Analytic group-theoretical form factors of hydrogenlike atoms for discrete and continuum transitions

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Some of the unsolved problems concerning the form factors of hydrogenic atoms for discrete-discrete, discrete-continuum, and continuum-continuum transitions are solved by a new group-theoretical method. We give exact analytic formulas for these form factors and their summation over angular momenta and indicate analytic continuation procedures among the three cases.

I. INTRODUCTION

Inelastic transition form factors of atoms play an important role in collision theory. They are related to generalized oscillator strengths and are important in all phenomena involving excited states, and more recently in above-threshold ionization problems. Many of their properties have been reviewed.[1] Recently, Inokuti, Shimamura, and Itikawa[2] have drawn attention to "our surprisingly incomplete knowledge about form factors" of H-like atoms from the ground state to higher discrete and continuous states. They list a number of unsolved problems and state that "the knowledge concerning transitions from an excited state, either discrete or continuum, is even less satisfactory." These authors have corrected some of the mistakes in the literature and list among unsolved problems specifically the derivation of the Bethe's sum formula[3] for $\sum_{l=0}^{n-1}|\mathcal{E}_{nl}(\mathbf{K})|^2$, where $\mathcal{E}_{nl}(\mathbf{K})$ is the transition form factor from the ground state to a discrete state (nl) given by

$$\mathcal{E}_{nl}(\mathbf{K})=\int u_{nlm}^*(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{r}}u_0(\mathbf{r})d\mathbf{r}\ ,$$

and state that "we have been unable to derive" Bethe's sum formula from the general expression for $\mathcal{E}_{nl}(\mathbf{K})$. Bethe[3] derived the form factors, using the results of Wentzel,[4] in parabolic quantum numbers n_1,n_2, and carried out a summation over n_1-n_2 at fixed n.

We show in this work by an entirely independent group-theoretical calculation of the form factors that Bethe's final results are correct. We obtain the summation formula directly in spherical quantum numbers, which we believe to be new. This work is a continuation of our previous results on hydrogenic inelastic form factors,[5] which in turn were the limiting cases of the more general relativistic form factors and structure functions.[6] We shall also give a method of analytic continuation to obtain summation formulas for discrete-continuum and continuum-continuum transitions, this is another of the unsolved problems listed by Inokuti *et al.*:[2] "It is difficult to extend the method of Massey and Mohr[7] to the continuum, and attempts with other methods of evaluation have been so far unsuccessful. The desired expression for the continuum must be related to the discrete case through the analytic continuation over the E plane." This is indeed the case.

Our methodology is based on the group-theoretical properties of the form factors derived from the, by now much used, dynamical group SO(4,2). All the calculations can be made directly in the group space where the wave functions and the scalar product are much simpler and where the principal quantum number is the eigenvalue of a group generator instead being related, in a complicated way, to energy, $E_n \sim -1/n^2$. Furthermore, analytic continuation to continuum states can be done very naturally, so that one general formula encompasses all cases. This shows the power of the group-theoretical approach.

The calculation of the form factors can be reduced simply to an evaluation of the matrix elements of a symmetry-group operator element. In Sec. II we elaborate on how this comes about.

II. PHYSICAL BASIS OF THE GROUP-THEORETICAL METHOD

It is well known that for H-like atoms the subspace of bound states for a fixed energy E_n spans a representation space of dimension n^2 of the symmetry group SO(4). For scattering states the group is SO(3,1), an analytic continuation of SO(4). When the atom interacts with an external electromagnetic field, an energy-momentum is transferred to the atom, which then can make transitions to other energy subspaces. Thus we have to connect one representation space of SO(4) with another. It turns out that the totality of all states of the atom span again a representation space of a group—this time the dynamical group SO(4,2) containing all the SO(4) subspaces—with correct multiplicities. Moreover, the current operator is expressible in terms of the Lie algebra of SO(4,2), so that one can say that the dynamical-group representation contains all the information about the system and its coupling to the electromagnetic field. Let us explain first in physical terms how these symmetry and dynamical groups arise. The dynamical variables of the electron in the Coulomb field $\mathbf{r}$ and $\mathbf{p}$ (spinless case), and their enveloping algebra, $r, \mathbf{r}\times\mathbf{p}, r^2, p^2, \ldots$, contain not only the algebra of angular momentum, i.e., Lie algebra of the rotation group SO(3) and the bigger Lie algebra of SO(4), which commute with the Hamiltonian, but also a larger algebra which we shall give in the text whose irreducible representation contains all the states of the atom. Thus we have dynamical variables connecting different energy and angular-momentum values.

When the atom is at rest the group representation gives us the states $|nlm\rangle$. But when the atom is moving we have also to label the states by the momentum $\mathbf{p}$ of the state $|nlm;\mathbf{p}\rangle$ as well. The moving states form a representation of the Galilei group (or, in the relativistic case, of the Poincaré group). In group-theoretical language we *induce* a representation of the Galilei (or Poincaré) group by going from the rest-frame states to the moving states. This can be done by a so-called "boost" operator $\mathbf{M}$

$$|nlm;\mathbf{p}\rangle = e^{-i\mathbf{K}\cdot\mathbf{M}}|nlm;\mathbf{p}_0\rangle .$$

The boost operator $\mathbf{M}$ is also contained in the Lie algebra of the dynamical group. Therefore the transition form factors are essentially the scalar products between initial and final states:

$$\langle nlm;\mathbf{p}_f|nlm;\mathbf{p}_i\rangle = \langle nlm|e^{i\mathbf{K}\cdot\mathbf{M}}|nlm\rangle ,$$

where $\mathbf{K}=\mathbf{p}_f-\mathbf{p}_i$ is the momentum transfer. For the nonrelativistic case, the boost operator $\mathbf{M}$ is simply proportional to $\mathbf{x}$, the position operator, and we get, for $\mathbf{p}_i=0$, the form factor $\mathcal{E}_{nl}(\mathbf{K})$ given above, or more generally, the $F_\lambda(\mathbf{K})$ given further in Sec. IV, Eq. (4.1).

The expression for $\mathcal{E}_{nl}(\mathbf{K})$ admits three different interpretations: (i) Fourier transform of the charge and current distributions, (ii) the transition matrix element of an external electromagnetic plane wave $e^{i\mathbf{K}\cdot\mathbf{x}}$, and (iii) the transition between the atom in the rest frame and the moving atom with a momentum transfer $\mathbf{K}$. Accordingly one may express the physics in different ways, but their equivalence gives us some more insight into the meaning and calculation of form factors.

Although we deal here with the simplest form factors, more general form factors involving spin and inelastic electric and magnetic transitions can also be calculated by our method.[5,6] We begin with the group properties of states, then express the form factors as matrix elements of group generators in appropriate bases, evaluate these matrix elements and the summation formulas for all three kinds of transitions, and give the prescription for analytic continuation.

