# Continuous-variable representation of dynamic groups in quantum systems

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The continuous-variable representation of the dynamic group is discussed in general terms, based on the differential of the group manifold. The unified formalism of various continuous-variable representations of the dynamic group is presented, and the essential common features of different continuous-variable representation theories are exploited from the general mathematical-physical point of view. The right-hand isomorphic representation and the left-hand anti-isomorphic representation of the dynamic group, and their relationship and physical implication are discussed in detail. Specialization of the general formalism to the function spaces of the group, based on both the Hilbert space of quantum-mechanical states and the von Neumann space of the quantum-statistical states (density matrices), leads to the results of both the generator coordinate approach to the dynamic-group representation and the generalized quantum characteristic function method. Techniques for calculating the continuous-variable representation of the group algebra are proposed and illustrated with the Lipkin SU(2), the Elliott SU(3), and its extension Sp(6) symmetries as examples. The results obtained for these dynamic groups are instructive for both physicists and mathematicians.

## I. INTRODUCTION

The dynamic-group approach to the investigation of collective behavior of quantum many-body systems (especially nuclear many-body systems) is becoming more and more attractive and significant in quantum physics. In nuclear physics, we have the seniority theory of the quasispin model[1] and the Lipkin two-level model[2] [SU(2) dynamic symmetry], the Elliott SU(3) model[3] and its extension, the Sp(6) model,[4] the interacting-boson model[5] (IBM) [SU(6) dynamic symmetry], the SO(8) model of Ginocchio[6] and the fermion dynamical-symmetry model (FDSM) of Wu and Feng *et al.*[7] There are two basic problems in the dynamic-group approach to quantum many-body problems. Firstly, one has to identify the relevant dynamic symmetry dominating a specific quantum system. This is a dynamical problem. Secondly, given a quantum system with its dynamic symmetry identified, one should work out a proper representation of the dynamic group, solve the problem and gain physical information from this representation. This is a problem of representation and solution.

In the past decade, extensive studies of nuclear collective phenomena have shown that nuclear many-body systems have certain kinds of dynamic symmetries which dominate nuclear collective motion.[8] These nuclear collective modes can be described in terms of interacting bosons (algebraic model), or in terms of variable mean field (geometric model). Suppose the dynamic group of a nuclear system is identified on the microscopic (fermion) level, we still need the boson (algebraic) representation and the continuous-variable (geometric) representation of the group, because only in these representations the collective variables (the order parameters of the collective modes) are manifested explicitly. The continuous-variable representation of the dynamic group has recently become a powerful tool which has been used to explore the connection between macroscopic and microscopic nuclear collective models,[4,9] to establish the relationship among different phenomenological nuclear collective models.[10] Recently, it has also been used to study nuclear phase transitions.[11] In brief, to achieve a better understanding of nuclear collective phenomena, one has to study dynamic groups and their continuous-variable representations; extensive efforts have been made in this respect.

The theories of continuous-variable representations of dynamic groups can be classified into three main categories: (i) The coherent-state theory and boson representations (CS-BR).[12–19] (ii) The generator coordinate approach to the dynamic-group representation (DGR-GCM).[20,21] (iii) The generalized quantum characteristic function method (GQCF).[22] From a comparative investigation of the above three approaches, one has found out that the DGR-GCM is the generalization of the CS-BR.[21] Both the DGR-GCM and the CS-BR works in the Hilbert space of quantum-mechanical states, while the GQCF, different from the above two, works in the von

 

Neumann space of quantum-statistical states (density matrices). However, there exist common foundations and close links among the above three approaches. It is one of the purposes of this paper to explore the common features and to present a unified formalism for the continuous-variable representation of dynamic groups, which contains the above three approaches as special cases.

This paper and the previous paper[21] (on DGR-GCM) are sister papers. However, the present paper emphasizes the general aspects of the topic, and is different from and new in comparing with the DGR-GCM regarding the following points.

(1) Based on differential operation on the group manifold, the continuous-variable representations of the group generators are calculated. The right isomorphic representation and the left anti-isomorphic representation of the group are given. While the DGR-GCM only involves the right representation, the left representation is precisely the isomorphic representation of the corresponding intrinsic group $\bar{G}$.[23] The interesting relationship between these two representations is established.

(2) The detailed results are presented and discussed for the representations in the function spaces of the group (coset) based on both the Hilbert space and the von Neumann space which are the only two base spaces of physical importance and meaning. When the function space of the group is constructed from the Hilbert space, the results of the DGR-GCM are obtained and extended. If the function space of the group is constructed from the von Neumann space, the results of the GQCF are reproduced and generalized. In all the above cases, the right and left representations of the group and their relationship are presented in detail.

(3) The techniques for calculating the differential representation of the group algebra are illustrated with special emphasis on the case where the group element is written in noncanonical form. Most results are new compared to the previous paper.[21]

The paper is organized as follows: In Sec. II, after introducing the definition of differential operations on the group manifold, the right and the left representations are calculated and discussed, respectively. The differential representations of the group algebra on the coset submanifold are given in Sec. III in a version parallel to Sec. II. In Sec. IV, continuous-variable representations in function spaces of the group (coset) based on both the Hilbert space and the von Neumann space are presented. The results of both the DGR-GCM and the GQCF are reproduced and generalized from a more general point of view so that the common features of the two approaches are exploited. Techniques for calculating the differential operators of the group algebra are illustrated in Sec. V by taking the Lipkin SU(2), the Elliott SU(3) and its extension Sp(6) as examples.

## II. DIFFERENTIAL AND DIFFERENTIAL REPRESENTATION OF THE GROUP GENERATORS ON THE GROUP MANIFOLD

Let $G$ be a Lie group, the collection of the group elements $U(g)$ spans a topological space (called group space) $x_G$ which is a differentiable manifold. Any group element $U(g')$ can also be considered to be an operator acting on the group manifold $x_G$. It is evident that the group space $x_G$ is closed under the action of $U(g')$. In this manifold, one can define differential operation and thus calculate differential representations of the group generators—the Dyson representation[24] of the group.

### A. Differential operation on the group manifold

Let $g$ be the group parameters of $r$ dimensions, $g=(g^1,g^2,\ldots,g^r)$. We are now defining the differential $dU(g)$ and the partial derivative $\partial U(g)/\partial g^m$. For clarity, we take the canonical form of the group element

$$U(g)=\exp(ig\circ x),\quad g\circ x=\sum_m g^m x_m \ , \tag{2.1}$$

and $x_m$ are generators of the group.[25]

Construct an operator $\hat{U}(\lambda)$ such as

$$\hat{U}(\lambda)=\exp(-i\lambda g\circ x)\exp[i\lambda(g+dg)\circ x] \tag{2.2}$$

so that

$$\begin{aligned}\frac{d\hat{U}(\lambda)}{d\lambda}&=\exp(-i\lambda g\circ x)[i(-g\circ x+g\circ x+dg\circ x)]\\&\quad\times\exp[i\lambda(g+dg)\circ x]\\&=\exp(-i\lambda g\circ x)(idg\circ x)\exp[i\lambda(g+dg)\circ x]\end{aligned} \tag{2.3}$$

and

$$\hat{U}(0)=I \ . \tag{2.4}$$

In view of Eq. (2.4), we have the integral solution of Eq. (2.3),

$$\begin{aligned}\hat{U}(\lambda)=I+\int_0^\lambda &\exp(-i\mu g\circ x)(idg\circ x)\\&\times\exp[i\mu(g+dg)\circ x]d\mu \ .\end{aligned} \tag{2.5}$$

Keeping terms to first order in $dg$, making use of Eq. (2.2) and multiplying both sides by $\exp(i\lambda g\circ x)$ and then setting $\lambda=1$, we obtain

$$\exp[i(g+dg)\circ x]=\exp(ig\circ x)+\int_0^1\exp[i(1-\mu)g\circ x](idg\circ x)\exp(i\mu g\circ x)d\mu \tag{2.6}$$

or

$$\begin{aligned}dU(g)\equiv U(g+dg)-U(g)&=\exp[i(g+dg)\circ x]-\exp(ig\circ x)\\&=\int_0^1\exp[i(1-\mu)g\circ x](idg\circ x)\exp(i\mu g\circ x)d\mu=\sum_m\frac{\partial U(g)}{\partial g^m}dg^m \ ,\end{aligned} \tag{2.7}$$

where

$$\frac{\partial U(g)}{\partial g^m} \equiv \int_0^1 \exp[i(1-\mu)g\circ x]i\frac{\partial(g\circ x)}{\partial g^m}\exp(i\mu g\circ x)d\mu$$
$$= \int_0^1 \exp[i(1-\mu)g\circ x]ix_m \exp(i\mu g\circ x)d\mu \ . \qquad (2.8)$$

Equations (2.7) and (2.8) are the definitions of the differential $dU(g)$ and the partial derivatives $\partial U(g)/\partial g^m$ on the group manifold.

### B. Right and left differential representation of the generators in the group space

Define the right differential representation of the generator $x_m$ as

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]U(g)\equiv \lim_{\delta g^m\to 0}\frac{1}{\delta g^m}[U(g)U(\delta g^m)-U(g)]=U(g)(ix_m) \ . \qquad (2.9)$$

To calculate $D_{R,m}$ we form

$$U(g)U(\delta g^m)-U(g)=U(\varphi(g,\delta g^m))-U(g)$$
$$\equiv U(g+d^Rg)-U(g)=\sum_l \frac{\partial U(g)}{\partial g^l}d^Rg^l=\sum_l \Theta^l_{R,m}\frac{\partial U}{\partial g^l}\delta g^m \qquad (2.10)$$

with

$$\Theta^l_{R,m}(g)\equiv \left.\frac{\partial \varphi^l(g,b)}{\partial b^m}\right|_{b=0} \ . \qquad (2.11)$$

$\varphi(a,b)$ describes the composition law of the group $G$ in parameter space. Combining (2.9) and (2.10) leads to

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]U(g)=\sum_l \Theta^l_{R,m}(g)\frac{\partial}{\partial g^l}U(g) \ , \qquad (2.12)$$

i.e.,

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]=\sum_l \Theta^l_{R,m}(g)\frac{\partial}{\partial g^l} \qquad (2.13)$$

is the right representation of $ix_m$. It is easy to prove that $D_{R,m}(g,\partial/\partial g)$ is isomorphic to $ix_m$,

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]D_{R,n}\left[g,\frac{\partial}{\partial g}\right]U(g)=U(g)(ix_m)(ix_n) \ . \qquad (2.14)$$

Therefore $-iD_{R,m}$ is the continuous-variable representation of the Lie algebra $x_m$ in the group space.

