dx + e^{\beta y} \int_y^1 e^{-\beta x} f(x,y) dx . \tag{35}$$

Here the first term corresponds to the region $y > x$, and the second term corresponds to the region of $y < x$. Thus it is useful to define, for $n \geq 2$, an upper and a lower component of the integral of iteration:

$$L_k(x_{k+1}) = \int_0^{x_{k+1}} dx_k e^{\beta x_k} \int_0^1 dx_{k-1} \gamma(x_k, x_{k-1}) \cdots \int_0^1 dx_1 \gamma(x_2, x_1) , \tag{36}$$

$$U_k(x_{k+1}) = \int_{x_{k+1}}^1 dx_k e^{-\beta x_k} \int_0^1 dx_{k-1} \gamma(x_k, x_{k-1}) \cdots \int_0^1 dx_1 \gamma(x_2, x_1) . \tag{37}$$

L_k and U_k satisfy the iterative relations

$$L_k(x_{k+1}) = \int_0^{x_{k+1}} dx_k [L_{k-1}(x_k) + e^{2\beta x_k} U_{k-1}(x_k)] , \tag{38}$$

$$U_k(x_{k+1}) = \int_{x_{k+1}}^1 dx_k [e^{-2\beta x_k} L_{k-1}(x_k) + U_{k-1}(x_k)] . \tag{39}$$

In order to calculate $\overline{B}_{k+1}$, we notice that the integration x_{k+1} in Eq. (34) can also be broken up into two pieces:

$$\overline{B}_{k+1} = \int_0^1 dx_{k+1} \left[\int_0^{x_{k+1}} + \int_{x_{k+1}}^1 \right] dx_k \int_0^1 dx_{k-1} \cdots \gamma(x_k, x_{k-1}) \cdots \gamma(x_2, x_1) . \tag{40}$$

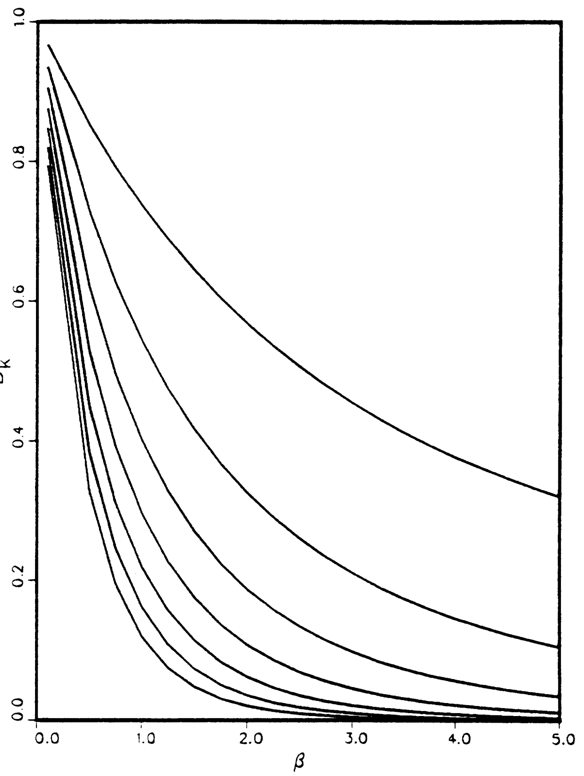

FIG. 1. Values of $\overline{B}_k(\beta)$ are given according to formulas in the Appendix. At any fixed β, the curves in descending order correspond to $k=2-8$.