III. CANONICAL DISCRETE AND CONTINUOUS BASES OF SO(4,2)

The most degenerate unitary irreducible representations of SO(4,2) have been discussed in detail in several papers.[8,9] These representations can be obtained by algebraically restricting the Lie algebra of SO(4,2) to the representation relation

$$\{L_{\alpha\beta}, L^{\alpha\gamma}\} = 2(l_0^2-1)g_\beta^\gamma, \quad \alpha,\beta,\gamma=1,2,\ldots,6 \tag{3.1}$$

where the parameter $l_0=0,\pm\frac{1}{2},\pm 1,\ldots$, fixes the spin of the representation, and $L_{\alpha\beta}=-L_{\beta\alpha}$ are the Lie generators of SO(4,2) satisfying the Lie product

$$[L_{\alpha\beta},L_{\gamma\delta}] = i(g_{\alpha\delta}L_{\beta\gamma}+g_{\beta\gamma}L_{\alpha\delta}-g_{\alpha\gamma}L_{\beta\delta}-g_{\beta\delta}L_{\alpha\gamma}) ,$$
$$g_{\alpha\alpha}=(-1,-1,-1,-1;+1,+1) . \tag{3.2}$$

The canonical discrete basis $\{\varphi_{nlm}^{l_0}\}$ of the most degenerate unitary irreducible representations of SO(4,2) is obtained from the reduction

$$\mathrm{SO}(4,2)\supset\mathrm{SO}(4,1)\supset\mathrm{SO}(4)\supset\mathrm{SO}(3)\supset\mathrm{SO}(2) ,$$

and the basis functions satisfy

$$L_{12}\varphi_{nlm}^{l_0} = m\varphi_{nlm}^{l_0}, \quad m=-l,\ldots,+l ,$$
$$\tfrac{1}{2}L_{ij}L^{ij}\varphi_{nlm}^{l_0} = l(l+1)\varphi_{nlm}^{l_0} ,$$
$$l=|l_0|,\ldots,n-1; i,j=1,2,3 , \tag{3.3}$$
$$L_{56}\varphi_{nlm}^{l_0} = n\varphi_{nlm}^{l_0}, \quad n=1+|l_0|,\ldots,\infty .$$

Under the restriction (3.1), the invariant products of SO(4,2) and SO(4,1) are expressed in terms of l_0 and those of SO(4) in terms of l_0 and n.

A canonical continuous basis $\{\varphi_{\nu lm}^{l_0}\}$ can be obtained from the reduction

$$\mathrm{SO}(4,2)\supset\mathrm{SO}(4,1)\supset\mathrm{SO}(3,1)\supset\mathrm{SO}(3)\supset\mathrm{SO}(2)$$

and the corresponding continuous basis functions satisfy,

$$L_{12}\varphi_{\nu lm}^{l_0} = m\varphi_{\nu lm}^{l_0}, \quad m=-l,\ldots,l$$
$$\tfrac{1}{2}L_{ij}L^{ij}\varphi_{\nu lm}^{l_0} = l(l+1)\varphi_{\nu lm}^{l_0}, \quad l=|l_0|,\ldots,\infty, \quad i,j=1,2,3$$
$$L_{46}\varphi_{\nu lm}^{l_0} = \nu\varphi_{\nu lm}^{l_0}, \quad \nu\in(-\infty,\infty) . \tag{3.4}$$

Again the condition (3.1) implies that the invariant products of SO(3,1) can be expressed in terms of ν and l_0. In the discussions that follow for the hydrogen atom we use the spin-zero representations of SO(4,2) for which $l_0=0$. The continuous basis states can be expanded in terms of discrete states as

$$\varphi_{\nu lm} = \sum_{n=1}^{\infty} V_{n\nu}^{l+1}(i\pi/2)\varphi_{nlm} , \tag{3.5}$$

where the V functions are SO(2,1) representation functions[9,10] given by

$$V_{n\nu}^{k}(\theta) = \frac{\exp\left[i\frac{\pi}{2}(i\nu-1)\right]}{\sqrt{2}\sqrt{\sin[\pi(i\nu)]}}\left[\begin{pmatrix}k-1+n\\-k+n\end{pmatrix}\begin{pmatrix}k-1+i\nu\\-k+i\nu\end{pmatrix}\right]^{1/2}[\tanh(\theta/2)]^{i\nu+n}[\sinh(\theta/2)]^{-2k}$$
$$\times{}_2F_1(k-i\nu,k-n;2k;-[\sinh(\theta/2)]^{-2}) . \tag{3.6}$$

IV. DISCRETE-DISCRETE TRANSITIONS

In an earlier publication[5] we computed the charge form factor function $I_{nlm}^{n'l'm'}(q)=F(q)$ from

$$F_\lambda(q)=\int \psi^*_{n'l'm'}(\mathbf{x})\exp\left[i\left[\frac{\mu}{m_e}\right]\mathbf{q}\cdot\mathbf{x}\right]J_\lambda\psi^x_{nlm}d^3x \ , \tag{4.1}$$

where μ is the reduced mass, m_e is the mass of the electron, $\mathbf{q}$ is the momentum transfer, and the current operator J is given by

$$J_\lambda=\left\{1;\frac{\mathbf{q}}{2m_e}\right\}$$
$$=\left\{\frac{1}{a}(L_{56}-L_{46});\frac{1}{m_e}L_{i6}+\frac{q_i}{2m_e}\frac{1}{a}(L_{56}-L_{46})\right\} ,\quad i=1,2,3 \ . \tag{4.2}$$

Here a is an arbitrary scale parameter, $\psi_{n'l'm'}$ and ψ_{nlm} are final and initial physical states, respectively. The physical states are defined in terms of discrete basis states as

$$\psi_{nlm}(\mathbf{x})=\frac{1}{N}\exp(-i\theta_nL_{45})\varphi_{nlm}(\mathbf{x}) \ , \tag{4.3}$$

where $\theta_n=\ln(aN)$; $N=n/Z$ is obtained by fixing the charge of the physical state to be Z (initial) and Z' (final). The physical state is boosted from rest to acquire a momentum $\mathbf{q}$ by the Lie operator $(-m_p/a)(L_{i5}-L_{i4})$; m_p being the mass of proton. By boosting the final state in the z direction we obtain from (4.1) the charge form factor function in spinor notation[5] as

$$I_{nlm}^{m'l'm'}=\frac{1}{N'}\langle n'l'm'|G(L_{56}-L_{46})|nlm\rangle \ ,$$
$$G=\exp(-i\theta_{n'n}L_{45})\exp[-iKN(L_{35}-L_{34})] \ , \tag{4.4}$$
$$\theta_{n'n}=\ln(N/N'),\quad N'=\frac{n'}{Z'},\quad N=\frac{n}{Z},\quad K=q_3m_p \ .$$

The action of L_{56} and L_{46} on $|nlm\rangle$ is given by

$$L_{56}|nlm\rangle=n|nlm\rangle \ ,$$
$$L_{46}|nlm\rangle=\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}|(n+1)lm\rangle+\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}|(n-1)lm\rangle \ .$$

Therefore (4.4) becomes

$$I_{nlm}^{n'l'm'}=\frac{1}{N'}\{n\langle n'l'm'|G|nlm\rangle-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\langle n'l'm'|G|(n+1)lm\rangle-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\langle n'l'm'|G|(n-1)lm\rangle\} \ . \tag{4.5}$$