The left differential representation of the generators in the group space is defined through

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]U(g)= \lim_{\delta g^m\to 0}\frac{1}{\delta g^m}[U(\delta g^m)U(g)-U(g)]$$
$$=(ix_m)U(g) \ . \qquad (2.15)$$

Repeating the steps used for calculating $D_{R,m}$, we find

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]=\sum_l \left.\frac{\partial\varphi^l(b,g)}{\partial b^m}\right|_{b=0}\frac{\partial}{\partial g^l}$$
$$\equiv \sum_l \Theta^l_{m,L}(g)\frac{\partial}{\partial g^l} \ . \qquad (2.16)$$

$D_{m,L}(g,\partial/\partial g)$ is anti-isomorphic to $ix_m$,

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]D_{n,L}\left[g,\frac{\partial}{\partial g}\right]U(g)=(ix_n)(ix_m)U(g) \ . \qquad (2.17)$$

It is known that the intrinsic group $\bar{G}$ is anti-isomorphic to[23] $G$. Therefore the left representation $D_{m,L}$ of $ix_m$ is precisely the isomorphic representation of the intrinsic group $\bar{G}$, with generators $\bar{x}_m$,

$$D_{m,L}D_{n,L}\leftrightarrow(i\bar{x}_m)(i\bar{x}_n) \ . \qquad (2.18)$$

It is easy to show that the right representation $D_{R,m}$ commutes with the left representation $D_{m,L}$. In fact, from

$$D_{R,m}U(g)=U(g)(ix_m), \quad D_{n,L}U(g)=(ix_n)U(g) \ , \qquad (2.19)$$

we obtain after multiplying with $(ix_m)$ from the right and with $(ix_n)$ from the left, respectively,

$$D_{R,m}D_{n,L}=D_{n,L}D_{R,m} \ . \qquad (2.20)$$

Equation (2.20) just reflects the fact that group $G$ and its intrinsic group $\bar{G}$ commute

$$[G,\bar{G}]=0, \quad [x_m,\bar{x}_n]=0 \ . \qquad (2.21)$$

### C. Differential representation of the generators with the group element in noncanonical form

The preceding discussions are confined to the canonical form of the group element and the differential representation of the generators is usually rather complicated. For practical use, the noncanonical (Eulerian) form is more convenient. For SO(3), the Euler form of the group element is

$$U(g)=\exp(i\alpha J_z)\exp(i\beta J_y)\exp(i\gamma J_z) \ . \qquad (2.22)$$

For the Eulerian form, every exponent contains only one generator and the partial derivative of the group element

$\partial U(g)/\partial g^m$ can be taken as usual functions except that the order of operators cannot be changed. For instance, from Eq. (2.22) we have

$$\frac{\partial U(g)}{\partial \alpha}=iJ_z U(g)\ ,$$
$$\frac{\partial U(g)}{\partial \beta}=\exp(i\alpha J_z)(iJ_y)\exp(-i\alpha J_z)U(g)\ , \quad (2.23)$$
$$\frac{\partial U(g)}{\partial \gamma}=U(g)(iJ_z)\ .$$

All the formulas of $D_{R,m}$ and $D_{m,L}$ are still valid, noting that the functions $\varphi(a,b)$, $\Theta^l_{R,m}$, and $\Theta^l_{m,L}$ should be calculated according to the noncanonical form. For the SO(3), it follows that

$$U(g+dg)=U(g)U(\delta g)$$
$$=\exp(i\varphi^\alpha J_z)\exp(i\varphi^\beta J_y)\exp(i\varphi^\gamma J_z)\ , \quad (2.24)$$

$$\varphi^\alpha(g,\delta g)=\alpha+d\alpha\ ,$$
$$\varphi^\beta(g,\delta g)=\beta+d\beta\ , \quad (2.25)$$
$$\varphi^\gamma(g,\delta g)=\gamma+d\gamma\ .$$

In practical cases, one usually avoids calculating $\varphi(a,b)$. A trick is used to obtain $D_{R,m}$ and $D_{m,L}$; examples will be found in Sec. V.

## III. DIFFERENTIAL AND DIFFERENTIAL REPRESENTATION OF THE GROUP GENERATORS ON THE COSET SUBMANIFOLD

Any group element can be factorized according to a certain subgroup $S$,

$$U(g)=S(g_S)\Omega_R(g_R),\quad U(g)=\Omega_L(g_L)S(g_S)\ , \quad (3.1)$$

where $\Omega_R(g_R)$ and $\Omega_L(g_L)$ are right and left coset elements with respect to the subgroup $S$ respectively. $g_S$, $g_R$, and $g_L$ are parameters of the subgroup manifold $S(g_S)$ and of coset submanifolds $\Omega_R(g_R)$ and $\Omega_L(g_L)$. It should be pointed out that $\Omega_R(g_R)$ and $\Omega_L(g_L)$ are different submanifolds in the group space and they are not closed under the action of the group element $U(g)$. In order to define the differential operation and calculate the corresponding differential representation of the group generators in the coset subspace, we first evaluate for, e.g., the right differential representation

$$\Omega_R(g_R)\exp(i\delta g^m x_m)=U(g+d^Rg)=S(d^Rg_S)\Omega_R(g_R+d^Rg)$$
$$=S(d^Rg_S)\left[\Omega_R(g_R)+\sum_l \Theta^l_{R,m}(g_R)\frac{\partial\Omega_R(g_R)}{\partial g^l_R}\delta g^m\right]\ , \quad (3.2)$$

where

$$\Theta^l_{R,m}(g_R)\equiv\left.\frac{\partial\varphi^l_R(g,b)}{\partial b^m}\right|_{b=0,g=g_R}\ . \quad (3.3)$$

In the limit $\delta g^m\to 0$, we have

$$\lim_{\delta g^m\to 0} S(d^Rg_S)=I\ . \quad (3.4)$$

The differential and derivative operations on the coset submanifold are defined in the following. If the coset elements are written in canonical forms, the differential and partial derivatives are defined according to Sec. II A, Eqs. (2.7) and (2.8), while for the Eulerian forms, they are defined according to Sec. II C, Eq. (2.24). Therefore we have

$$d\Omega_R(g_R)=\Omega_R(g_R+d^Rg)-\Omega_R(g_R)$$
$$=\delta g^m\sum_l\Theta^l_{R,m}(g_R)\frac{\partial\Omega_R(g_R)}{\partial g^l_R}\ . \quad (3.5)$$

The right differential representation of $x_m$ is defined as

$$D_{R,m}\left[g_R,\frac{\partial}{\partial g_R}\right]\Omega_R(g_R)$$
$$\equiv\lim_{\delta g^m\to 0}\frac{1}{\delta g^m}[\Omega_R(g_R)\exp(i\delta g^m x_m)-\Omega_R(g_R)]$$
$$=\Omega_R(g_R)(ix_m)\ , \quad (3.6)$$

which according to (3.2) and (3.4) gives

$$D_{R,m}\left[g_R,\frac{\partial}{\partial g_R}\right]=\sum_l\Theta^l_{R,m}(g_R)\frac{\partial}{\partial g^l_R}\ . \quad (3.7)$$

Equation (3.7) has the same structure as (2.13) but is defined in coset space, with the number of parameters reduced. Similarly, for the left representation, we have

$$D_{m,L}\left[g_L,\frac{\partial}{\partial g_L}\right]=\sum_l\Theta^l_{m,L}(g_L)\frac{\partial}{\partial g^l_L}\ . \quad (3.8)$$

It is easy to show that $D_{R,m}$ are the continuous-variable representation of the group generators in the right coset subspace and are isomorphic to $x_m$, while $D_{m,L}$ are the continuous-variable representation of the intrinsic algebra $\bar{x}_m$ and anti-isomorphic to $x_m$,

$$D_{R,n}D_{R,m}\Omega_R(g_R)=\Omega_R(g_R)(ix_n)(ix_m)\ ,$$
$$D_{n,L}D_{m,L}\Omega_L(g_L)=(ix_m)(ix_n)\Omega_L(g_L)\ . \quad (3.9)$$

Since $\Omega_R(g_R)$ acted upon by $D_{R,m}$ and $\Omega_L(g_L)$ acted upon by $D_{m,L}$ are different subspaces of the group manifold, $D_{R,m}$ and $D_{m,L}$ are defined in different spaces. However, if the connection between $\Omega_R$ and $\Omega_L$ is known, the relationship between $D_{R,m}$ and $D_{m,L}$ can be established accordingly. In Sec. V, we shall give examples concerning this point.

## IV. DIFFERENTIAL AND DIFFERENTIAL REPRESENTATION OF THE GENERATORS IN THE FUNCTION SPACE OF THE GROUP

As the object on which the group element $U(g)$ and the coset element $\Omega(g)$ act is the Hilbert space of quantum-mechanical states $|\psi\rangle$, or the von Neumann space of quantum-statistical states (the density matrices) $\rho$, we can construct the function space of the group (coset) by means of the inner-product operation in the Hilbert space or the trace operation in the von Neumann space. In the former case, we obtain the continuous-variable representation of quantum states in the parameter space of the dynamical group, which can be used to describe the collective motions induced by the dynamical group. In the latter case, we obtain the generalized quantum characteristic function which can be used to calculate the ensemble averages of physical quantities.

### A. Function space of the group (coset) based on the Hilbert space

The results of Secs. II and III can be transplanted to the function space of the group (coset) based on the Hilbert space, if the starting-state vector $|\phi_0\rangle$ in the inner product is properly chosen.