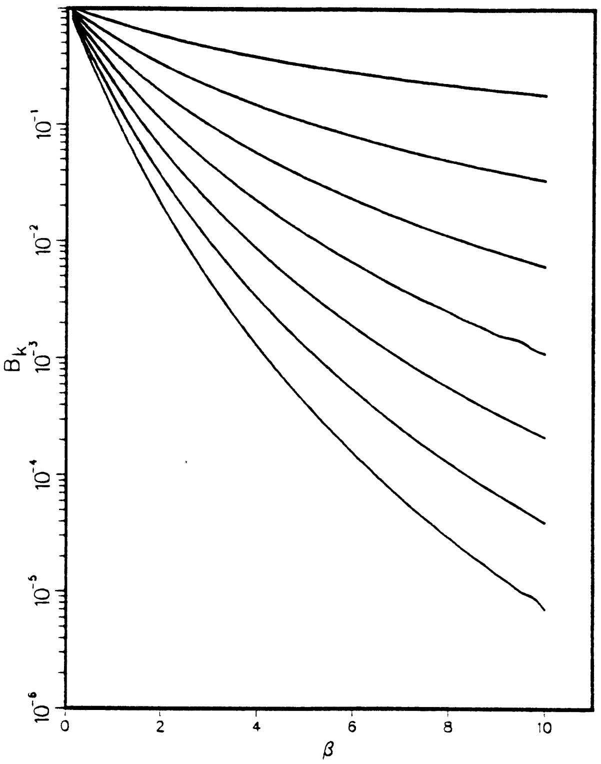

FIG. 2. Values of $\overline{B}_k(\beta)$ in $\log_{10}$ scale are given according to formulas in the Appendix. At any fixed β, the curves in descending order correspond to $k=2-8$.

Using the transformation

$$x_j'=1-x_j,\quad j=1,\ldots,n+1$$

the second term can be folded into the first term. We finally obtain

$$\overline{B}_{k+1}=2\int_0^1 e^{-\beta x_{k+1}}L_k(x_{k+1})dx_{k+1}\ . \tag{41}$$

Starting with $L_2(x)$ and $U_2(x)$ higher order $\overline{B}_k$ can be generated. Explicit calculation shows that we may formally start at $n=0$ with

$$L_0(x_1)=e^{\beta x_1}\ , \tag{42}$$

$$U_0(x_1)=0\ , \tag{43}$$

$$\overline{B}_1=1\ . \tag{44}$$

This leads to

$$L_1(x_2)=(1/\beta)(e^{\beta x_2}-1)\ , \tag{45}$$

$$U_1(x_2)=(1/\beta)(e^{-\beta x_2}-e^{-\beta})\ , \tag{46}$$

$$\overline{B}_2=(2/\beta^2)(e^{-\beta}+\beta-1)\ , \tag{47}$$

which gives $L_2(x)$ and $U_2(x)$ in agreement with explicit evaluations. Further iterations can easily be carried out to $n\geq 3$. However, the algebra of integration soon becomes rather complicated. Since each step of the iteration involves only one variable, the formulation is very suitable for symbolic computer integration. In the Appendix we have listed the explicit expressions up to $\overline{B}_8$. They are also plotted in Figs. 1 and 2.

B. Evaluation of B_k

In this section we shall discuss the corresponding iterative procedure for the evaluation of B_{k+1} for the same case of the Lorentzian correlation function, and $\omega_c=\omega_T$. B_{k+1} is now given by

$$B_{k+1}=\int_0^1 dx_{k+1}\int_0^1 dx_k\gamma(x_{k+1},x_k)\cdots\int_0^1 dx_1\gamma(x_2,x_1)\gamma(x_1,x_{k+1})\ . \tag{48}$$

It differs from the expression of $\overline{B}_{k+1}$ for having the additional factor of $\gamma(x_1,x_{k+1})$. This additional complication can be bypassed in the following fashion. We shall first break up the total expression of Eq. (48) into $(n+1)!$ pieces so that within each piece I_k the set of (x_j) possesses a unique ordering in magnitude. For example, the first piece may satisfy

$x_{k+1} < x_k < \cdots < x_2 < x_1$.