The matrix elements of G in (4.5) can be computed[9,5] explicitly by using Euler rotations in SO(2,1) and using SO(4) and SO(2,1) representation functions.[10,11] We obtain[5]

$$I_{nlm}^{n'l'm'}=\frac{1}{N'}\left[n\sum_{l_0}D^{[n'-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n'n}(\beta)D^{[n-1,0]}_{l_0ml}(\gamma)-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\sum_{l_0}D^{[n'-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n'n+1}(\beta)D^{[n,0]}_{l_0ml}(\gamma)-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\sum_{l_0}D^{[n'-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n'n-1}(\beta)D^{[n-2,0]}_{l_0ml}(\gamma)\right],\quad l_0=0,1,2,\ldots,\min\{(n'-1),(n-1)\} \ , \tag{4.6}$$

$$\sinh(\beta/2)=\frac{1}{2}\frac{1}{\sqrt{NN'}}[(N'-N)^2+K^2(N')^2N^2]^{1/2},\quad N=\frac{n}{Z},\quad N'=\frac{n'}{Z'}$$
$$\cosh(\beta/2)=\frac{1}{2}\frac{1}{\sqrt{NN'}}[(N'+N)^2+K^2(N')^2N^2]^{1/2} \ ,$$
$$\sinh\beta\sin\alpha=-KN \ ,$$
$$\sinh\beta\cos\alpha=\frac{1}{2NN'}[(N')^2-N^2+K^2(N')^2N^2] \ ,$$
$$\sinh\beta\sin\gamma=KN' \ , \tag{4.7}$$
$$\sinh\beta\cos\gamma=-\frac{1}{2NN'}[N^2-(N')^2+K^2(N')^2N^2] \ ,$$
$$V^k_{n'n}(\theta)=(-1)^{-k+n}\left[\begin{pmatrix}k-1+n'\\-k+n'\end{pmatrix}\begin{pmatrix}k-1+n\\-k+n\end{pmatrix}\right]^{1/2}[\tanh(\theta/2)]^{n'+n}[\sinh(\theta/2)]^{-2k}$$
$$\times{}_2F_1\{k-n',k-n;2k;-[\sinh(\theta/2)]^{-2}\},\quad n'\geq n$$

$$V^{k}_{n'n}(\theta)=V^{k}_{nn'}(-\theta)\ ,$$
$$V^{1}_{n'1}(\beta)=\sqrt{n}\,[\tanh(\beta/2)]^{n'-1}[\cosh(\beta/2)]^{-2}\ ,$$
$$V^{1}_{n'2}(\beta)=\frac{1}{\sqrt{2}}\sqrt{n'}(n'-1)[\tanh(\beta/2)]^{n'-2}[\cosh(\beta/2)]^{-4}-\sqrt{2}\sqrt{n'}[\tanh(\beta/2)]^{n'}[\cosh(\beta/2)]^{-2}\ ,$$
$$V^{2}_{n'2}(\beta)=\frac{1}{\sqrt{6}}[(n'-1)n'(n'+1)]^{1/2}[\tanh(\beta/2)]^{n'-2}[\cosh(\beta/2)]^{-4}\ ,$$
$$D^{[l_+,l_0]}_{l'ml}(\theta)=\sqrt{(2l'+1)(2l+1)}\sum_{m_1,m_2}\begin{bmatrix}\frac{1}{2}(l_++l_0) & \frac{1}{2}(l_+-l_0) & l'\\ m_1 & m_2 & m\end{bmatrix}\begin{bmatrix}\frac{1}{2}(l_++l_0) & \frac{1}{2}(l_+-l_0) & l\\ m_1 & m_2 & m\end{bmatrix}\exp[-i\theta(m_1-m_2)]\ ,$$
$$[D^{[l_+,l_0]}_{l'ml}(\theta)]^*=D^{[l_+,l_-]}_{lml'}(-\theta)\ ,$$
$$\sum_l D^{[l_+,l_0]}_{l'm'l}(\theta)D^{[l_+,l_-]}_{lm''l''}(-\theta)=\delta_{l'l''}\delta_{m'm''}\ ,$$
$$\sum_l D^{[l_+,l_0]}_{l'ml}(\theta_1)D^{[l_+,l_-]}_{lml''}(\theta_2)=D^{[l_+,l_-]}_{l'ml''}(\theta_1+\theta_2)\ ,$$
$$D^{[0,0]}_{000}(\gamma)=1,\quad D^{[1,0]}_{000}(\gamma)=\cos\gamma,\quad D^{[1,0]}_{100}(\gamma)=-i\sin\gamma\ ,\tag{4.8}$$
$$D^{[l_+,0]}_{l'll}(\theta)=\left[\begin{bmatrix}l_++l'+1\\ l_+-l'\end{bmatrix}\begin{bmatrix}l_++l+1\\ l_+-l\end{bmatrix}^{-1}\begin{bmatrix}l'+l\\ l'-l\end{bmatrix}\begin{bmatrix}2l\\ l\end{bmatrix}\begin{bmatrix}2l'\\ l'\end{bmatrix}^{-1}\right]^{1/2}\begin{bmatrix}l_++l'+1\\ l_+-l'\end{bmatrix}^{-1}(-2i\sin\theta)^{l'-l}C^{l'+1}_{l_+-l'}(\cos\theta)\ ,$$
$$D^{[l_+,0]}_{l',l,l+1}(\theta)=-i\left[\frac{(2l+3)}{(l_++l+2)(l_+-l)}\right]^{1/2}\frac{d}{d\theta}D^{[l_+,0]}_{l'll}(\theta)\ .\tag{4.9}$$

If the initial state is the ground state $|100\rangle$, then Eq. (4.6) becomes the expressions derived by Massey and Mohr.[7] For simplicity we take the charges $Z=Z'=1$; and after using the following identities of the Gegenbauer polynomial

$$(n+m)C^m_n(\cos\alpha)=mC^{m+1}_n(\cos\alpha)-mC^{m+1}_{n-2}(\cos\alpha),\quad C^m_0(\cos\alpha)=1$$
$$(n+2m)C^m_n(\cos\alpha)=2mC^{m+1}_n(\cos\alpha)-2mC^{m+1}_{n-1}(\cos\alpha)\ ,\tag{4.9a}$$
$$\frac{dC^m_n(\cos\alpha)}{d\alpha}=-2m\sin\alpha C^{m+1}_{n-1}(\cos\alpha)\ ,$$

we obtain

$$I^{n'l'_0}_{100}=\frac{1}{n'}\left[D^{[n'-1,0]}_{l'00}(\alpha)V^1_{n'1}(\beta)-\frac{1}{\sqrt{2}}D^{[n'-1,0]}_{l'00}(\alpha)V^1_{n'2}(\beta)\cos\gamma-\frac{1}{\sqrt{2}}D^{[n'-1,0]}_{l'01}(\alpha)V^2_{n'2}(\beta)(-i\sin\gamma)\right]\ ,$$
$$I^{n'l'_0}_{100}=(-i)^{l'}2^{2l'+3}(n')^{l'+1}\sqrt{(2l'+1)}(l'+1)!\left[\frac{(n'-l'-1)!}{(n'+l')!}\right]^{1/2}K^{l'}\frac{[(n'-1)^2+K^2(n')^2]^{(n'-l'-3)/2}}{[(n'+1)^2+K^2(n')^2]^{(n'+l'+3)/2}}$$
$$\times\{(n'+1)[(n'-1)^2+K^2(n')^2]C^{l'+2}_{n'-l'-1}(\cos\alpha)-2n'[(n'-1)^2+K^2(n')^2]^{1/2}[(n'+1)^2+K^2(n')^2]^{1/2}C^{l'+2}_{n'-l'-2}(\cos\alpha)$$
$$+(n'-1)[(n'+1)^2+K^2(n')^2]C^{l'+2}_{n'-l'-3}(\cos\alpha)\}\ ,\tag{4.10}$$
$$\cos\alpha=[(n')^2-1+K^2(n')^2]/[(n'-1)^2+K^2(n')^2]^{1/2}[(n'+1)^2+K^2(n')^2]^{1/2}\ .$$