#### 1. Functions of the group in the Hilbert space

In order to contain all the information of the group representations in the function space, the starting-state vector $|\phi_0\rangle$ should be chosen such that it has not any symmetry with respect to the group $G$, i.e., it should contain components of all irreducible representations of $G$. Let $|\psi\rangle$ be an arbitrary state vector of the Hilbert space; the functions of the group can be written as

$$F^R(g)\equiv\langle\phi_0|U(g)|\psi\rangle,\quad F^L(g)\equiv\langle\psi|U(g)|\phi_0\rangle\ , \tag{4.1}$$

for the right and left representation, respectively. The representations of the generators are, in obvious notation,

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]F^R(g)\equiv\langle\phi_0|U(g)ix_m|\psi\rangle = \sum_l\Theta^l_{R,m}(g)\frac{\partial}{\partial g_l}F^R(g)\ , \tag{4.2}$$

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]F^L(g)\equiv\langle\psi|ix_m U(g)|\phi_0\rangle = \sum_l\Theta^l_{m,L}(g)\frac{\partial}{\partial g_l}F^L(g)\ . \tag{4.3}$$

#### 2. Functions of the coset in the Hilbert space

Sometimes the physical problem is confined to a subspace of the whole Hilbert space, which contains a state vector $|\phi_0\rangle$ invariant with respect to a subgroup $S(g_S)$ of $G$,

$$S(g_S)|\phi_0\rangle=|\phi_0\rangle\ . \tag{4.4}$$

The functions of the coset $G/S$ are then defined as

$$F^R(g_R)\equiv\langle\phi_0|S(g_S)\Omega_R(g_R)|\psi\rangle =\langle\phi_0|\Omega_R(g_R)|\psi\rangle\ , \tag{4.5a}$$

$$F^L(g_L)\equiv\langle\psi|\Omega_L(g_L)S(g_S)|\phi_0\rangle =\langle\psi|\Omega_L(g_L)|\phi_0\rangle\ , \tag{4.5b}$$

where $|\psi\rangle$ is an arbitrary state in the subspace. Only those irreducible representations which contain states invariant with respect to the subgroup $S$ are considered in this case, i.e., some information has been lost.

The continuous-variable representations of $ix_m$ are

$$D_{R,m}\left[g_R,\frac{\partial}{\partial g_R}\right]F^R(g_R)=\langle\phi_0|\Omega_R(g_R)ix_m|\psi\rangle\ , \tag{4.6}$$

$$D_{m,L}\left[g_L,\frac{\partial}{\partial g_L}\right]F^L(g_L)=\langle\psi|ix_m\Omega_L(g_L)|\phi_0\rangle\ , \tag{4.7}$$

with $D_{R,m}$ and $D_{m,L}$ from Eqs. (3.7) and (3.8). Since $\Omega_R(g_R)$ and $\Omega_L(g_L)$ are different, $D_{R,m}$ and $D_{m,L}$ act on different function spaces. However, the relationship between $F^R$ and $F^L$, $D_{R,m}$ and $D_{m,L}$ can be established as will be shown in Sec. V.

#### 3. Equations of motion and expectation values of physical variables

Suppose the Hamiltonian of a nuclear system be $H(ix_m)$ and the Schrödinger equation is

$$H(ix_m)|\psi\rangle=E|\psi\rangle\ , \tag{4.8a}$$

$$\langle\psi|H(ix_m)=E\langle\psi|\ . \tag{4.8b}$$

Multiplying Eq. (4.8a) from the left with $\langle\phi_0|U(g)$ and Eq. (4.8b) from the right with $U(g)|\phi_0\rangle$, we obtain

$$H^R(D_{R,m})F^R(g)=EF^R(g)\ , \tag{4.9a}$$

$$H^L(D_{m,L})F^L(g)=EF^L(g)\ . \tag{4.9b}$$

It should be noted that $H^R(D_{R,m},\ldots,D_{R,n},\ldots)$ is isomorphic to $H(ix_m,\ldots,ix_n,\ldots)$ and $H^L(D_{m,L},\ldots,D_{n,L},\ldots)$ is anti-isomorphic to $H(ix_n,\ldots,ix_m\ldots)$.

For the function space of the coset, Eqs. (4.9a) and (4.9b) are still valid, but the parameters should be restricted to the coset space $g\to g_R$ (or $g\to g_L$).

If $U(g)$ is unitary, we have the closure relation and the orthogonality relation as follows:[21]

$$\int U^\dagger(g)|\phi_0\rangle\langle\phi_0|U(g)d\mu(g)=I\ , \tag{4.10}$$

$$\langle\phi_0|U(g)U^\dagger(g')|\phi_0\rangle=\Delta(g-g')\ , \tag{4.11}$$

where the $\Delta$ function is subject to

$$\int\Delta(g-g')d\mu(g')=1\ , \tag{4.12a}$$

$$\int f(g')\Delta(g-g')d\mu(g')=f(g)\ . \tag{4.12b}$$

It is easy to obtain

$$|\psi\rangle = \int U^{\dagger}(g)|\phi_0\rangle\langle\phi_0|U(g)|\psi\rangle d\mu(g)$$
$$= \int U^{\dagger}(g)|\phi_0\rangle F^R(g) d\mu(g) . \quad (4.13)$$

In the above equation, $|\psi\rangle$ is a state vector in fermion space, while $F^R(g)$ is the corresponding state in the continuous-variable function space. Therefore Eq. (4.13) is the transformation from the continuous-variable function space $F^R(g)$ to the fermion space $|\psi\rangle$. The expectation value of a physical variable $O(ix_m)$ is evaluated as follows:

$$\langle\psi|O(ix_m)|\psi\rangle = \langle F^R(g)|O^R(D_{R,m})|F^R(g)\rangle$$
$$= \int [F^R(g)]^* O^R(D_{R,m}) F^R(g) d\mu(g) . \quad (4.14)$$

Here $O^R(D_{R,m})$ is isomorphic to $O(ix_m)$. For the left representation we have similar results, but $O^L(D_{m,L})$ is anti-isomorphic to $O(ix_m)$.

#### 4. Representation in the coherent-state space

The coherent-state space is a special coset space in which the subgroup $S$ is the maximal stability subgroup of $G$ and $|\phi_0\rangle$ is the stability state of $S$ (the extreme state of an irreducible representation). Let $x_l$ be generators of $S$, $x_\mu$ be raising operators, $x_{-\mu}$ the lowering operators, and $|\phi_0\rangle$ be the lowest-weight state; then

$$x_l|\phi_0\rangle = \Lambda_l|\phi_0\rangle, \quad x_l \in S \quad (4.15a)$$

$$x_{-\mu}|\phi_0\rangle = 0 , \quad (4.15b)$$

$$S(g_S)|\phi_0\rangle = \exp\left[i\sum_l g_S^l x_l\right]|\phi_0\rangle$$
$$= \exp\left[i\sum_l g_S^l \Lambda_l\right]|\phi_0\rangle . \quad (4.15c)$$

Let

$$U^R(g) = \exp\left[i\sum_l g_S^l x_l\right] \exp\left[i\sum_\tau g_+^\tau x_\tau\right] \times \exp\left[i\sum_\mu g_-^{-\mu} x_{-\mu}\right] , \quad (4.16a)$$

$$U^L(g) = \exp\left[i\sum_\mu g_+^\mu x_\mu\right] \exp\left[i\sum_\tau g_-^{-\tau} x_{-\tau}\right] \times \exp\left[i\sum_l g_S^l x_l\right] , \quad (4.16b)$$

and define

$$F_c^R(g_-) = \langle\phi_0|U^R(g)|\psi\rangle|_{g_S=0} = \langle\phi_0|\Omega_R(g_-)|\psi\rangle , \quad (4.17a)$$

$$\Omega_R(g_-) = \exp\left[i\sum_\mu g_-^{-\mu} x_{-\mu}\right] ,$$

$$F_c^L(g_+) = \langle\psi|U^L(g)|\phi_0\rangle|_{g_S=0} = \langle\psi|\Omega_L(g_+)|\phi_0\rangle , \quad (4.17b)$$

$$\Omega_L(g_+) = \exp\left[i\sum_\mu g_+^\mu x_\mu\right] .$$

The right coherent-state representation of $ix_m$ is defined as

$$D_{R,m}\left[g_-, \frac{\partial}{\partial g_-}\right] F_c^R(g_-) \equiv \langle\phi_0|\Omega_R(g_-)(ix_m)|\psi\rangle$$
$$= \lim_{\delta g^m \to 0} \frac{1}{\delta g^m}\langle\phi_0|\Omega_R(g_-)\exp(i\delta g^m x_m) - \Omega_R(g_-)|\psi\rangle$$
$$= \lim_{\delta g^m \to 0} \frac{1}{\delta g^m}\langle\phi_0|\exp\left[i\sum_l dg_S^l x_l\right]\Omega_R(g_- + d^R g_-) - \Omega_R(g_-)|\psi\rangle|_{\delta g^m \neq 0, \delta g^l = 0(\neq m)}$$
$$= \left[i\sum_l \Theta_{R,m}^l(g_-)\Lambda_l + \sum_\mu \Theta_{R,m}^{-\mu}(g_-)\frac{\partial}{\partial g_-^{-\mu}}\right] F_c^R(g_-) , \quad (4.18a)$$

$$D_{R,m}\left[g_-, \frac{\partial}{\partial g_-}\right] = i\sum_l \Theta_{R,m}^l(g_-)\Lambda_l + \sum_\mu \Theta_{R,m}^{-\mu}(g_-)\frac{\partial}{\partial g_-^{-\mu}} . \quad (4.18b)$$

For the left representation, we have

$$D_{m,L}\left[g_+, \frac{\partial}{\partial g_+}\right] = i\sum_l \Theta_{m,L}^l(g_+)\Lambda_l + \sum_\tau \Theta_{m,L}^\tau(g_+)\frac{\partial}{\partial g_+^\tau} . \quad (4.19)$$

The relationship between $D_{R,m}$ and $D_{m,L}$, $F_c^R(g_-)$ and $F_c^L(g_+)$ can be found as will be seen in Sec. V C.