We then relabel the integration variables, so that x_{k+1} is always the largest variable. Pieces of the integration can then be combined to the form

$$I_k' = \int_0^1 dx_{k+1} \int dx_k \int dx_{k-1} \cdots dx_1 \gamma(x_{k+1}, x_k) \cdots \gamma(x_2, x_1) \gamma(x_1, x_{k+1}) \quad (x_1, \ldots, x_k, < x_{k+1}) \; . \tag{49}$$

However, the ordering between x_k and x_{k-1} is not unique, we need to consider the separate cases of $x_k < x_{k-1}$ and $x_k > x_{k-1}$. This leads to the introduction of an upper and a lower component of the integral similar to those in the preceding section except for replacing 1 by x_{k+1}. Thus the new iterative relationship is given by

$$L_{j+1}(x; x_{k+1}) = \int_0^x dx [L_j(x; x_{k+1}) + e^{2\beta x} U_j(x; x_{k+1})] \; , \tag{50}$$

$$U_{j+1}(x; x_{k+1}) = \int_x^{x_{k+1}} dx [e^{-2\beta x} L_j(x; x_{k+1}) + U_j(x; x_{k+1})] \; . \tag{51}$$

Since there are $(k+1)$ identical terms of relabeling, and since there is the additional factor of

$$\gamma(x_1, x_{k+1}) = e^{-\beta x_{k+1}} e^{\beta x_1} \; ,$$

B_{k+1} is given by

$$B_{k+1} = (k+1) \int_0^1 e^{-2\beta x_{k+1}} L_k(x_{k+1}; x_{k+1}) dx_{k+1} \; . \tag{52}$$

Even though the $L_k(x)$ and U_x function are defined only for $n \geq 2$, explicit calculation indicates that we may formally start for $k=0$ with

$$L_0(x_1; x) = e^{2\beta x} \; , \tag{53}$$

$$U_0(x_1; x) = 0 \; , \tag{54}$$

$$B_1 = 1 \; . \tag{55}$$

This leads to

$$L_1(x_2; x) = (1/2\beta)(e^{2\beta x_2} - 1) \; , \tag{56}$$

$$U_1(x_2; x) = x - x_2 \; , \tag{57}$$

$$B_2 = (1/2\beta^2)(e^{-2\beta} + 2\beta - 1) \; . \tag{58}$$

This iterative procedure can easily be carried out to $k \geq 3$. The algebra of integration is again rather compli-

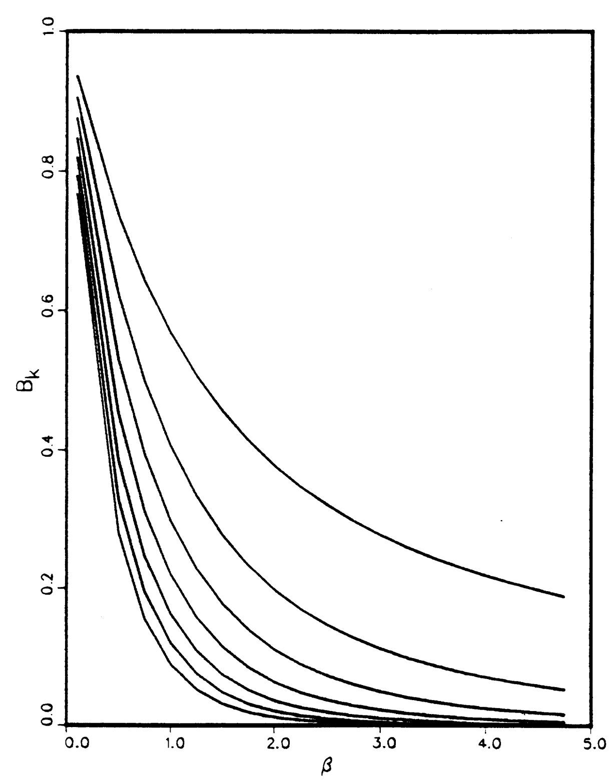

FIG. 3. Values of $B_k(\beta)$ are given according to formulas in the Appendix. At any fixed β, the curves in descending order correspond to $k=2-8$.