The form factor $I^{n'l'm}_{nlm}$ is singular at $\cosh(\beta/2)=0$. This implies that

$$K^2=-\left[\frac{n'+n}{n'n}\right]^2=2(\sqrt{B_n}+\sqrt{B_{n'}})^2\ ,\tag{4.11}$$

where $B_n=-1/2n^2$ is the binding energy of the state $|nlm\rangle$. The expression for K^2 given by (4.11) is in exact agreement with the results of the perturbation theory. We now square (4.6) and sum over l' and obtain

$$\sum_{l'=0}^{n'-1} |I_{nlm}^{n'l'm}|^2$$

$$=\frac{1}{(N')^2}\Bigg\{ n^2 \sum_{l_0} |V_{n'n}^{l_0+1}(\beta) D_{l_0 ml}^{[n-1,0]}(\gamma)|^2$$

$$-\frac{n}{2}[(n-l)(n+l+1)]^{1/2} \sum_{l_0} [V_{n'n}^{l_0+1}(\beta) V_{n'n+1}^{l_0+1}(\beta)]$$

$$\times \{ D_{l_0ml}^{[n-1,0]}(\gamma)[D_{l_0ml}^{[n,0]}(\gamma)]^* + [D_{l_0ml}^{[n-1,0]}(\gamma)]^* D_{l_0ml}^{[n,0]}(\gamma)\}$$

$$-\frac{n}{2}[(n+l)(n-l-1)]^{1/2} \sum_{l_0} [V_{n'n}^{l_0+1}(\beta) V_{n'n-1}^{l_0+1}(\beta)]$$

$$\times \{ D_{l_0ml}^{[n-1,0]}(\gamma)[D_{l_0ml}^{[n-2,0]}(\gamma)]^* + [D_{l_0ml}^{[n-1,0]}(\gamma)]^* D_{l_0ml}^{[n-2,0]}(\gamma)\}$$

$$+\tfrac{1}{4}[(n-l)(n+l+1)] \sum_{l_0} |V_{n'n+1}^{l_0+1}(\beta) D_{l_0ml}^{[n,0]}(\gamma)|^2$$

$$+\tfrac{1}{4}[(n-l)(n+l+1)(n+l)(n-l-1)]^{1/2}$$

$$\times \sum_{l_0} (V_{n'n+1}^{l_0+1}(\beta) V_{n'n-1}^{l_0+1}(\beta)) \{ D_{l_0ml}^{[n,0]}(\gamma)[D_{l_0ml}^{[n-2,0]}(\gamma)]^* + [D_{l_0ml}^{[n,0]}(\gamma)]^* D_{l_0ml}^{[n-2,0]}(\gamma)\}$$

$$+\tfrac{1}{4}[(n+l)(n-l-1)] \sum_{l_0} |V_{n'n-1}^{l_0+1}(\beta) D_{l_0ml}^{[n-2,0]}(\gamma)|^2 \Bigg\} . \tag{4.12}$$

If the initial state is the ground state $|100\rangle$ we obtain the simpler result

$$\sum_{l'=0}^{n'-1} |I_{100}^{n'l'0}|^2 = \frac{1}{(N')^2}\{ |V_{n'1}^{1}(\beta)|^2 - \sqrt{2}[V_{n'1}^{1}(\beta) V_{n'2}^{1}(\beta)]\cos\gamma + \tfrac{1}{2}|V_{n'2}^{1}(\beta)|^2\cos^2\gamma + \tfrac{1}{2}|V_{n'2}^{2}(\beta)|^2 \sin^2\gamma \} . \tag{4.13}$$

We now use (4.8) and (4.7) with the charges $Z=Z'=1$ and obtain

$$\sum_{l'=0}^{n'-1} |I_{100}^{n'l'0}|^2 = \frac{n'}{(N')^2}[\tanh(\beta/2)]^{2n'}[\sinh(\beta/2)\cosh(\beta/2)]^{-6}\tfrac{1}{16}$$

$$\times \Bigg[\frac{1}{16(n')^4}\{[(n'-1)^2+(Kn')^2]^2[(n'+1)^2+(Kn')^2]^2$$

$$+4n'(n'-1)[1-(n')^2+(Kn')^2][(n'-1)^2+(Kn')^2][(n'+1)^2+(Kn')^2]$$

$$-2[1-(n')^2+(Kn')^2][(n'-1)^2+(Kn')^2]^2[(n'+1)^2+(Kn')^2]$$

$$+4(n')^2(n'-1)^2[1-(n')^2+(Kn')^2]^2 - 4n'(n'-1)[1-(n')^2+(Kn')^2]^2[(n'-1)^2+(Kn')^2]$$

$$+[1-(n')^2+(Kn')^2]^2[(n'-1)^2+(Kn')^2]^2\} + \tfrac{1}{3}[(n')^2-1](Kn')^2 \Bigg] . \tag{4.14}$$

The term within the braces simplifies to be equal to $16(n')^4(Kn')^4$; and using again (4.7), we obtain

$$\sum_{l'} |I_{100}^{n'l'0}|^2 = 2^8 K^2 (n')^7 \{\tfrac{1}{3}[(n')^2-1]+(Kn')^2\} \frac{[(n'-1)^2+(Kn')^2]^{n'-3}}{[(n'+1)^2+(Kn')^2]^{n'+3}} . \tag{4.15}$$

Equation (4.15) agrees exactly with the result obtained by Bethe[3] for $K=q/\alpha$. If we define

$$\varphi_{n'}(K) = \frac{1}{K^2}\sum_{l'} |I_{100}^{n'l'0}|^2 ,$$

then

$$\varphi_{n'}(0) = \frac{2^8 (n')^7 (n'-1)^{2n'-5}}{3(n'+1)^{2n'+5}} , \tag{4.16}$$

which is essentially the intensity formula for the Layman series. Bethe's formula given by (4.15) can also be obtained from (4.10) by using the Gegenbauer addition formula[12]

$$C^{m}_{n_1+n_2}(\cos\theta)=\frac{n_1!n_2!(2m+n_1+n_2-1)!}{(n_1+n_2)![(m-1)!]^2}\sum_k \frac{(-4)^k[(m+k-1)!]^2(2m+k-2)!(2m+2k-1)}{k!(2m+n_1+k-1)!(2m+n_2+k-1)!}(\sin\theta)^{2k}$$
$$\times C^{m+k}_{n_1-k}(\cos\theta)C^{m+k}_{n_2-k}(\cos\theta),\quad k=0,1,\ldots,\min\{n_1,n_2\}\ . \tag{4.17}$$

V. DISCRETE-CONTINUUM TRANSITIONS

In this case the final state in (4.1) is given by the physical continuum states $\psi_{\nu lm}(\mathbf{x})$ which are defined in terms of continuum basis states as

$$\psi_{\nu lm}(\mathbf{x})=\frac{1}{N_\nu}\exp(-i\theta_\nu L_{45})\varphi_{\nu lm}(\mathbf{x})\ , \tag{5.1}$$

where $\theta_\nu=\ln(a\nu)$. Consequently, the charge-form-factor function in spinor notation is given by

$$I^{\nu l'm'}_{nlm}=\frac{1}{N_\nu}\langle \nu l'm'|G(L_{56}-L_{46})|nlm\rangle\ ,$$
$$G=\exp(-i\theta_{\nu n}L_{45})\exp[-iKN(L_{35}-L_{34})]\ , \tag{5.2}$$
$$\theta_{\nu n}=\ln(N/\nu),\quad N=n/Z,\quad K=q_3m_p\ .$$