### B. Function space of the group (coset) based on the von Neumann space

To describe the quantum-statistical state we need the density matrices which span the von Neumann space. When the dynamic group acts on this space, by virtue of the trace operation, we can construct the generalized quantum characteristic functions of the group (coset) and define the continuous-variable representation in the constructed function space. This procedure leads to the results of the GQCF and its generalization.

#### 1. GQCF on the group space (Ref. 22)

Suppose $\rho$ is an arbitrary state vector in the von Neumann space, which is not symmetric at all with respect the group $G$. Then the generalized quantum characteristic functions are defined as

$$W^R(g) \equiv \mathrm{Tr}\rho U(g) = W^L(g) \equiv \mathrm{Tr} U(g)\rho = W(g) . \quad (4.20)$$

The right and left representations of $ix_m$ are readily obtained as follows:

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]W^R(g)\equiv \mathrm{Tr}[\rho U(g)(ix_m)]$$
$$=\sum_l \Theta^l_{R,m}(g)\frac{\partial}{\partial g^l}W^R(g) , \qquad (4.21)$$

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]W^L(g)\equiv \mathrm{Tr}[(ix_m)U(g)\rho]$$
$$=\sum_l \Theta^l_{m,L}(g)\frac{\partial}{\partial g^l}W^L(g) . \qquad (4.22)$$

Here $D_{R,m}$ and $D_{m,L}$ take the same form and possess the same properties as given in Sec. II B.

The von Neumann equation

$$i\hbar\dot{\rho}=[H,\rho],\quad H=H(ix_m) , \qquad (4.23)$$

can be easily transformed into the corresponding equation of the continuous-variable representation as follows:

$$i\hbar\dot{W}(g,t)=\mathrm{Tr}[H\rho U(g)]-\mathrm{Tr}[\rho H U(g)]$$
$$=[H^R(D_{R,m})-H^L(D_{m,L})]W(g,t) . \qquad (4.24)$$

Let

$$\rho(t)=\sum_n \omega_n(0)|\psi_n(t)\rangle\langle\psi_n(t)| , \qquad (4.25a)$$

with

$$i\hbar\frac{\partial|\psi_n(t)\rangle}{\partial t}=H|\psi_n(t)\rangle , \qquad (4.25b)$$

then

$$W(g,t)=\sum_n \omega_n(0)W_n(g,t) , \qquad (4.26)$$

$$W_n(g,t)=\langle\psi_n(t)|U(g)|\psi_n(t)\rangle , \qquad (4.27)$$

$$i\hbar\dot{W}_n(g,t)=[H^R(D_{R,m})-H^L(D_{m,L})]W_n(g,t) . \qquad (4.28)$$

Equations (4.24) and (4.28) indicate that the time evolution of the GQCF $W(g,t)$ and $W_n(g,t)$ is governed by the continuous-variable representation of the Hamiltonian $H^R(D_{R,m})$ of the dynamic group $G$ and of $H^L(D_{m,L})$ of the intrinsic dynamic group $\bar{G}$.

For the stationary case, we have

$$\rho=\sum_n \omega_n(0)|\psi_n(0)\rangle\langle\psi_n(0)| , \qquad (4.29)$$

$$H|\psi_n\rangle=E_n|\psi_n\rangle , \qquad (4.30)$$

$$W(g)=\sum_n \omega_n(0)W_n(g) , \qquad (4.31)$$

$$W_n(g)=\langle\psi_n(0)|U(g)|\psi_n(0)\rangle , \qquad (4.32)$$

where $W_n(g)$ is a bilinear function of $|\psi_n\rangle$ and $\langle\psi_n|$, and obeys the equations

$$H^R(D_{R,m})W_n(g)=E_nW_n(g) ,$$
$$H^L(D_{m,L})W_n(g)=E_nW_n(g) . \qquad (4.33)$$

The above equations show that $W_n(g)$ contains information of the dynamic symmetry $H(G)$ and its intrinsic counterpart $H(\bar{G})$, i.e., $W_n(g)$ contains information about both $|\psi_n\rangle$ and $\langle\psi_n|$.

For the average values of physical quantities, we have

$$\langle O(ix_m)\rangle=\mathrm{Tr}O(ix_m)\rho=[\mathrm{Tr}\rho U(g)O(ix_m)]_{g=0}=[\mathrm{Tr}O(ix_m)U(g)\rho]_{g=0}=O^R(D_{R,m})W(g)|_{g=0}$$
$$=O^L(D_{m,L})W(g)|_{g=0} , \qquad (4.34)$$

$$\langle\psi_n|O(ix_m)|\psi_n\rangle=\langle\psi_n|U(g)O(ix_m)|\psi_n\rangle|_{g=0}=\langle\psi_n|O(ix_m)U(g)|\psi_n\rangle|_{g=0}=O^R(D_{R,m})W_n(g)|_{g=0}$$
$$=O^L(D_{m,L})W_n(g)|_{g=0} , \qquad (4.35)$$

$$\langle O(ix_m)\rangle=\sum_n \omega_n O^R(D_{R,m})W_n(g)|_{g=0}=\sum_n \omega_n O^L(D_{m,L})W_n(g)|_{g=0} . \qquad (4.36)$$

### 2. *GQCF on the coset subspace*

Suppose $\rho_c$ is a state vector in a subspace, which is invariant with respect to the subgroup $S$,

$$S\rho_c=\rho_c S=\rho_c . \qquad (4.37)$$

In this case, a special quantum-statistical ensemble with symmetry $S$ is treated and the GQCF degenerates to the one in the coset space $G/S$. From Eqs. (3.1), (4.20), and (4.37), we have

$$W^R_c(g_R)=\mathrm{Tr}\rho_c\Omega_R(g_R) , \qquad (4.38a)$$

$$W^L_c(g_L)=\mathrm{Tr}\Omega_L(g_L)\rho_c , \qquad (4.38b)$$

the subscript $c$ denoting "coset." In general, $\Omega_R(g_R)\neq\Omega_L(g_L)$ and hence $W^R_c(g_R)\neq W^L_c(g_L)$. The representations of $ix_m$ are readily obtained from the results of Sec. III,

$$D_{R,m}\left[g_R,\frac{\partial}{\partial g_R}\right]W^R_c(g_R)\equiv\mathrm{Tr}[\rho_c\Omega_R(g_R)ix_m]$$
$$=\sum_l \Theta^l_{R,m}(g_R)\frac{\partial}{\partial g^l_R}W^R_c(g_R) , \qquad (4.39)$$

$$D_{m,L}\left[g_L,\frac{\partial}{\partial g_L}\right]W_c^L(g_L)\equiv \mathrm{Tr}[ix_m\Omega_L(g_L)\rho_c] = \sum_l \Theta^l_{m,L}(g_L)\frac{\partial}{\partial g_L^l}W_c^L(g_L)\ . \tag{4.40}$$

In general, the von Neumann equations for $W_c^R(g_R)$ and $W_c^L(g_L)$ do not exist. Only in the case $\Omega_R(g_R)=\Omega_L(g_L)$ and $W_c^R(g_R)=W_c^L(g_L)$, we can derive the following equation:

$$i\hbar\dot{W}_c(g_c,t)=[H^R(D_{R,m})-H^L(D_{m,L})]W_c(g_c,t)\ . \tag{4.41}$$

For the stationary case, we have the following results:

$$\rho_c=\sum_n \omega_n^c|\psi_n^c\rangle\langle\psi_n^c|\ , \tag{4.42}$$

$$H|\psi_n^c\rangle=E_n^c|\psi_n^c\rangle\ , \tag{4.43}$$

$$S|\psi_n^c\rangle=|\psi_n^c\rangle\ , \tag{4.44}$$

$$W_c(g_c)=\sum_n \omega_n^c W_n^c(g_c)\ , \tag{4.45}$$

$$W_n^c(g_c)=\langle\psi_n^c|\Omega(g_c)|\psi_n^c\rangle\ , \tag{4.46}$$

$$H^R(D_{R,m})W_n^c(g_c)=E_n^c W_n^c(g_c)\ , \tag{4.47a}$$

$$H^L(D_{m,L})W_n^c(g_c)=E_n^c W_n^c(g_c)\ , \tag{4.47b}$$

$$\langle O(ix_m)\rangle_c=\mathrm{Tr}\rho_c O(ix_m)=O^R(D_{R,m})W_c(g_c)|_{g_c=0} = O^L(D_{m,L})W_c(g_c)|_{g_c=0}\ , \tag{4.48}$$

$$\langle\psi_n^c|O(ix_m)|\psi_n^c\rangle=O^R(D_{R,m})W_n^c(g_c)|_{g_c=0} = O^l(D_{m,L})W_n^c(g_c)|_{g_c=0}\ . \tag{4.49}$$

#### 3. GQCF in the coherent-state space

To define the GQCF in the coherent-state space, we need state vectors with the following properties:

$$x_l\rho=\Lambda_l\rho=\rho x_l,\quad x_l\in S \tag{4.50}$$

$$x_{-\mu}\rho=0,\quad \rho x_\mu=0\ . \tag{4.51}$$

Now let

$$U(g)=\exp\left[i\sum_l g_S^l x_l\right]\exp\left[i\sum_\tau g_+^\tau x_\tau\right]\exp\left[i\sum_\mu g_-^{-\mu}x_{-\mu}\right] \tag{4.52}$$

and calculate

$$\mathrm{Tr}\rho U(g)=\mathrm{Tr}U(g)\rho=\exp\left[i\sum_l g_S^l\Lambda_l\right]\mathrm{Tr}\left[\rho\exp\left[i\sum_\mu g_-^{-\mu}x_{-\mu}\right]\right] = \exp\left[i\sum_l g_S^l\Lambda_l\right]\mathrm{Tr}\left[\exp\left[i\sum_\mu g_-^{-\mu}x_{-\mu}\right]\rho\right]=\exp\left[i\sum_l g_S^l\Lambda_l\right]\ . \tag{4.53}$$

The last step in Eq. (4.53) is based on Eq. (4.51) and indicates that $\mathrm{Tr}\rho U(g)$ is not a functional of $\rho$. The above result means that, because of the cyclic invariance of the trace operation, we cannot define any generalized quantum characteristic function by means of lowering or raising operators. In other words, the coherent-state space demands $\rho$ to be the extreme state vector, which is too restrictive to define a generalized quantum characteristic function according to Eq. (4.53).

