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# Generalized canonical transformations and path integrals

L. Chetouani and L. Guechi
*Département de Physique Théorique, Institut de Physique, Université de Constantine, Constantine, Algeria*

Théophile F. Hammann
*Laboratoire de Mathématiques, Physique Mathématique et Informatique, Faculté des Sciences et Techniques, Université de Haute Alsace, 4 rue des Frères Lumière, F 68093 Mulhouse, France*
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Some time-dependent physical systems do not admit, in the general case, either an invariant or auxiliary equation. The study of these systems is then in general made easier by space-time transformation of coordinates. This is true for the case for a rectangular well with a moving wall, which generalized canonical transformations reduce, in the path-integral formalism, to the case of a variable-frequency oscillator with fixed walls. The variable-frequency harmonic oscillator is an example where the invariant and the auxiliary equations can be calculated. All calculations are done in phase space.

## I. INTRODUCTION

It is well known that an exact and analytical solution of the Schrödinger equation can only be found for a limited number of potentials. If the potentials are, in addition, time dependent, it is very rare to be able to find exact solutions, as the Schrödinger equation then becomes a partial-derivative equation with two variables. Thus, for example, for the variable-frequency harmonic oscillator, one can find an analytical solution through different methods, among which the invariant approach.[1-3]

The variable frequency or variables mass oscillator, as well as an infinite rectangular potential well of variable width, belong to the class of potentials which can be resolved through the generalized canonical transformations (GCT) method. These canonical transformations, followed by a time transformation,[4] are defined as

$$x=Q\rho(t/t_0)\ ,\qquad p=P/\rho(t/t_0)\ , \tag{1}$$

$$\frac{ds}{dt}=\rho^{-2}(t/t_0)\ , \tag{2}$$

where $\rho(t/t_0)$ is an arbitrary function without dimensions, and $t_0$ the time unit.

The subject of this paper is the study of certain time-dependent physical systems which do not admit, in the general case, either invariant or auxiliary equations. When the coordinates of such a system are submitted to the space-time transformations (1) and (2), the exact analytical, perturbative, or numerical description, is made easier. This is the case of a particle in a rectangular well with variable width, where the boundary conditions are thus time dependent. Through the space-time transformations (1) and (2), this system becomes equivalent to a variable-frequency harmonic oscillator which has to move on a segment with constant boundaries.

Through a merely classical calculation, utilizing the Hamiltonian formalism that is known to be well adapted to canonical transformations, we establish the general relation which exists between the propagators when one change the coordinate system via the GCT. This relation is valid for all time-dependent potentials.

The calculations are made within the framework of a path-integral approach which avoids the use of any auxiliary equations (Sec. II). The simple case of a variable-frequency harmonic oscillator is then treated as an example. The invariant and the auxiliary equation are rigorously established (Sec. III) and not admitted from the beginning, as in other formalisms.[3]

Finally, the variable-width rectangular wall is analyzed for the first time, as far as we known, in the framework of a path-integral approach. For an arbitrary law of motion of the moving wall there is neither an invariant nor an auxiliary equation. For certain particular motions, the GCT transforms this case with time-dependent boundary conditions into a problem with constant boundary conditions (Sec. IV).

In all the following equations we use the notation as follows:

$$\bar\rho=\bar\rho\left[\frac{s}{t_0}\right]=\rho\left[\frac{t}{t_0}\right]=\rho,\quad \dot\rho=\frac{d\rho}{d(t/t_0)},\quad \dot{\bar\rho}=\frac{d\bar\rho}{d(s/t_0)}\ ,$$

$$\tilde u_j=\frac{u_j+u_{j-1}}{2},\quad \Delta u_j=u_j-u_{j-1}\ .$$

## II. PROPAGATOR FOR A TIME-DEPENDENT POTENTIAL

In the canonical formulation of the path integrals, the propagator is written formally as follows, in standard notation,

$$K(x_f,t_f;x_i,t_i)=\int \mathcal{D}x\,\mathcal{D}p\,\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}(p\dot x-H)dt\right]\ . \tag{3}$$

Or, in a time-graded representation, and by using the midpoint prescription[5] in conformity with the Weyl correspondence rules,[6]

 

$$K(x_f,t_f;x_i,t_i)=\lim_{N\to\infty}\int\prod_{j=1}^{N-1}dx_j\prod_{j=1}^{N}dp_j\frac{1}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[p_j(x_j-x_{j-1})-\epsilon H\left[p_j,\frac{x_j+x_{j-1}}{2},\frac{t_j+t_{j-1}}{2}\right]\right]\right\}, \tag{4}$$

where

$$x_j=x(t_j),\quad \epsilon=t_j-t_{j-1}=\frac{t_f-t_i}{N},$$
$$x_i=x(t_i)=x(t_0),\quad x_f=x(t_f)=x(t_N).$$