Using the action of L_{56} and L_{46} on $|nlm\rangle$ and the expansion (3.5), we obtain

$$I^{\nu l'm'}_{nlm}=\frac{1}{N_\nu}\{n\langle \nu l'm'|G|nlm\rangle-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\langle \nu l'm'|G|(n+1)lm\rangle$$
$$-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\langle \nu l'm'|G|(n-1)lm\rangle\} \tag{5.3}$$

$$I^{\nu l'm'}_{nlm}=\frac{1}{N_\nu}\Bigg[n\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\langle n_0l'm'|G|nlm\rangle-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\langle n_0l'm'|G|(n+1)lm\rangle$$
$$-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\langle n_0l'm'|G|(n-1)lm\rangle\Bigg],\quad n_0=1,2,\ldots,\infty\ . \tag{5.4}$$

As in the discrete case, the matrix elements of G in (5.4) can be computed[9] by using Euler rotations in SO(2,1) and using SO(4) and SO(2,1) representation functions:[10,11]

$$I^{\nu l'm}_{nlm}=\frac{1}{N_\nu}\Bigg[n\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\sum_{l_0}D^{[n_0-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n_0n}(\beta)D^{[n-1,0]}_{l_0ml}(\gamma)$$
$$-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\sum_{l_0}D^{[n_0-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n_0n+1}(\beta)D^{[n,0]}_{l_0ml}(\gamma)$$
$$-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)\sum_{l_0}D^{[n_0-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n_0n-1}(\beta)D^{[n-2,0]}_{l_0ml}(\gamma)\Bigg]\ ,$$
$$l_0=0,1,2,\ldots,\infty \tag{5.5}$$

where the Euler angles are given by (for $Z=1$)

$$\begin{aligned}
\sinh(\beta/2)&=\frac{1}{2}\frac{1}{\sqrt{\nu n}}[(\nu-n)^2+(K\nu n)^2]^{1/2}\ ,\\
\cosh(\beta/2)&=\frac{1}{2}\frac{1}{\sqrt{\nu n}}[(\nu+n)^2+(K\nu n)^2]^{1/2}\ ,\\
\sinh\beta\sin\alpha&=-Kn\ ,\\
\sinh\beta\cos\alpha&=\frac{1}{2\nu n}[\nu^2-n^2+(K\nu n)^2]\ ,\\
\sinh\beta\sin\gamma&=K\nu\ ,\\
\sinh\beta\cos\gamma&=-\frac{1}{2\nu n}[n^2-\nu^2+(K\nu n)^2]\ .
\end{aligned}\tag{5.6}$$

The summation over n_0 can be performed using again Euler rotation in SO(2,1) and the identity[9]

$$\sum_{n_0}V^{l'+1}_{\nu n_0}(-i\pi/2)D^{[n_0-1,0]}_{l'ml_0}(\alpha)V^{l_0+1}_{n_0n}(\beta)$$
$$=\sum_L D^{[n-1,0]}_{l_0mL}(\alpha_2)V^{L+1}_{n\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)\ ,$$
$$L=0,1,\ldots,n-1 \tag{5.7}$$

$$\begin{aligned}
2\cosh^2(\beta_2/2)&=i\sinh\beta\cos\alpha+1\ ,\\
2\sinh^2(\beta_2/2)&=i\sinh\beta\cos\alpha-1\ ,\\
\sinh\beta_2\sin\alpha_2&=-i\sin\alpha\ ,\\
\sinh\beta_2\cos\alpha_2&=i\cosh\beta\cos\alpha\ ,\\
\sinh\beta_2\sinh\gamma_2&=-i\sinh\beta\sin\alpha\ ,\\
\sinh\beta_2\cosh\gamma_2&=i\cosh\beta\ .
\end{aligned}\tag{5.8}$$

The SO(2,1) representation functions $V^{L+1}_{n\nu}(\beta_2)$ are given by (3.6). The special values we will need are the following:

$$V^1_{1\nu}(\beta_2)=\mu(i\nu)\sqrt{i\nu}[\tanh(\beta_2/2)]^{i\nu-1}[\cosh(\beta_2/2)]^{-2}\ ,$$

$$V^1_{2\nu}(\beta_2)=\mu(i\nu)\left[\frac{1}{\sqrt{2}}\sqrt{i\nu}(i\nu-1)[\tanh(\beta_2/2)]^{i\nu-2}[\cosh(\beta_2/2)]^{-4}-\sqrt{2}\sqrt{i\nu}[\tanh(\beta_2/2)]^{i\nu}[\cosh(\beta_2/2)]^{-2}\right]\ , \tag{5.9}$$

$$V^2_{2\nu}(\beta_2)=-\mu(i\nu)\frac{1}{\sqrt{6}}[(i\nu-1)(i\nu)(i\nu+1)]^{1/2}[\tanh(\beta_2/2)]^{i\nu-2}[\cosh(\beta_2/2)]^{-4}\ ,$$

$$\mu(i\nu)=\frac{\exp[-i(\pi/2)(i\nu-1)]}{\sqrt{2}\sqrt{\sin[\pi(i\nu)]}}\ .$$

Also in (5.7) the SO(3,1) representation functions are given by[13]

$$D^{[i\nu-1,0]}_{Lml'}(\gamma_2)=\sqrt{(2L+1)(2l'+1)}\sum_{m_1,m_2}\begin{bmatrix}\frac{1}{2}(i\nu-1) & \frac{1}{2}(i\nu-1) & L\\ m_1 & m_2 & m\end{bmatrix}\begin{bmatrix}\frac{1}{2}(i\nu-1) & \frac{1}{2}(i\nu-1) & l'\\ m_1 & m_2 & m\end{bmatrix}\exp[-i\gamma_2(m_1-m_2)]\ ,$$

$$D^{[i\nu-1,0]}_{00l'}(\gamma_2)=(2i)^{l'}\sqrt{2l'+1}\,l'!\left[\frac{(i\nu-l'-1)!}{(i\nu)(i\nu+l')!}\right]^{1/2}(i\sinh\gamma_2)^{l'}C^{l'+1}_{i\nu-l'-1}(\cosh\gamma_2)\ .$$

$$\begin{aligned}D^{[i\nu-1,0]}_{10l'}(\gamma_2)&=i\left[\frac{3}{(i\nu+1)(i\nu-1)}\right]^{1/2}\frac{d}{d\gamma_2}D^{[i\nu-1,0]}_{00l'}(\gamma_2)\\&=(2i)^{l'}\sqrt{2l'+1}\,l'!\left[\frac{-3(i\nu-l'-1)!}{(i\nu+1)(i\nu-1)(i\nu)(i\nu+l')!}\right]^{1/2}\\&\quad\times[l'(i\sinh\gamma_2)^{l'-1}\cosh\gamma_2C^{l'+1}_{i\nu-l'-1}(\cosh\gamma_2)-2(l'+1)(i\sinh\gamma_2)^{l'+1}C^{l'+2}_{i\nu-l'-2}(\cosh\gamma_2)]\ .\end{aligned}\tag{5.10}$$

Equation (5.5) becomes after the summation over n_0

$$\begin{aligned}I^{\nu l'm}_{nlm}=\frac{1}{N_\nu}\Bigg[&n\sum_L\sum_{l_0}D^{[n-1,0]}_{l_0mL}(\alpha_2)V^{L+1}_{n\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)D^{[n-1,0]}_{l_0ml}(\gamma)\\&-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\sum_L\sum_{l_0}D^{[n,0]}_{l_0mL}(\alpha_2)V^{L+1}_{n+1\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)D^{[n,0]}_{l_0ml}(\gamma)\\&-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\sum_L\sum_{l_0}D^{[n-2,0]}_{l_0mL}(\alpha_2)V^{L+1}_{n-1\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)D^{[n-2,0]}_{l_0ml}(\gamma)\Bigg]\ .\end{aligned}\tag{5.11}$$