## V. TECHNIQUES FOR CALCULATING THE CONTINUOUS-VARIABLE REPRESENTATION OF THE GROUP GENERATORS: ILLUSTRATIONS

The formalism presented in the above sections will be applied to practical cases where the various continuous-variable representations of the Lipkin SU(2), the Elliott SU(3), and its extension Sp(6) are calculated as illustrations. Techniques for calculating various differential representations of the generators are given in the examples and most results obtained here are new.

### A. Representations in the group space

#### 1. Group element in canonical form

Suppose the group element is written in canonical form (2.1). Let us calculate the commutator

$$\left[\frac{\partial}{\partial g^m},U(g)\right]=\exp(ig\circ x)\left[\exp(-ig\circ x)\frac{\partial}{\partial g^m}\exp(ig\circ x)-\frac{\partial}{\partial g^m}\right] = \exp(i\circ gx)\sum_l V_m^l(g)(ix_l)=\sum_l V_m^l(g)D_{R,l}\left[g,\frac{\partial}{\partial g}\right]U(g)\ , \tag{5.1a}$$

where the c-numbers $V^l_m(g)$ are defined through

$$V^l_m(g)(ix_l)=\exp(-ig\circ x)\frac{\partial}{\partial g^m}\exp(ig\circ x)-\frac{\partial}{\partial g^m} . \tag{5.1b}$$

Since det $\|V^l_m(g)\|\neq 0$, the inverse of $V$, $V^{-1}$, exists.[21] Hence

$$D_{R,m}\left[g,\frac{\partial}{\partial g}\right]=\sum_l V^{-1}(g)^l_m\frac{\partial}{\partial g^l} , \tag{5.2}$$

which is the right representation. In a similar way, we obtain the left representation of $ix_m$,

$$D_{m,L}\left[g,\frac{\partial}{\partial g}\right]=\sum_l [V^{-1}(-g)]^l_m\frac{\partial}{\partial g^l} . \tag{5.3}$$

*Example: SU(2).* Let $K_\alpha$ ($\alpha=1,2,3$) be generators of the SU(2) satisfying

$$[K_\alpha,K_\beta]=i\sum_\gamma \epsilon_{\alpha\beta\gamma},K_\gamma . \tag{5.4}$$

The group element in canonical form is

$$U(g)=\exp(ig\circ K),\quad g\circ K=\sum_\alpha g^\alpha K_\alpha . \tag{5.5}$$

It is not difficult to calculate

$$[K_\alpha,ig\circ K]=\sum_\gamma C_{\alpha\gamma}(g)K_\gamma=(CK)_\alpha , \tag{5.6a}$$

where the matrix $C(g)$ is defined as

$$C=\left[C_{\alpha\gamma}\right]=\left[-\sum_\beta g^\beta\epsilon_{\alpha\beta\gamma}\right]=\begin{bmatrix}0 & g^3 & -g^2\\ -g^3 & 0 & g^1\\ g^2 & -g^1 & 0\end{bmatrix} . \tag{5.6b}$$

By means of the identity

$$\exp(-ig\circ K)\frac{\partial}{\partial g^\sigma}\exp(ig\circ K)-\frac{\partial}{\partial g^\sigma}=\sum_{N=1}^{\infty}\frac{1}{N!}\mathcal{C}_N , \tag{5.7a}$$

where

$$\mathcal{C}_0=\frac{\partial}{\partial g^\sigma},\quad \mathcal{C}_N=[\mathcal{C}_{n-1},ig\circ K]=i(C^{N-1}K)_\sigma , \tag{5.7b}$$

we have

$$\exp(-ig\circ K)\frac{\partial}{\partial g^\sigma}\exp(ig\circ K)-\frac{\partial}{\partial g^\sigma}=\sum_{N=1}^{\infty}\frac{i}{N!}(C^{N-1}K)_\sigma=\sum_\nu V^\nu_\sigma(g)(iK_\nu) , \tag{5.8}$$

which is linear in the generators $K_\nu$, with

$$V(g)=\frac{1}{C}(e^C-1) . \tag{5.9}$$

From Eqs. (5.2) and (5.9), we obtain the right representation of $iK_\sigma$,

$$D_{R,\sigma}\left[g,\frac{\partial}{\partial g}\right]=\sum_\nu\left[\frac{1}{e^{C-1}}C\right]^\nu_\sigma\frac{\partial}{\partial g^\nu} . \tag{5.10}$$

For the left representation, we have

$$D_{\sigma,L}\left[g,\frac{\partial}{\partial g}\right]=\sum_\nu\left[\frac{-1}{e^{-C}-1}C\right]^\nu_\sigma\frac{\partial}{\partial g^\nu} . \tag{5.11}$$

### 2. Group element in noncanonical form

We still use the SU(2) as an example. Let $U(g)$ be in the Eulerian form,

$$U(g)=\exp(i\gamma K_3)\exp(i\beta K_2)\exp(i\alpha K_3) ,\quad g=(\alpha,\beta,\gamma) . \tag{5.12}$$

To calculate the right representation we evaluate, expanding the exponentials and using commutator (5.4) together with (2.9):

$$\frac{\partial}{\partial\alpha}U(g)=U(g)iK_3=D_{R,3}\left[g,\frac{\partial}{\partial g}\right]U(g) , \tag{5.13}$$

$$\frac{\partial}{\partial\beta}U(g)=U(g)\exp(-i\alpha K_3)(iK_2)\exp(i\alpha K_3)=\left[(\cos\alpha)D_{R,2}\left[g,\frac{\partial}{\partial g}\right]-(\sin\alpha)D_{R,1}\left[g,\frac{\partial}{\partial g}\right]\right]U(g) , \tag{5.14}$$

$$\frac{\partial}{\partial\gamma}U(g)=U(g)\exp(-i\alpha K_3)\exp(-i\beta K_2)(iK_3)\exp(i\beta K_2)\exp(i\alpha K_3)=[(\cos\alpha)(\sin\beta)D_{R,1}+(\sin\alpha)(\sin\beta)D_{R,2}+(\cos\beta)D_{R,3}]U(g) . \tag{5.15}$$

The solution of the above equations is

$$D_{R,1}\left[g,\frac{\partial}{\partial g}\right]=i\mathcal{K}_1^D=-(\cos\alpha)(\cot\beta)\frac{\partial}{\partial\alpha}-(\sin\alpha)\frac{\partial}{\partial\beta}+\frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\ , \tag{5.16a}$$

$$D_{R,2}\left[g,\frac{\partial}{\partial g}\right]=i\mathcal{K}_2^D=-(\sin\alpha)(\cot\beta)\frac{\partial}{\partial\alpha}+(\cos\alpha)\frac{\partial}{\partial\beta}+\frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\ , \tag{5.16b}$$

$$D_{R,3}\left[g,\frac{\partial}{\partial g}\right]=i\mathcal{K}_3^D=\frac{\partial}{\partial\alpha}\ . \tag{5.16c}$$

These are precisely the angular momentum operators in the Eulerian angle form.

For the left representation, we obtain

$$D_{1,L}\left[g,\frac{\partial}{\partial g}\right]=(\cos\gamma)(\cot\beta)\frac{\partial}{\partial\gamma}+(\sin\gamma)\frac{\partial}{\partial\beta}-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\ , \tag{5.17a}$$

$$D_{2,L}\left[g,\frac{\partial}{\partial g}\right]=-(\sin\gamma)(\cot\beta)\frac{\partial}{\partial\gamma}+(\cos\gamma)\frac{\partial}{\partial\beta}+\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}\ , \tag{5.17b}$$

$$D_{3,L}\left[g,\frac{\partial}{\partial g}\right]=\frac{\partial}{\partial\gamma}\ . \tag{5.17c}$$

This is precisely the continuous-variable representation of the intrinsic group $\overline{\mathrm{SU}(2)}$ in the Eulerian angle form. It is well known that $D_{\alpha,L}$ is anti-isomorphic to $K_\alpha$ and isomorphic to $\overline{K}_\alpha$, and that SU(2) and $\overline{\mathrm{SU}(2)}$ commute.

### B. Representation in the coset space

To calculate the differential representation of the generators in the coset space, it is convenient to work in the function space of the coset where the invariant state vector $|\phi_0\rangle$ with respect to $S$ plays the role of the projectors onto the coset space.

#### *1. Example: SU(2)/SO(2)*

Let the group element be

$$U(g)=\exp(i\gamma K_3)\exp(i\beta K_2)\exp(i\alpha K_3)\ ,$$

where the SO(2) generator is $K_3$. The projector $P_{\Omega_R}$ onto the right coset space is equivalent to putting $\gamma=0$, and the projector $P_{\Omega_L}$ onto the left coset space is equivalent to putting $\alpha=0$,

$$P_{\Omega_R}U(g)=\exp(i\beta K_2)\exp(i\alpha K_3)=\Omega_R(g_R)\ ,\quad g_R=(\alpha,\beta) \tag{5.18a}$$

$$P_{\Omega_L}U(g)=\exp(i\gamma K_3)\exp(i\beta K_2)=\Omega_L(g_L)\ ,\quad g_L=(\gamma,\beta)\ . \tag{5.18b}$$

This procedure can be easily realized by introducing an SO(2) invariant state $|\phi_0\rangle$ in the Hilbert space,

$$K_3|\phi_0\rangle=0\ . \tag{5.19}$$

The functions of the coset SU(2)/SO(2) are defined as follows:

$$F^R(g_R)=\langle\phi_0|U(g)|\psi\rangle=\langle\phi_0|e^{i\beta K_2}e^{i\alpha K_3}|\psi\rangle\ , \tag{5.20a}$$

$$F^L(g_L)=\langle\psi|U(g)|\phi_0\rangle=\langle\psi|e^{i\gamma K_3}e^{i\beta K_2}|\phi_0\rangle\ . \tag{5.20b}$$

It is obvious that the right representation can only be defined in the function space $F^R(g_R)$ and the left one in the $F^L(g_L)$. By using the results of Sec. V A, we can easily obtain the representation of $K_\alpha$ in the coset space SU(2)/SO(2).