The dynamics of the physical system is governed by the Hamiltonian $H=p^2/2m+V(x,t)$, in the space-time system $(x,p,t)$. In a space-time system $(Q,P,t)$ the dynamics is ruled by

$$\mathcal{H}(Q,P;t)=\frac{P^2}{2m\rho^2}-\frac{PQ\dot\rho}{t_0\rho}+V(\rho Q,t),$$

which is easily deduced from the classical-mechanics equations[7]

$$p=\frac{\partial}{\partial x}F_2(x,P;t),\quad Q=\frac{\partial}{\partial P}F_2(x,P;t), \tag{5a}$$

$$\mathcal{H}=H+\frac{\partial F_2}{\partial t},\quad p\dot x-H=P\dot Q-\mathcal{H}+\frac{dF}{dt},\quad F=-PQ+F_2. \tag{5b}$$

The generating function responsible for the transformation is

$$F_2(x,P;t)=PQ=P\frac{x}{\rho}.$$

Thus $F=0$. Let us show that through the transformation (1) and (2) the measure is transformed according to the law

$$\mathcal{D}x\,\mathcal{D}p=\frac{1}{(\rho_f\rho_i)^{1/2}}\mathcal{D}Q\,\mathcal{D}P, \tag{6}$$

where $\rho_f=\rho(t_f/t_0)$ and $\rho_i=\rho(t_i/t_0)$. Let us first recall that, according to Eq. (4), the particle is moving from the position $x_{j-1}$ to the position $x_j$, in the time interval $[j-1,j]$, while its impulsion is constant and equal to $p_j$. In the following time interval $[j,j+1]$, the constant impulsion of the particle will be $p_{j+1}\neq p_j$. The trajectory $x(t)$ followed by the particle is thus a continuous broken line, whereas its impulsion is discontinuous (piece-wise continuous).

Let us then make the following remarks.

(i) If the impulsion is $p_j$ in the interval $[j-1,j]$, the position of the particle is known only to the extent of an uncertainty $\Delta x=x_j-x_{j-1}$.

(ii) If, at a time $t_j$ the position is $x_j$, it is the impulsion which is indeterminate. One recognizes here Heisenberg's uncertainty principle. In other words, the variables $x_j$ and $p_j$ are not really canonical coordinates. Therefore it is not clear how to write, in a general manner, the canonical transformations in their discretized version. For the particular transformation (1) we use the first principles by replacing the partial derivatives $\partial$ by the variations $\Delta$ in Eqs. (5a). In the discrete version and according to the midpoint prescription, we have the following relations for the momenta:

$$p_j\simeq\frac{\Delta F_2(j)}{\Delta x_j}=\frac{F_2(x_j,P_j;\tilde t_j)-F_2(x_{j-1},P_j;\tilde t_j)}{\Delta x_j}$$
$$=\frac{P_j}{\rho(\tilde t_j/t_0)}, \tag{7a}$$

and the following equations for the coordinates:

$$Q_j=\frac{\Delta F_2(j)}{\Delta P_j}=\frac{F_2(x_j,P_{j+1};t_j)-F_2(x_j,P_j;t_j)}{P_{j+1}-P_j}$$
$$=\frac{x_j}{\rho(t_j/t_0)}. \tag{7b}$$

But it is still easy to see that

$$4(\rho_j\rho_{j-1})=(\rho_j+\rho_{j-1})^2-(\rho_j-\rho_{j-1})^2$$
$$=(\rho_j+\rho_{j-1})^2+O(\epsilon^2)$$
$$=4\rho^2\left[\frac{t_j+t_{j-1}}{2t_0}\right]+O(\epsilon^2),$$

and thus

$$p_j\approx\frac{P_j}{(\rho_j\rho_{j-1})^{1/2}}[1+O(\epsilon^2)].$$

The measure then becomes

$$\prod_{j=1}^{N-1}dx_j\prod_{j=1}^{N}dp_j\frac{1}{2\pi\hbar}=\frac{1+O(\epsilon)}{(\rho_N\rho_0)^{1/2}}\prod_{j=1}^{N-1}dQ_j\prod_{j=1}^{N}dP_j\frac{1}{2\pi\hbar}, \tag{8}$$

and, at the limit where $\epsilon\to 0$ ($N\to\infty$), we obtain the result we looked for (6). Expressed as a function of the new variables $(Q,P,t)$, the action is written as

$$A=\int_{t_i}^{t_f}[p\dot x-H(x,p;t)]dt$$
$$=\int_{t_i}^{t_f}[P\dot Q-\mathcal{H}(Q,P;t)]dt$$
$$=\int_{t_i}^{t_f}\left[P\dot Q+\frac{PQ\dot\rho}{t_0\rho}-\left[\frac{P^2}{2m\rho^2}+V(\rho Q,t)\right]\right]dt. \tag{9a}$$