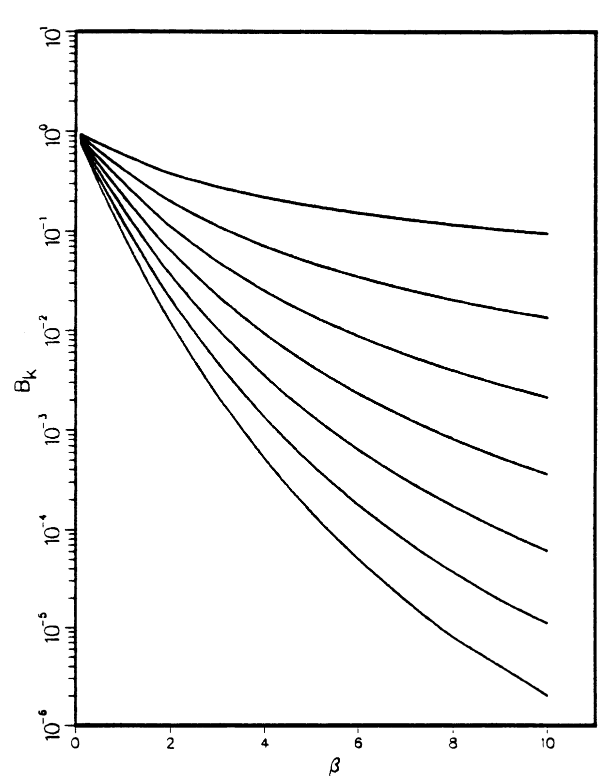

FIG. 4. Values of $B_k(\beta)$ in $\log_{10}$ scale are given according to formulas in the Appendix. At any fixed β, the curves in descending order correspond to $k=2-8$.

cated, but it is very suitable for symbolic computer integration. In the Appendix we have also listed explicit expressions up to B_8. They are also plotted in Figs. 3 and 4.

IV. LIMITING CASES

A. Chaotic limit

In the case of a totally chaotic source $[V_c(y)=0]$ and with a Lorentzian spectrum for $V_T(y)$, an analytic expression of the generating function

$$G(x)=\sum_{n=0}^{\infty} P_n(1-x)^n$$

is obtained. Here

$$G(x)=e^{\beta}\left[\cosh(z)+\frac{1}{2}\left[\frac{\beta}{z}+\frac{z}{\beta}\right]\sinh(x)\right]^{-1}, \qquad (59)$$

where

$$z=[\beta\langle n_T\rangle x+(\beta)^2]^{1/2} .$$

It is well known that in the limit of vanishing Δ_T the above expression takes the form

$$G(x)=(1+\langle n_T\rangle x)^{-1} . \qquad (60)$$

Thus for a very short time interval, the natural limit of the P_n is a negative binomial of $k=1$. This is to be compared with the limit of the classical processes, where the P_n is constructed from the Poisson distribution as the fundamental subprocess. In the limit of vanishing time interval, the classical processes tends to a Poisson distribution for P_n instead.

B. Total coherent limit

In the limit of vanishing chaotic component, $\langle n_T\rangle=0$, each component of ϕ_j contributes as a Poisson distribution. Since the convolution of Poisson distributions is again a Poisson distribution, the overall distribution of P_n is Poissonian for any given $\langle n_c\rangle$.