For the right representation, setting $\alpha=\varphi$, $\beta=\theta$, and $\gamma=0$, from Eqs. (5.20a) and (5.16), we obtain

$$F^R(g_R)=\langle\phi_0|e^{i\theta K_2}e^{i\varphi K_3}|\psi\rangle,\quad g_R=(\theta,\varphi) \tag{5.21}$$

and

$$D_{R,1}\left[g_R,\frac{\partial}{\partial g_R}\right]=i\mathcal{K}_1^D=-(\cos\varphi)(\cot\theta)\frac{\partial}{\partial\varphi}-(\sin\varphi)\frac{\partial}{\partial\theta}\ , \tag{5.22a}$$

$$D_{R,2}\left[g_R,\frac{\partial}{\partial g_R}\right]=i\mathcal{K}_2^D=-(\sin\varphi)(\cot\theta)\frac{\partial}{\partial\varphi}+(\cos\varphi)\frac{\partial}{\partial\theta}\ , \tag{5.22b}$$

$$D_{R,3}\left[g_R,\frac{\partial}{\partial g_R}\right]=i\mathcal{K}_3^D=\frac{\partial}{\partial\varphi}\ , \tag{5.22c}$$

i.e., the orbital-angular-momentum operators of a particule with spin zero.

For the left representation, setting $\beta=\theta$, $\gamma=\varphi$, and $\alpha=0$, from Eqs. (5.20b) and (5.17), we obtain

$$F^L(g_L)=\langle\psi|e^{i\varphi K_3}e^{i\theta K_2}|\phi_0\rangle,\quad g_L=(\theta,\varphi) \tag{5.23}$$

$$D_{1,L}\left[g_L,\frac{\partial}{\partial g_L}\right]=(\cos\varphi)(\cot\theta)\frac{\partial}{\partial\varphi}+(\sin\varphi)\frac{\partial}{\partial\theta}\ , \tag{5.24a}$$

$$D_{2,L}\left[g_L,\frac{\partial}{\partial g_L}\right]=-(\sin\varphi)(\cot\theta)\frac{\partial}{\partial\varphi}+(\cos\varphi)\frac{\partial}{\partial\theta}\ , \tag{5.24b}$$

$$D_{3,L}\left[g_L,\frac{\partial}{\partial g_L}\right]=\frac{\partial}{\partial\varphi}\ . \tag{5.24c}$$

This is the differential representation of the intrinsic group $\overline{\mathrm{SU}(2)}$ in the coset space. However, $D_{R,\alpha}(g_R,\partial/\partial g_R)$ and $D_{\alpha,L}(g_L,\partial/\partial g_L)$ have different definitions since they act on the different function spaces $F^R(g_R)$ and $F^L(g_L)$ which are related as follows:

$$F^L(g_L)=[F^R(-g_R)]^* . \tag{5.25}$$

It is easy to check that Eqs. (5.22) and (5.24) satisfy the relation

$$D^*_{\alpha,L}\left[-g_L,-\frac{\partial}{\partial g_L}\right]=-D_{R,\alpha}\left[g_R,\frac{\partial}{\partial g_R}\right] . \tag{5.26}$$

### 2. *Example: SU(3)/SO(3) in canonical form*

The Elliott SU(3) algebra is $SU(3)=\{Q_\mu,L_q\}$, where $Q_\mu$ are the Elliott quadrupole operators, and the angular momentum operators $L_q$ constitute the SO(3) algebra. The SU(3) group element can be written as

$$U^L(g)=\exp(i\alpha Q)\exp(i\Omega L) , \tag{5.27a}$$

$$U^R(g)=\exp(i\Omega L)\exp(i\alpha Q) , \tag{5.27b}$$

where

$$\alpha Q=\sum_\mu \alpha_\mu Q_{\bar\mu},\quad Q_{\bar\mu}=(-1)^\mu Q_{-\mu},\quad \mu=-2,-1,0,1,2$$

$$\Omega L=\sum_q \Omega_q L_{\bar q},\quad L_{\bar q}=(-1)^q L_{-q},\quad q=-1,0,1 .$$

Choosing $|\phi_0\rangle$ as the SO(3) invariant state vector, $L_q|\phi_0\rangle=0$, we have the functions of the coset SU(3)/SO(3) as follows:

$$F^R(g_R)=\langle\phi_0|U^R(g)|\psi\rangle=\langle\phi_0|e^{i\alpha Q}|\psi\rangle , \tag{5.28a}$$

$$F^L(g_L)=\langle\psi|U^L(g)|\phi_0\rangle=\langle\psi|e^{i\alpha Q}|\phi_0\rangle . \tag{5.28b}$$

It is obvious that

$$F^L(g_L)=[F^R(-g_R)]^* . \tag{5.28c}$$

In a previous paper,[21] we presented the right representation of the Elliott SU(3) in the coset SU(3)/SO(3) space,

$$\mathcal{L}^R_q=\left[B\frac{\partial}{\partial\alpha^*}\right]_q , \tag{5.29a}$$

$$\mathcal{Q}^R_\mu=-i\left[\left[3B^\dagger B\right]^{1/2}\cot(3B^\dagger B)^{1/2}\frac{\partial}{\partial\alpha^*}\right]_\mu , \tag{5.29b}$$

where

$$B=(B_{\nu\mu}),\quad B_{\nu\mu}=\sqrt{30}\,\alpha_{\bar d}(-1)^\nu\begin{bmatrix}2&1&2\\ \mu-\nu&\nu&-\mu\end{bmatrix} , \tag{5.29c}$$

where the subscript $d=\mu-\nu$, and

$$\frac{\partial}{\partial\alpha^*}=\left[\frac{\partial}{\partial\alpha_{\bar\mu}}\right],\quad Q=(Q_\mu),\quad L=(L_\mu)$$
$$(L_{-2}=L_2=0) . \tag{5.29d}$$

The left representation can be obtained in a similar way as that for the derivation of Eq. (5.26). It reads

$$\mathcal{L}^L_{\bar q}=\mathcal{L}^{R*}_q(-\alpha) , \tag{5.30a}$$

$$\mathcal{Q}^L_{\bar\mu}=\mathcal{Q}^{R*}_\mu(-\alpha) . \tag{5.30b}$$

### 3. *Example: SU(3)/SU(2) in noncanonical form*

Choose $|\phi_0\rangle$ to be a scalar of $SU(2)=\{Q_2+Q_{-2},L_z,Q_2-Q_{-2}\}$ subject to

$$Q_{\pm2}|\phi_0\rangle=L_z|\phi_0\rangle=0 , \tag{5.31a}$$

$$Q_0|\phi_0\rangle=\mathcal{E}|\phi_0\rangle . \tag{5.31b}$$

The group element of the SU(3) can be written as

$$U^R(g)=e^{-i\alpha'(Q_2+Q_{-2})}e^{-i\beta' L_z}e^{-i\gamma'(Q_2+Q_{-2})}e^{i\nu L_y}\times e^{i\gamma(Q_2+Q_{-2})}e^{i\beta L_z}e^{i\alpha(Q_2+Q_{-2})}e^{i\theta Q_0} , \tag{5.32a}$$

$$U^L(g)=e^{i\theta Q_0}e^{i\alpha(Q_2+Q_{-2})}e^{i\beta L_z}e^{i\gamma(Q_2+Q_{-2})}e^{i\nu L_y}\times e^{-i\gamma'(Q_2+Q_{-2})}e^{-i\beta' L_z}e^{-i\alpha'(Q_2+Q_{-2})} . \tag{5.32b}$$

The functions of the coset SU(3)/SU(2) are

$$F^R(g_R)=\langle\phi_0|U^R(g)|\psi\rangle=\langle\phi_0|e^{i\nu L_z}e^{i\gamma(Q_2+Q_{-2})}e^{i\beta L_z}e^{i\alpha(Q_2+Q_{-2})}\times e^{i\theta Q_0}|\psi\rangle , \tag{5.33a}$$

$$F^L(g_R)=\langle\psi|U^L(g)|\phi_0\rangle=\langle\psi|e^{i\theta Q_0}e^{i\alpha(Q_2+Q_{-2})}e^{i\beta L_z}e^{i\gamma(Q_2+Q_{-2})}\times e^{i\nu L_y}|\phi_0\rangle , \tag{5.33b}$$

where $g_R=g_L=\{\theta,\alpha,\beta,\gamma,\nu\}$. The generalization of the method for deriving SU(2) representation (5.12) and (5.16) leads to the following results for the right representation:

$$\mathcal{L}^R_x=i\left[\tfrac{3}{2}\sin(\theta+\alpha-\gamma)(\sin\beta)(\tan\nu)\frac{\partial}{\partial\theta}-\tfrac{1}{2}\sin(\theta+\alpha-\gamma)(\csc\beta)(\cot\nu)\frac{\partial}{\partial\alpha}+\cos(\theta+\alpha-\gamma)(\cos\beta)(\cot\nu)\frac{\partial}{\partial\beta}+\tfrac{1}{2}\sin(\theta+\alpha-\gamma)[(\csc\beta)(\cot\nu)+(\sin\beta)(\tan\nu)]\frac{\partial}{\partial\gamma}+\cos(\theta+\alpha-\gamma)(\sin\beta)\frac{\partial}{\partial\nu}\right] , \tag{5.34a}$$