Now, the summation over l_0 is immediate in view of the addition theorem of SO(4) representation functions given in (4.9). Hence we obtain a simple form for charge factors as

$$\begin{aligned}I^{\nu l'm}_{nlm}=\frac{1}{N_\nu}\Bigg\{&n\sum_L D^{[n-1,0]}_{lmL}(\alpha_2-\gamma)V^{L+1}_{n\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)\\&-\tfrac{1}{2}[(n-l)(n+l+1)]^{1/2}\sum_L D^{[n,0]}_{lmL}(\alpha_2-\gamma)V^{L+1}_{n+1\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)\\&-\tfrac{1}{2}[(n+l)(n-l-1)]^{1/2}\sum_L D^{[n-2,0]}_{lmL}(\alpha_2-\gamma)V^{L+1}_{n-1\nu}(\beta_2)D^{[i\nu-1,0]}_{Lml'}(\gamma_2)\Bigg]\ ,\end{aligned}\tag{5.12}$$

where $L=0,1,2,\ldots,n-1$ or n or $n-2$, as the case may be. If the initial state is the ground state $|100\rangle$, then equation (5.12) becomes

$$I^{\nu l'0}_{100}=\frac{1}{N_\nu}\left[V^1_{1\nu}(\beta_2)D^{[i\nu-1,0]}_{00l'}(\gamma_2)-\frac{1}{\sqrt{2}}V^1_{2\nu}(\beta_2)D^{[i\nu-1,0]}_{00l'}(\gamma_2)\cos(\alpha_2-\gamma)-\frac{1}{\sqrt{2}}V^2_{2\nu}(\beta_2)D^{[i\nu-1,0]}_{10l'}(\gamma_2)[i\sin(\alpha_2-\gamma)]\right]\ . \tag{5.13}$$

We now use (5.6), (5.8), (5.9), and (5.10) and rewrite (5.13) as

$$\begin{aligned}I^{\nu l'0}_{100}=\frac{1}{N_\nu}\mu(i\nu)(2i)^{l}\frac{\sqrt{2l'+1}(l'+1)!}{(8i\nu)}&\left[\frac{(i\nu-l'-1)!}{(i\nu+l')!}\right]^{1/2}(i\sinh\gamma_2)^{l'}[\tanh(\beta_2/2)]^{i\nu}[\cosh(\beta_2/2)\sinh(\beta_2/2)]^{-3}\\&\times\{4(i\nu)(i\nu+1)[\sinh(\beta_2/2)]^2C^{l'+2}_{i\nu-l'-1}(\cosh\gamma_2)+8\nu^2[\sinh(\beta_2/2)\cosh(\beta_2/2)]C^{l'+2}_{i\nu-l'-2}(\cosh\gamma_2)\\&+4(i\nu)(i\nu-1)[\cosh(\beta_2/2)]^2C^{l'+2}_{i\nu-l'-3}(\cosh\gamma_2)\}\ ,\end{aligned}\tag{5.14}$$

where we have used, as in the discrete case, the identities of the Gegenbauer polynomials (4.9a) in terms of the hyperbolic cosine. Furthermore, we have for arbitrary n

$$\sinh(\beta_2/2)=\frac{1}{2}\frac{1}{\sqrt{n(i\nu)}}\{(i\nu-n)^2+[K(i\nu)n]^2\}^{1/2},$$
$$\cosh(\beta_2/2)=\frac{1}{2}\frac{1}{\sqrt{n(i\nu)}}\{(i\nu+n)^2+[K(i\nu)n]^2\}^{1/2},$$
$$i\sinh\gamma_2=[2(i\nu)n](Kn)\{(i\nu-n)^2+[K(i\nu)n]^2\}^{-(1/2)}\{(i\nu+n)^2+[K(i\nu)n]^2\}^{-(1/2)},$$
$$\cosh\gamma_2=\{(i\nu)^2-n^2+[K(i\nu)n]^2\}\{(i\nu-n)^2+[K(i\nu)n]^2\}^{-(1/2)}\{(i\nu+n)^2+[K(i\nu)n]^2\}^{-(1/2)}. \tag{5.15}$$

We now substitute (5.15) into (5.14) and using $N_\nu=i\nu$ and the definition of $\mu(i\nu)$ given in (5.9) we obtain with

$$\cosh\gamma_2=\{(i\nu)^2-1+[K(i\nu)]^2\}\{(i\nu-1)^2+[K(i\nu)]^2\}^{-(1/2)}\{(i\nu+1)^2+[K(i\nu)]^2\}^{-(1/2)}, \tag{5.16}$$

the final expression for the charge-form-factor function as

$$I_{100}^{\nu l'0}=i^{l'}2^{2l'+3}(i\nu)^{l'+1}\sqrt{2l'+1}(l'+1)!\frac{\exp[-i(\pi/2)(i\nu-1)]}{\sqrt{2}\sqrt{\sin[\pi(i\nu)]}}\left[\frac{(i\nu-l'-1)!}{(i\nu+l')!}\right]^{1/2}K^{l'}\frac{\{(i\nu-1)^2+[K(i\nu)]^2\}^{(i\nu-l'-3)/2}}{\{(i\nu+1)^2+[K(i\nu)]^2\}^{(i\nu+l'+3)/2}}$$
$$\times\Big[(i\nu+1)[(i\nu-1)^2+(K(i\nu))^2]C_{i\nu-l'-1}^{l'+2}(\cosh\gamma_2)-2(i\nu)\{(i\nu-1)^2+[K(i\nu)]^2\}^{1/2}$$
$$\times\{(i\nu+1)^2+[K(i\nu)]^2\}^{1/2}C_{i\nu-l'-2}^{l'+2}(\cosh\gamma_2)+(i\nu-1)\{(i\nu+1)^2+[K(i\nu)]^2\}C_{i\nu-l'-3}^{l'+2}(\cosh\gamma_2)\Big]. \tag{5.17}$$

Comparing (4.10) and (5.17) we see that the analytic continuation is given by

$$I_{100}^{\nu l'0}=\frac{\exp[-i(\pi/2)(i\nu-1)]}{\sqrt{2}\sqrt{\sin[\pi(i\nu)]}}I_{100}^{n'l'0}\Big|_{n'\to i\nu}. \tag{5.18}$$