C. $\beta=0$ short time limit

When the coherent component is present, a closed expression for P_n is not known. We can, however, examine the limiting case of very small β. To the first order of β, it is possible to show that

$$B_k=1-\tfrac{1}{3}k\beta+O(\beta^2) ,$$
$$\overline{B}_k=1-\tfrac{1}{3}(k-1)\beta+O(\beta^2) . \qquad (61)$$

Substituting the above expression into Eq. (1), we obtain

$$H(x)=\sum_{k=1}^{\infty}(-x\langle n_T\rangle)^k\left[\frac{1}{k}\left[1-\frac{1}{3}k\beta\right]\right]+\frac{\langle n_c\rangle}{\langle n_T\rangle}\left[1+\frac{1}{3}(k-1)\beta\right] ,$$
$$G(x)=\frac{1}{1+x\langle n_T\rangle}\exp\left[-\frac{x}{1+x\langle n_T\rangle}\left[\langle n_c\rangle-\frac{\beta}{3}\langle n_T\rangle-\frac{\beta\langle n_c\rangle}{3(1+x\langle n_T\rangle)}\right]\right] . \qquad (62)$$

And for the limit of a short time interval, the P_n distribution is very similar to a GL distribution.

D. $\beta=\infty$ long correlation limit

In the case of a finite correlation length but a very long time interval, we obtain the limit $\beta=\infty$. Since the factorial cumulant moments have the limiting behavior

$$B_k=\frac{1\times3\times5\times\cdots\times(2k-3)}{1\times2\times3\times\cdots\times(k-1)}\beta^{1-k} , \qquad (63)$$

$$\overline{B}_k=\frac{2\times4\times6\times\cdots\times(2k-2)}{1\times2\times3\times\cdots\times(k-1)}\beta^{1-k}=\left[\frac{2}{\beta}\right]^{k-1} , \qquad (64)$$

the corresponding generating function can be summed up exactly as

$$\ln G(x)=\beta[1-(1+2x\langle n_T\rangle/\beta)]^{1/2}$$
$$-x\langle n_c\rangle/(1+2x\langle n_T\rangle/\beta) . \qquad (65)$$

Up to the order of β^{-1},

$$G(x)=\exp[-(1/\beta)(\langle n_T\rangle+\langle n_c\rangle)x]+O(\beta^{-2}) \qquad (66)$$

and the corresponding P_n distribution is Poissonian.

E. Interpolating expressions

Given the limiting behaviors of the functions, it is possible to construct a variety of interpolation formulas for the factorial-cumulant moments $\overline{B}_k$ and B_k. To illustrate this, we use an interpolating function of β^2

$$M(\beta)=[1+(\beta/\beta_0)^2]^{-1} . \qquad (67)$$

This is one of the simplest examples, $M(\beta)$ approaches 1 up to the second order of β at $\beta=0$, approaches zero up to $1/\beta^2$ at $\beta=\infty$, and is 0.5 at $\beta=\beta_0$. β_0 is therefore a control parameter, that we can tune the relative importance of the two limiting behaviors at $\beta=0$ and ∞. Using this function, we can construct an interpolating formula for the functions $B_k(\beta)$ and $\bar{B}_k(\beta)$ as

$$B_k(\beta)'=B_k^0(\beta)M(\beta)+B_k^\infty(\beta)[1-M(\beta)] , \quad (68)$$

$$\bar{B}_k(\beta)'=\bar{B}_k^0(\beta)M(\beta)+\bar{B}_k^\infty(\beta)[1-M(\beta)] , \quad (69)$$

where $B_k^0(\beta)$ has the correct threshold behavior at $\beta=0$, but vanishes faster than $B_k(\beta)$ at $\beta=\infty$. B_k^∞ has the correct threshold behavior at $\beta=\infty$, but vanishes faster than $B_k(\beta)$ at $\beta=0$. The same applies to the $\bar{B}_k$ functions. For example, an appropriate choice with proper limiting behavior at $\beta=0$ and $\beta=\infty$ is given by