$$\mathcal{L}_y^R = i\left[-\tfrac{3}{2}\sin(\theta-\alpha-\gamma)(\cos\beta)(\tan\nu)\frac{\partial}{\partial\theta} - \tfrac{1}{2}\sin(\theta-\alpha-\gamma)(\sec\beta)(\cot\nu)\frac{\partial}{\partial\alpha} + \cos(\theta-\alpha-\gamma)(\sin\beta)(\cot\nu)\frac{\partial}{\partial\beta} - \tfrac{1}{2}\sin(\theta-\alpha-\gamma)[(\sec\beta)(\cot\nu)+(\cos\beta)(\tan\nu)]\frac{\partial}{\partial\gamma} - \cos(\theta-\alpha-\gamma)(\cos\beta)\frac{\partial}{\partial\nu}\right] . \quad (5.34b)$$

$$\mathcal{L}_z^R = i\left[(\cot2\beta)(\sin2\alpha)\frac{\partial}{\partial\alpha} - (\cos2\alpha)\frac{\partial}{\partial\beta} - (\csc2\beta)(\sin2\beta)\frac{\partial}{\partial\gamma}\right] , \quad (5.34c)$$

$$(Q_2+Q_{-2})^R = -\sqrt{6}i\frac{\partial}{\partial\alpha}, \quad Q_0^R = -3i\frac{\partial}{\partial\theta} , \quad (5.34d)$$

$$(Q_2-Q_{-2})^R = \sqrt{6}\left[-(\cot2\beta)(\cos2\beta)\frac{\partial}{\partial\alpha} - (\sin2\alpha)\frac{\partial}{\partial\beta} + (\csc2\beta)(\cos2\alpha)\frac{\partial}{\partial\gamma}\right] , \quad (5.34e)$$

$$(Q_1+Q_{-1})^R = +(\tfrac{3}{2})^{1/2}\left[-3\cos(\theta+\alpha-\gamma)(\sin\beta)(\tan\nu)\frac{\partial}{\partial\theta} + \cos(\theta+\alpha-\gamma)(\csc\beta)(\cot\nu)\frac{\partial}{\partial\alpha} + 2\sin(\theta+\alpha-\gamma)(\cos\beta)(\cot\nu)\frac{\partial}{\partial\beta} - \cos(\theta+\alpha-\gamma)[(\csc\beta)(\cot\nu)+(\sin\beta)(\tan\nu)]\frac{\partial}{\partial\gamma} + 2\sin(\theta+\alpha-\gamma)(\sin\beta)\frac{\partial}{\partial\nu}\right] , \quad (5.34f)$$

$$(Q_1-Q_{-1})^R = (\tfrac{3}{2})^{1/2}i\left[3\cos(\theta-\alpha-\gamma)(\cos\beta)(\tan\nu)\frac{\partial}{\partial\theta} + \cos(\theta-\alpha-\gamma)(\sec\beta)(\cot\nu)\frac{\partial}{\partial\alpha} + 2\sin(\theta-\alpha-\gamma)(\sin\beta)(\cot\nu)\frac{\partial}{\partial\beta} + \cos(\theta-\alpha-\gamma)[(\sec\beta)(\cot\nu)+(\cos\beta)(\tan\nu)]\frac{\partial}{\partial\gamma} - 2\sin(\theta-\alpha-\gamma)(\cos\beta)\frac{\partial}{\partial\nu}\right] , \quad (5.34g)$$

which allow analytical solutions of all the SU(3) irreducible representation bases.[26,27]

For the left representation, the following relations hold:

$$F^L(g_L) = [F^R(-g_R)]^* , \quad (5.35a)$$

$$\mathcal{L}_i^L(g_L) = [\mathcal{L}_i^R(-g_R)]^*, \quad i=x,y,z \quad (5.35b)$$

$$Q_\mu^L(g_L) = [Q_\mu^R(-g_R)]^*, \quad \mu=\pm2,\pm1,0 . \quad (5.35c)$$

### C. Representation in the coherent-state space. Example: Sp(6)/$U(3)$

Let $\mathbf{r}_i$, $\mathbf{p}_i$ be the coordinate and momentum of the $i$th particle; the corresponding creation and annihilation operators can be constructed as follows:

$$a_{i\alpha}^\dagger = (r_{i\alpha} - ip_{i\alpha})/\sqrt{2}, \quad a_{i\alpha} = (r_{i\alpha}+ip_{i\alpha})/\sqrt{2} , \quad \alpha=x,y,z . \quad (5.36)$$

As an extension of the Elliott SU(3) group, the Sp(6) group consists of the following generators:

$$A_{\alpha\beta}^\dagger = \frac{1}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{i=1}^{A} a_{i\alpha}^\dagger a_{i\beta}^\dagger , \quad (5.37a)$$

$$A_{\alpha\beta} = \frac{1}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{i=1}^{A} a_{i\alpha}a_{i\beta} , \quad (5.37b)$$

$$C_{\alpha\beta} = \sum_{i=1}^{A}(a_{i\alpha}^\dagger a_{i\beta} + \tfrac{1}{2}\delta_{\alpha\beta}) , \quad (5.37c)$$

where the generators $C_{\alpha\beta}$ constitute the $U(3)$ subalgebra. Their commutators are

$$[C_{\alpha\beta}, C_{\gamma\epsilon}] = C_{\alpha\epsilon}\delta_{\beta\gamma} - C_{\gamma\beta}\delta_{\alpha\epsilon} . \quad (5.38a)$$

$$[C_{\alpha\beta}, A_{\gamma\epsilon}] = -\frac{1}{\sqrt{1+\delta_{\gamma\epsilon}}}(\delta_{\alpha\gamma}\sqrt{1+\delta_{\epsilon\beta}}A_{\epsilon\beta} + \delta_{\alpha\epsilon}\sqrt{1+\delta_{\gamma\beta}}A_{\gamma\beta}) , \quad (5.38b)$$

$$[A_{\alpha\beta}^\dagger, A_{\gamma\epsilon}] = -\frac{1}{\sqrt{(1+\delta_{\alpha\beta})(1+\delta_{\gamma\epsilon})}}(C_{\beta\gamma}\delta_{\alpha\epsilon} + C_{\alpha\gamma}\delta_{\beta\epsilon} + C_{\beta\epsilon}\delta_{\alpha\gamma} + C_{\alpha\epsilon}\delta_{\beta\gamma}) , \quad (5.38c)$$

$$[C_{\alpha\beta},A^\dagger_{\gamma\epsilon}]=\frac{1}{\sqrt{1+\delta_{\gamma\epsilon}}}[\delta_{\beta\gamma}\sqrt{1+\delta_{\alpha\epsilon}}A^\dagger_{\alpha\epsilon}+\delta_{\beta\epsilon}\sqrt{1+\delta_{\alpha\gamma}}A^\dagger_{\alpha\gamma}]\ , \tag{5.38d}$$

$$[A_{\alpha\beta},A_{\gamma\epsilon}]=[A^\dagger_{\alpha\beta},A^\dagger_{\gamma\epsilon}]=0\ . \tag{5.38e}$$

$A^\dagger_{\alpha\beta}$ are the cross-shell two-phonon excitation operators and responsible for the giant quadrupole and monopole resonance excitations. $C_{\alpha\beta}$ are the phonon-conversion operators and responsible for the surface oscillations. Suppose $|\phi_0\rangle=|N,N,N\rangle$ is the ground state of a nucleus with a closed shell in harmonic-oscillator basis, then

$$A_{\alpha\beta}|\phi_0\rangle=0\ , \tag{5.39a}$$

$$C_{\alpha\beta}|\phi_0\rangle=\Lambda_{\alpha\beta}|\phi_0\rangle,\quad \Lambda_{\alpha\beta}=\left[N+\frac{A}{2}\right]\delta_{\alpha\beta}\ , \tag{5.39b}$$

where $N$ is the total phonon number in each direction. Obviously $|\phi_0\rangle$ is the lowest-weight state of a certain irreducible representation of the Sp(6). To construct the coherent state of the Sp(6), we write

$$U^R(g)=\exp\left[\sum_{\alpha,\beta}g_S^{\alpha\beta}C_{\alpha\beta}\right]\exp(g^*_+A^\dagger)\exp(g_-A)\ , \tag{5.40a}$$

$$U^L(g)=\exp(g^*_+A^\dagger)\exp(g_-A)\exp\left[\sum_{\alpha,\beta}g_S^{\alpha\beta}C_{\alpha\beta}\right]\ , \tag{5.40b}$$

where

$$g^*_+A^\dagger=\sum_{\gamma,\epsilon}\frac{1+\delta_{\gamma\epsilon}}{\sqrt{2}}(g_+^{\gamma\epsilon})^*A^\dagger_{\gamma\epsilon}\ ,\qquad g_-A=\sum_{\gamma,\epsilon}\frac{1+\delta_{\gamma\epsilon}}{\sqrt{2}}g_-^{\gamma\epsilon}A_{\gamma\epsilon}\ . \tag{5.40c}$$

The functions of the coherent state are defined as

$$F^R(g_-)=\langle\phi_0|U^R(g)|\psi\rangle|_{g_S=0}=\langle\phi_0|e^{g_-A}|\psi\rangle\ , \tag{5.41a}$$

$$F^L(g_+)=\langle\psi|U^L(g)|\phi_0\rangle|_{g_S=0}=\langle\psi|e^{g^*_+A^\dagger}|\phi_0\rangle\ . \tag{5.41b}$$

We first calculate the right representation. From the equation

$$\frac{\partial}{\partial g_-^{\alpha\beta}}F^R(g_-)=\sqrt{2}(1+\delta_{\alpha\beta})\langle\phi_0|e^{g_-A}A_{\alpha\beta}|\psi\rangle =\sqrt{2}(1+\delta_{\alpha\beta})\mathcal{A}^R_{\alpha\beta}\left[g_-,\frac{\partial}{\partial g_-}\right]F^R(g_-) \tag{5.42a}$$

we have

$$\mathcal{A}^R_{\alpha\beta}\left[g_-,\frac{\partial}{\partial g_-}\right]=\frac{1}{\sqrt{2}(1+\delta_{\alpha\beta})}\frac{\partial}{\partial g_-^{\alpha\beta}}\ . \tag{5.42b}$$