We now square the general charge form factor (5.12) and sum over l'. The summation over l' can be trivially done using the orthonormality properties of SO(3,1) representation functions. Consequently, we obtain a result similar to the discrete case (4.12):

$$\sum_{l'=0}^{\infty}|I_{nlm}^{\nu l'}|^2=\frac{1}{N_\nu^2}\Big\{n^2\sum_L|D_{lmL}^{[n-1,0]}(\delta_2)V_{n\nu}^{L+1}(\beta_2)|^2$$
$$-\frac{n}{2}[(n-l)(n+l+1)]^{1/2}\sum_L\{D_{lmL}^{[n-1,0]}(\delta_2)V_{n\nu}^{L+1}(\beta_2)[D_{lmL}^{[n,0]}(\delta_2)]^*[V_{n+1\nu}^{L+1}(\beta_2)]^*$$
$$+[D_{lmL}^{[n-1,0]}(\delta_2)]^*[V_{n\nu}^{L+1}(\beta_2)]^*D_{lmL}^{[n,0]}(\delta_2)V_{n+1\nu}^{L+1}(\beta_2)\}$$
$$-\frac{n}{2}[(n+l)(n-l-1)]^{1/2}\sum_L\{D_{lmL}^{[n-1,0]}(\delta_2)V_{n\nu}^{L+1}(\beta_2)[D_{lmL}^{[n-2,0]}(\delta_2)]^*[V_{n-1\nu}^{L+1}(\beta_2)]^*$$
$$+[D_{lmL}^{[n-1,0]}(\delta_2)]^*[V_{n\nu}^{L+1}(\beta_2)]^*D_{lmL}^{[n-2,0]}(\delta_2)V_{n-1\nu}^{L+1}(\beta_2)\}$$
$$+\tfrac{1}{4}[(n-l)(n+l+1)]\sum_L|D_{lmL}^{[n,0]}(\delta_2)V_{n+1\nu}^{L+1}(\beta_2)|^2$$
$$+\tfrac{1}{4}[(n-l)(n+l+1)(n+l)(n-l-1)]^{1/2}$$
$$\times\sum_L\{D_{lmL}^{[n,0]}(\delta_2)V_{n+1\nu}^{L+1}(\beta_2)[D_{lmL}^{[n-2,0]}(\delta_2)]^*[V_{n-1\nu}^{L+1}(\beta_2)]^*$$
$$+[D_{lmL}^{[n,0]}(\delta_2)]^*[V_{n+1\nu}^{L+1}(\beta_2)]^*D_{lmL}^{[n-2,0]}(\delta_2)V_{n-1\nu}^{L+1}(\beta_2)\}$$
$$+\tfrac{1}{4}[(n+l)(n-l-1)]^{1/2}\sum_L|D_{lmL}^{[n-2,0]}(\delta_2)V_{n-1\nu}^{L+1}(\beta_2)|^2\Big\},\quad \delta_2=\alpha_2-\gamma. \tag{5.19}$$

However, if the initial state is the ground state $|100\rangle$, we obtain a simple expression as in the discrete case (4.13):

$$\sum_{l'=0}^{\infty}|I_{100}^{\nu l'0}|^2=\frac{1}{N_\nu^2}\Big[|V_{1\nu}^1(\beta_2)|^2-\frac{1}{\sqrt{2}}\{V_{1\nu}^1(\beta_2)[V_{2\nu}^1(\beta_2)]^*+[V_{1\nu}^1(\beta_2)]^*V_{2\nu}^1(\beta_2)\}\cos\delta_2$$
$$+\tfrac{1}{2}|V_{2\nu}^1(\beta_2)|^2\cos^2\delta_2+\tfrac{1}{2}|V_{2\nu}^2(\beta_2)|^2\sin^2\delta_2\Big], \tag{5.20}$$

where, for arbitrary n, the Euler angle $\delta_2=\alpha_2-\gamma$ is given by [from (5.8) and (5.15)]

$$\sinh\beta_2\cos\delta_2=\frac{1}{2(i\nu)n}\{(i\nu)^2-n^2+[K(i\nu)n]^2\}\ ,$$
$$\sinh\beta_2\sin\delta_2=K(i\nu)\ . \tag{5.21}$$

Now, substituting (5.9) into (5.20) and simplifying exactly as in the discrete case, we obtain

$$\sum_{l'=0}^{\infty}|I_{100}^{\nu l'0}|^2=\frac{il^{\pi\nu}}{2\sinh(\pi\nu)}2^8(i\nu)^7K^2\left[\frac{[(i\nu)^2-1]}{3}+[K(i\nu)]^2\right]\frac{\{(i\nu-1)^2+[K(i\nu)]^2\}^{i\nu-3}}{\{(i\nu+1)^2+[K(i\nu)]^2\}^{i\nu+3}}\ . \tag{5.22}$$

This equation (5.22) is essentially obtained from the discrete case (4.15) by using the analytic continuation (5.18). In order to obtain Bethe's result,[3] we take $K=q/\alpha$ and $\nu=\alpha/\kappa$. Since (5.22) is integrable in ν, the integrability in κ requires a measure $d\nu/d\kappa=-\alpha/\kappa^2$. Using Bethes' variables we see

$$2\sinh(\pi\nu)=\exp\left[\pi\frac{\alpha}{\kappa}\right]\left[1-\exp\left[-2\pi\frac{\alpha}{\kappa}\right]\right]\ ,$$

$$(i\nu)^7K^2=-i\frac{\alpha^5q^2}{\kappa}\ ,$$

$$\frac{(i\nu)^2-1}{3}+[K(i\nu)]^2=-\frac{1}{\kappa^2}\left[\frac{(\alpha^2+\kappa^2)}{3}+q^2\right]\ ,$$

$$\frac{\{(i\nu-1)^2+[K(i\nu)]^2\}^{i\nu}}{\{(i\nu+1)^2+[K(i\nu)]^2\}^{i\nu}}=\exp\left[-2\frac{\alpha}{\kappa}\tan^{-1}\left[\frac{2\alpha\kappa}{\alpha^2+q^2-\kappa^2}\right]\right]\ ,$$

Since

$$\ln\left[\frac{1+ix}{1-ix}\right]=i2\tan^{-1}(x)\ ,$$

$$\{(i\nu-1)^2+[K(i\nu)]^2\}^{-3}\{(i\nu+1)^2+[K(i\nu)]^2\}^{-3}=\kappa^{12}[(q+\kappa)^2+\alpha^2]^{-3}[(q-\kappa)^2+\alpha^2]^{-3}\ . \tag{5.23}$$

We finally obtain

$$\sum_{l'=0}^{\infty}|I_{100}^{\alpha/\kappa l'0}|^2=\frac{2^8\kappa q^2\alpha^6[\frac{1}{3}(\alpha^2+\kappa^2)+q^2]}{[(q+\kappa)^2+\alpha^2]^3[(q-\kappa)^2+\alpha^2]^3}\times\frac{\exp\left[-2\frac{\alpha}{\kappa}\tan^{-1}\left[\frac{2\alpha\kappa}{\alpha^2+q^2-\kappa^2}\right]\right]}{\left[1-\exp\left[-2\pi\frac{\alpha}{\kappa}\right]\right]}\equiv|\epsilon_\kappa(q)|^2\ . \tag{5.24}$$

The square of the coordinate matrix element is given by[3]

$$|x_{0\kappa}|^2=\frac{1}{q^2}\lim_{q\to0}|\epsilon_\kappa(q)|^2\ ,$$
$$=\frac{2^8}{3}\frac{\alpha^6\kappa}{(\alpha^2+\kappa^2)^5}\frac{\exp\left[-4\frac{\alpha}{\kappa}\tan^{-1}\left[\frac{\kappa}{\alpha}\right]\right]}{\left[1-\exp\left[-2\pi\frac{\alpha}{\kappa}\right]\right]} \tag{5.25}$$

since

$$\tan^{-1}\left[\frac{2\alpha\kappa}{\alpha^2\kappa^2}\right]=2\tan^{-1}\left[\frac{\kappa}{\alpha}\right]\ .$$

Equations (5.24) and (5.25) are in exact agreement with the results of Bethe.[3] Equation (5.24) may also be obtained from (5.20) by using the Gegenbauer addition formula (4.17) in terms of hyperbolic cosines with the summation over k in (4.17) ranging from 0 to ∞.