$$B_k^0(\beta)=e^{-\beta k/3} , \quad (70)$$

$$B_k^\infty(\beta)=\frac{(2k-3)!!}{(k-1)!(\beta+1)^{k-1}} , \quad (71)$$

$$\bar{B}_k^0(\beta)=e^{-\beta(k-1)/3} , \quad (72)$$

$$\bar{B}_k^\infty(\beta)=2/(\beta+1)^{k-1} . \quad (73)$$

In the range between $\beta=0$ and 10, both $B_k(\beta)$ and $\bar{B}_k(\beta)$ very by five orders of magnitude. However, with an arbitrary choice of $\beta_0=2$, the above expression gives an accuracy of up to 84% for B_2 and $\bar{B}_2$. The accuracy becomes somewhat less for higher moments and may even reduce to 40% for the intermediate region of β, where neither limiting function gives a good approximation to the real expressions. The approximate generating function $G(x)$ distribution can be expressed in closed form. We have

$$G^0(x)=\frac{1}{1+x\langle n_T\rangle e^{\beta/3}} \times\exp[-s\langle n_c\rangle/(1+x\langle n_T\rangle e^{\beta/3})] , \quad (74)$$

$$G^\infty(x)=\exp\{(\beta+1)[1-(1+2x\langle n_T\rangle/\beta)^{1/2} -x\langle n_c\rangle/(1+2x\langle n_T\rangle/\beta)]\} , \quad (75)$$

so that

$$G(x)=G^0(x)M(\beta)+G^\infty(x)[1-M(\beta)] . \quad (76)$$

Since the algebra tends to be complicated, we shall omit further analysis along this direction.

V. CONCLUSION

In this paper, we have derived very simple iterative equations, which allow us to calculate factorial-cumulant moments of the photon-counting distribution of partial coherent light with Lorentzian noise spectrum. The explicit expressions display simple limiting behavior at short and long correlation length. We have also set up a simple approximate interpolation formula between the limiting expressions for $\beta=0$ and ∞. For a region of correlation where the cumulant moments vary by five orders of magnitude, the approximate formula remains satisfactory. Depending on the region of interest for the probability P_n the relative weighting between the coherent strength and the chaotic strength may also be important, and better accuracy of the approximation of the interpolation formulation may be needed in some regions.

Although we have only presented analytic expressions for the cumulant moments for the Lorentzian type of spectrum profile, the general iterative approach can be generated to a wide range of other spectra. The structure of the iterative relations remain same. As long as the integrations can be carried out, the analytic approach is more efficient than numerical evaluations or Monte Carlo simulations for the moments. On the other hand, the algebra involved in this approach easily becomes rather complicated. Symbolic programs are ideal for bookkeeping of the explicit expressions. Limiting behavior of the new coefficients can be similarly discussed.

The work on analytic formulations presented here may be extended to photon counting in several detectors. Numerical results on the joint probability distributions have also been discussed elsewhere separately.[9]

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APPENDIX

The iterative relationships in Sec. III are particularly suitable for symbolic analysis. We have used a DOE–MACSYMA program to perform the integrals. Final results are given below for B_j and $\bar{B}_j$ with $j\le 8$:

$$B_2=\tfrac{1}{2}[e^{-2\beta}+(2\beta-1)]\beta^{-2} , \quad (A1)$$

$$B_3=\tfrac{3}{2}[(\beta+1)e^{-2\beta}+(\beta-1)]\beta^{-3} , \quad (A2)$$

$$B_4=\tfrac{1}{8}[e^{-4\beta}+4(4\beta^2+10\beta+7)e^{-2\beta}+(20\beta-29)]\beta^{-4} , \quad (A3)$$

$$B_5=\tfrac{5}{24}[3(\beta+1)e^{-4\beta}+4(2\beta^3+9\beta^2+15\beta+9)e^{-2\beta}+(21\beta-39)]\beta^{-5} , \quad (A4)$$

$$B_6=\tfrac{1}{32}[e^{-6\beta}+(48\beta^2+108\beta+66)e^{-4\beta}+(32\beta^4+224\beta^3+648\beta^2+900\beta+495)e^{-2\beta}+(252\beta-562)]\beta^{-6}\ , \tag{A5}$$