Furthermore, from the equation

$$\begin{aligned}\mathcal{C}^R_{\alpha\beta}\left[g_-,\frac{\partial}{\partial g_-}\right]F^R(g_-)&=\langle\phi_0|e^{g_-A}C_{\alpha\beta}e^{-g_-A}e^{g_-A}|\psi\rangle\\&=\langle\phi_0|\left[C_{\alpha\beta}e^{g_-A}+\sum_\epsilon\sqrt{(1+\delta_{\alpha\epsilon})(1+\delta_{\beta\epsilon})}\sqrt{2}g_-^{\alpha\epsilon}e^{g_-A}A_{\epsilon\beta}\right]|\psi\rangle\\&=\left[\Lambda_{\alpha\beta}+\sum_\epsilon\sqrt{(1+\delta_{\alpha\epsilon})(1+\delta_{\beta\epsilon})}\sqrt{2}g_-^{\alpha\epsilon}\mathcal{A}^R_{\epsilon\beta}\left[g_-,\frac{\partial}{\partial g_-}\right]\right]F^R(g_-)\end{aligned} \tag{5.43a}$$

we get

$$\begin{aligned}\mathcal{C}^R_{\alpha\beta}\left[g_-,\frac{\partial}{\partial g_-}\right]&=\Lambda_{\alpha\beta}+\sum_\epsilon\sqrt{(1+\delta_{\alpha\epsilon})(1+\delta_{\beta\epsilon})}\sqrt{2}g_-^{\alpha\epsilon}\mathcal{A}^R_{\epsilon\beta}\\&=\Lambda_{\alpha\beta}+\sum_\epsilon\sqrt{(1+\delta_{\alpha\epsilon})/(1+\delta_{\beta\epsilon})}g_-^{\alpha\epsilon}\frac{\partial}{\partial g_-^{\beta\epsilon}}\ .\end{aligned} \tag{5.43b}$$

To get $(\mathcal{A}^R_{\alpha\beta})^\dagger$, we need to calculate

$$\begin{aligned}0&=\langle\phi_0|A^\dagger_{\alpha\beta}e^{g_-A}|\psi\rangle\\&=\langle\phi_0|e^{g_-A}\left[A^\dagger_{\alpha\beta}-(2N+A)\sqrt{2}g_-^{\alpha\beta}-\frac{2}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{\gamma,\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})(1+\delta_{\gamma\epsilon})}g_-^{\alpha\gamma}g_-^{\beta\epsilon}A_{\epsilon\gamma}\right]|\psi\rangle\\&=\left[(\mathcal{A}^R_{\alpha\beta})^\dagger-(2N+A)\sqrt{2}g_-^{\alpha\beta}-\frac{2}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{\gamma\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})(1+\delta_{\gamma\epsilon})}g_-^{\alpha\gamma}g_-^{\beta\epsilon}\mathcal{A}^R_{\epsilon\gamma}\right]F^R(g_-)\end{aligned} \tag{5.44a}$$

and obtain

$$(\mathcal{A}^{R}_{\alpha\beta})^{\dagger}\left[g_{-},\frac{\partial}{\partial g_{-}}\right]=(2N+A)\sqrt{2}g_{-}^{\alpha\beta}+\frac{2}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{\gamma,\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})(1+\delta_{\gamma\epsilon})}g_{-}^{\alpha\gamma}g_{-}^{\beta\epsilon}\mathcal{A}^{R}_{\epsilon\gamma}$$

$$=(2N+A)\sqrt{2}g_{-}^{\alpha\beta}+\left[\frac{2}{1+\delta_{\alpha\beta}}\right]^{1/2}\sum_{\gamma,\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})(1+\delta_{\gamma\epsilon})}g_{-}^{\alpha\gamma}g_{-}^{\beta\epsilon}\frac{1}{1+\delta_{\epsilon\gamma}}\frac{\partial}{\partial g_{-}^{\epsilon\gamma}}\ . \quad (5.44b)$$

In summary, we have obtained the right coherent-state representation of the Sp(6) as follows:

$$\mathcal{A}^{R}_{\alpha\beta}=\frac{1}{\sqrt{2}(1+\delta_{\alpha\beta})}\frac{\partial}{\partial g_{-}^{\alpha\beta}}\ , \quad (5.45a)$$

$$(\mathcal{A}^{R}_{\alpha\beta})^{\dagger}=(2N+A)\sqrt{2}g_{-}^{\alpha\beta}+\left[\frac{2}{1+\delta_{\alpha\beta}}\right]^{1/2}\sum_{\gamma,\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})/(1+\delta_{\gamma\epsilon})}g_{-}^{\alpha\gamma}g_{-}^{\beta\epsilon}\frac{\partial}{\partial g_{-}^{\beta\epsilon}}\ , \quad (5.45b)$$

$$\mathcal{C}^{R}_{\alpha\beta}=(N+A/2)\delta_{\alpha\beta}+\sum_{\epsilon}\sqrt{(1+\delta_{\alpha\epsilon})/(1+\delta_{\beta\epsilon})}g_{-}^{\alpha\epsilon}\frac{\partial}{\partial g_{-}^{\beta\epsilon}}\ . \quad (5.45c)$$

The left representation can be obtained from the following relations:

$$F^{L}(g_{+})=[F^{R}(g_{-})]^{*}\quad (g_{+}=g_{-}^{*})\ , \quad (5.46)$$

$$\mathcal{A}^{L}_{\alpha\beta}\left[g_{+},\frac{\partial}{\partial g_{+}}\right]=\left[(\mathcal{A}^{R}_{\alpha\beta})^{\dagger}\left[g_{-},\frac{\partial}{\partial g_{-}}\right]\right]^{*}\ , \quad (5.47a)$$

$$(\mathcal{A}^{L}_{\alpha\beta})^{\dagger}\left[g_{+},\frac{\partial}{\partial g_{+}}\right]=\left[\mathcal{A}^{R}_{\alpha\beta}\left[g_{-},\frac{\partial}{\partial g_{-}}\right]\right]^{*}\ , \quad (5.47b)$$

$$\mathcal{C}^{L}_{\alpha\beta}\left[g_{+},\frac{\partial}{\partial g_{+}}\right]=\left[\mathcal{C}^{R}_{\beta\alpha}\left[g_{-},\frac{\partial}{\partial g_{-}}\right]\right]^{*}\ . \quad (5.47c)$$

By using the DGR-GCM,[21] the coherent-state representation can be transformed into boson form. For the right representation it amounts to the substitutions

$$\frac{1}{\sqrt{2}(1+\delta_{\alpha\beta})}\frac{\partial}{\partial g_{-}^{\alpha\beta}}\rightarrow b_{\alpha\beta}\ , \quad (5.48a)$$

$$\sqrt{2}g_{-}^{\alpha\beta}\rightarrow b^{\dagger}_{\alpha\beta}\ , \quad (5.48b)$$

where the boson operators obey

$$[b_{\alpha\beta},b^{\dagger}_{\alpha'\beta'}]=(\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\delta_{\alpha\beta'}\delta_{\beta\alpha'})/(1+\delta_{\alpha\beta})\ , \quad (5.49a)$$

$$[b_{\alpha\beta},b_{\alpha'\beta'}]=[b^{\dagger}_{\alpha\beta},b^{\dagger}_{\alpha'\beta'}]=0\ . \quad (5.49b)$$

The right representation in the boson form is thus[28]

$$\mathcal{A}^{R}_{\alpha\beta}=b_{\alpha\beta}\ , \quad (5.50a)$$

$$(\mathcal{A}^{R}_{\alpha\beta})^{\dagger}=(2N+A)b^{\dagger}_{\alpha\beta}+\frac{1}{\sqrt{1+\delta_{\alpha\beta}}}\sum_{\gamma,\epsilon}\sqrt{(1+\delta_{\alpha\gamma})(1+\delta_{\beta\epsilon})(1+\delta_{\epsilon\gamma})}\times b^{\dagger}_{\alpha\gamma}b^{\dagger}_{\beta\epsilon}b_{\gamma\epsilon}\ , \quad (5.50b)$$

$$\mathcal{C}^{R}_{\alpha\beta}=(N+A/2)\delta_{\alpha\beta}+\sum_{\epsilon}\sqrt{(1+\delta_{\alpha\epsilon})(1+\delta_{\beta\epsilon})}b^{\dagger}_{\alpha\epsilon}b_{\epsilon\beta}\ . \quad (5.50c)$$

## VI. DISCUSSION

In this paper we have presented a unified formalism for the continuous-variable representation of dynamic groups. Our general formalism is based on the differential on the group manifold. The advantage of the formalism is that it unifies different continuous-variable representation theories. For instance, the DGR-GCM and the GQCF at first glance appear to be quite different approaches. After the above lengthy investigation we finally find out that these two approaches are just two specializations of a general formalism in two different physical spaces (the Hilbert space and the von Neumann space). Their common features are thus unveiled clearly. The unification of different continuous-variable representation theories of the dynamic groups will strengthen their mathematical foundation and make the theories more transparent.

Another interesting point of the present formalism is to display the duality of the right and the left representations of the dynamic group. This duality of representations is universal and reflects the following profound and well-known fact: According to group theory, any group $G$ has its anti-isomorphic counterpart, its intrinsic group $\bar{G}$. This universal property of the group is of course preserved in its representation theory. We have known for a long time the situation for the rotation group SO(3) where the anti-isomorphic counterpart of the rotation group $\overline{\text{SO}(3)}$ is just the same group viewed from the intrinsic coordinate system (for example, the body-fixed system of a rotating top). From the SO(3) example we conjecture that any intrinsic dynamic group and its representation have some physical meaning. In this paper we have seen that the duality of representations in the SO(3) case exists in general. The questions are therefore raised about the physical meaning of the left representation of the dynamic group and its usage in physics. This problem deserves further investigation.

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