VI. CONTINUUM-CONTINUUM TRANSITIONS

In this case the initial and final states in (4.1) are given by the physical continuum states $\psi_{\nu lm}(\mathbf{x})$, as defined in (5.1). The charge-form-factor function in spinor notation is therefore given by

$$I_{\nu lm}^{\nu'l'm'}=\frac{1}{N_{\nu'}}\langle\nu'l'm'|G|L_{56}-L_{46}|\nu lm\rangle\ ,$$
$$G=\exp[-i\Theta_{\nu'\nu}L_{45}]\exp[-iK\nu(L_{35}-L_{34})]\ , \tag{6.1}$$
$$\Theta_{\nu'\nu}=\ln(\nu/\nu')\ .$$

The action of L_{56} and L_{46} on the continuum basis state $|\nu lm\rangle$ is given by

$$L_{46}|\nu lm\rangle=\nu|\nu lm\rangle\ ,$$
$$L_{56}|\nu lm\rangle=\tfrac{1}{2}[(-i\nu-l)(-i\nu+l+1)]^{1/2}|\nu+1lm\rangle+\tfrac{1}{2}[(-i\nu+l)(-i\nu-l-1)]^{1/2}|\nu-1lm\rangle\ . \tag{6.2}$$

Therefore the charge form factor becomes

$$I_{\nu lm}^{\nu'l'm'}=\frac{1}{N_{\nu'}}\{-\nu\langle\nu'l'm'|G|\nu lm\rangle+\tfrac{1}{2}[(-i\nu-l)(-i\nu+l+1)]^{1/2}\langle\nu'l'm'|G|\nu+1lm\rangle+\tfrac{1}{2}[(-i\nu+l)(-i\nu-l-1)]^{1/2}\langle\nu'l'm'|G|\nu-1lm\rangle\}\ . \tag{6.3}$$

The matrix elements of G can be computed as before using Euler rotations in SO(2,1) and using SO(3,1) representation functions.[13] We obtain

$$I_{\nu l m}^{\nu' l' m} = \frac{-1}{N_{\nu'}} \left[\nu \sum_{l_0} D_{l' m l_0}^{[i\nu'-1,0]}(\alpha) V_{\nu'\nu}^{l_0+1}(\beta) D_{l_0 m l}^{[i\nu-1,0]}(\gamma) - \tfrac{1}{2}[(-i\nu - l)(-i\nu + l + 1)]^{1/2} \sum_{l_0} D_{l' m l_0}^{[i\nu'-1,0]}(\alpha) V_{\nu'\nu+1}^{l_0+1}(\beta) D_{l_0 m l}^{[i\nu,0]}(\gamma) - \tfrac{1}{2}[(-i\nu + l)(-i\nu - l - 1)]^{1/2} \sum_{l_0} D_{l' m l_0}^{[i\nu'-1,0]}(\alpha) V_{\nu'\nu-1}^{l_0+1}(\beta) D_{l_0 m l}^{[i\nu-2,0]}(\gamma) \right], \quad l_0 = 0, 1, \ldots, \infty \tag{6.4}$$

$$\begin{aligned}
\sinh(\beta/2) &= \frac{1}{2} \frac{1}{\sqrt{(i\nu')(-i\nu)}} \{[i\nu' - (-i\nu)]^2 + [K(i\nu')(-i\nu)]^2\}^{1/2}, \\
\cosh(\beta/2) &= \frac{1}{2} \frac{1}{\sqrt{(i\nu')(-i\nu)}} \{[i\nu' + (-i\nu)]^2 + [K(i\nu')(-i\nu)]^2\}^{1/2}, \\
\sinh\beta \sinh\alpha &= -K(-i\nu), \\
\sinh\beta \cosh\alpha &= \frac{1}{2(i\nu')(-i\nu)} \{(i\nu')^2 - (-i\nu)^2 + [K(i\nu)(-i\nu)]^2\}, \\
\sinh\beta \sinh\gamma &= K(i\nu'), \\
\sinh\beta \cosh\gamma &= -\frac{1}{2\sqrt{(i\nu')(-i\nu')}} \{(-i\nu)^2 - (i\nu')^2 + [K(i\nu)(-i\nu')]^2\}.
\end{aligned} \tag{6.5}$$

The SO(2,1) representation function is given by[3]

$$\begin{aligned}
V_{\nu'\nu}^{l+1}(\beta) = {} & \frac{1}{2\sin[\pi(i\nu - i\nu')]} \left[\begin{bmatrix} l - i\nu \\ l - i\nu' \end{bmatrix} \begin{bmatrix} l + i\nu' \\ l + i\nu \end{bmatrix} \right]^{1/2} [i \tanh(\beta/2)]^{i\nu - i\nu'} [\cosh(\beta/2)]^{2l} \\
& \times {}_2F_1[-l + i\nu', -l - i\nu; i\nu' - i\nu + 1; \tanh^2(\beta/2)] \\
& + \frac{1}{2\sin[\pi(-i\nu + i\nu')]} \left[\begin{bmatrix} l + i\nu \\ l + i\nu' \end{bmatrix} \begin{bmatrix} l - i\nu' \\ l - i\nu \end{bmatrix} \right]^{1/2} [-i \tanh(\beta/2)]^{-i\nu + i\nu'} [\cosh(\beta/2)]^{2l} \\
& \times {}_2F_1[-l - i\nu', -l + i\nu; -i\nu' + i\nu + 1; \tanh^2(\beta/2)], \quad V_{\nu'\nu\pm1}^{l+1}(\beta) = V_{\nu'\nu}^{l+1}(\beta)|_{i\nu \to i\nu \pm 1}.
\end{aligned} \tag{6.6}$$

By comparing (6.4) with the discrete case (4.6) and also the SO(2,1) representation functions given in (4.9) and (6.6), we establish the analytic continuation

$$I_{\nu l m}^{\nu' l' m'} = \frac{1}{2} \left[\frac{1}{2\sin[\pi(i\nu - i\nu')]} I_{nlm}^{n'l'm'}|_{\substack{n \to i\nu \\ n' \to i\nu'}} + \text{c.c.} \right]. \tag{6.7}$$

Furthermore, as in the previous two cases, we can square (6.4) and sum over l' using the orthonormality properties of SO(3,1) representation functions.[13] When $K \to 0$, the Euler angles α and γ go to zero and hence the SO(3,1) representation functions give $\delta_{l'l_0}$ and δ_{ll_0}, and consequently the charge form factor (6.4) becomes

$$I_{\nu l m}^{\nu' l m}(K=0) = -\frac{1}{N_{\nu'}} \{\nu V_{\nu'\nu}^{l+1}(\beta) - \tfrac{1}{2}[(-i\nu - l)(-i\nu + l + 1)]^{1/2} V_{\nu'\nu+1}^{l+1}(\beta) - \tfrac{1}{2}[(-i\nu + l)(-i\nu - l - 1)]^{1/2} V_{\nu'\nu-1}^{l+1}(\beta)\}. \tag{6.8}$$

VII. CONCLUSIONS

We have obtained exact analytic results by a group-theoretical method for the unsolved problems concerning the form factors mentioned in the Introduction. The final results are (4.6) and sum formula (4.12) for discrete-discrete transitions, the special case (4.15) being the Bethe's result; Eqs. (5.11) and (5.19), the corresponding formulas for discrete-continuum transitions with the special case (5.22) again being the Bethe's result; and, finally, Eq. (6.4), the result for the continuum-continuum transitions. Now that analytic formulas for the form factors from the ground state are obtained, arbitrary form factors can be evaluated by computer if need be. The above form factors in these cases are the analytic continuation of each other.

Besides the classic Massey-Mohr paper,[7] there are a number of other results dealing with the same or related topics.[14]

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