$$B_7=\tfrac{1}{1920}[420(\beta+1)e^{-6\beta}+(4480\beta^3+16\,800\beta^2+22\,680\beta+10\,920)e^{-4\beta}$$
$$+(896\beta^5+8960\beta^4+39\,200\beta^3+92\,400\beta^2+115\,500\beta+60\,060)e^{-2\beta}+(27\,720\beta-71\,400)]\beta^{-7}\ , \tag{A6}$$

$$B_8=\tfrac{1}{11\,520}[90e^{-8\beta}+(8640\beta^2+18\,720\beta+10\,800)e^{-6\beta}$$
$$+(30\,720\beta^4+168\,960\beta^3+374\,400\beta^2+393\,120\beta+163\,800)e^{-4\beta}$$
$$+(2048\beta^6+27\,648\beta^5+168\,960\beta^4+591\,360\beta^3+1\,235\,520\beta^2+1\,441\,440\beta+720\,720)e^{-2\beta}$$
$$+(308\,880\beta-895\,410)]\beta^{-8}\ , \tag{A7}$$

$$\bar{B}_2=2(e^{-\beta}+\beta-1)\beta^{-2}\ , \tag{A8}$$

$$\bar{B}_3=[-e^{-2\beta}+2(\beta+4)e^{-\beta}+(4\beta-7)]\beta^{-3}\ , \tag{A9}$$

$$\bar{B}_4=\tfrac{1}{2}[e^{-3\beta}+(-4\beta-10)e^{-2\beta}+(2\beta^2+18\beta+47)e^{-\beta}+(16\beta-38)]\beta^{-4}\ , \tag{A10}$$

$$\bar{B}_5=\tfrac{1}{12}[-3e^{-4\beta}+(18\beta+36)e^{-3\beta}+(-24\beta^2-132\beta-204)e^{-2\beta}+(4\beta^3+60\beta^2+342\beta+732)e^{-\beta}+(192\beta-561)]\beta^{-5}\ , \tag{A11}$$

$$\bar{B}_6=\tfrac{1}{24}[3e^{-5\beta}+(-24\beta-42)e^{-4\beta}+(54\beta^2+234\beta+279)e^{-3\beta}+(-32\beta^3-288\beta^2-960\beta-1176)e^{-2\beta}$$
$$+(2\beta^4+44\beta^3+408\beta^2+1872\beta+3558)e^{-\beta}+(768\beta-2622)]\beta^{-6}\ , \tag{A12}$$

$$\bar{B}_7=\tfrac{1}{240}[-15e^{-6\beta}+(150\beta+240)e^{-5\beta}+(-480\beta^2-1800\beta-1830)e^{-4\beta}$$
$$+(540\beta^3+3780\beta^2+9630\beta+8800)e^{-3\beta}+(-160\beta^4-2080\beta^3-11\,160\beta^2-29\,100\beta-30\,945)e^{-2\beta}$$
$$+(4\beta^5+120\beta^4+1600\beta^3+11\,760\beta^2+47\,340\beta+83\,040)e^{-\beta}+(15\,360\beta-59\,370)]\beta^{-7}\ , \tag{A13}$$

$$\bar{B}_8=\tfrac{1}{1440}[45e^{-7\beta}+(-540\beta-810)e^{-6\beta}+(2250\beta^2+7650\beta+6975)e^{-5\beta}$$
$$+(-3840\beta^3-23\,040\beta^2-49\,680\beta-38\,340)e^{-4\beta}+(2430\beta^4+24\,300\beta^3+98\,820\beta^2+192\,780\beta+151\,605)e^{-3\beta}$$
$$-(384\beta^5+6720\beta^4+51\,360\beta^3+213\,120\beta^2+477\,900\beta+461\,430)e^{-2\beta}$$
$$+(4\beta^6+156\beta^5+2790\beta^4+29\,100\beta^3+185\,670\beta^2+683\,910\beta+1\,131\,615)e^{-\beta}+(184\,320\beta-789\,660)]\beta^{-8}\ . \tag{A14}$$

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