With the specific form of the transformation (1), we can verify that the elementary action $A(j,j-1)$ leading to Eq. (9a) can be obtained directly by using Eqs. (7a) and (7b),

$$A(j,j-1)=p_j\Delta x_j-H(\tilde x_j,p_j;\tilde t_j)\Delta t_j$$
$$\approx P_j\Delta Q_j-\mathcal{H}(\tilde Q,P_j;\tilde t_j)\Delta t_j. \tag{9b}$$

The limits $j-1$ and $j$ of the time interval play the same part. The evolution of the physical system is then described in the coordinates $(Q,P,t)$ by

$$K(x_f,t_f;x_i,t_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\int \mathcal{D}Q\,\mathcal{D}P\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f}\left[P\dot{Q}+\frac{PQ\dot{\rho}}{t_0\rho}-\left[\frac{P^2}{2m\rho^2}+V(\rho Q,t)\right]\right]dt\right\}. \tag{10}$$

The utilization of the relations (7a) and (7b) has thus permitted us to obtain, for the measure and the propagator, expressions in which $t_i$ and $t_f$ play the same part.

Let us utilize the time transformation (2) in order to bring to a constant the mass appearing in the kinetic term $P^2/2m\rho^2$,

$$K(x_f,t_f;x_it_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\int \mathcal{D}Q\,\mathcal{D}P\exp\left[\frac{i}{\hbar}\int_{s_i}^{s_f}\left\{P\dot{Q}-\left[\frac{P^2}{2m}-\frac{PQ\dot{\bar{\rho}}}{t_0\bar{\rho}}+\bar{\rho}^2V\left[\bar{\rho}Q,\int^s\bar{\rho}^2(\sigma/t_0)d\sigma\right]\right]\right\}ds\right], \tag{11}$$

where

$$s_i=\int^{t_i}\frac{d\sigma}{\rho^2(\sigma/t_0)},\quad s_f=\int^{t_f}\frac{d\sigma}{\rho^2(\sigma/t_0)}.$$

$$\Delta s_j=s_j-s_{j-1}=\frac{\Delta t}{\bar{\rho}(j)\bar{\rho}(j-1)}=\frac{\Delta t}{\bar{\rho}^2}(\bar{s}_j/t_0).$$

We notice at this stage that, because of the presence of the additional term $PQ\dot{\bar{\rho}}/(t_0\bar{\rho})$, the propagators (3) and (12) do not have the same structure. As

$$\frac{P^2}{2m}-\frac{PQ\dot{\bar{\rho}}}{t_0\bar{\rho}}=\frac{1}{2m}\left[\left[P-\frac{mQ\dot{\bar{\rho}}}{t_0\bar{\rho}}\right]^2-\left[\frac{mQ\dot{\bar{\rho}}}{t_0\bar{\rho}}\right]^2\right], \tag{12a}$$

the additional term can be eliminated through the following canonical transformation:

$$\mathcal{P}=P-\frac{mQ\dot{\bar{\rho}}}{t_0\bar{\rho}},\quad \mathcal{Q}=Q. \tag{12b}$$

Replacing again the partial derivatives of Eq. (5) by finite variations, a midpoint discretization of the transformation (12a) yields

$$\mathcal{P}_j=P_j-\frac{m}{t_0}\bar{Q}_j\frac{\dot{\bar{\rho}}}{\bar{\rho}}\bigg|_{s=\bar{s}_j},\quad \mathcal{Q}_j=Q_j. \tag{12c}$$

The generating function $F_2'(Q,\mathcal{P},s)$ responsible for the transformation $(Q,P,s)\to(\mathcal{Q},\mathcal{P},s)$ is

$$F_2'(Q,\mathcal{P},s)=\mathcal{P}Q+\frac{mQ^2\dot{\bar{\rho}}}{2t_0\bar{\rho}}.$$

The particle movement in the system $(\mathcal{Q},\mathcal{P},s)$ is ruled by the new Hamiltonian

$$\mathcal{H}'(\mathcal{Q},\mathcal{P},s)=\frac{\mathcal{P}^2}{2m}+\frac{1}{2}m\Omega^2\mathcal{Q}^2+\bar{\rho}^2V\left[\bar{\rho}\mathcal{Q},\int^s\bar{\rho}^2(\sigma/t_0)d\sigma\right], \tag{13a}$$

where

$$\Omega^2=\frac{1}{t_0^2}\left[\frac{\ddot{\bar{\rho}}}{\bar{\rho}}-2\left[\frac{\dot{\bar{\rho}}}{\bar{\rho}}\right]^2\right]=\frac{\rho^3\ddot{\rho}}{t_0^2}. \tag{13b}$$

Taking into account the invariance of the measure,

$$\mathcal{D}Q\,\mathcal{D}P=\mathcal{D}\mathcal{Q}\,\mathcal{D}\mathcal{P}, \tag{13c}$$

the propagator is written

$$K(x_f,t_f;x_i,t_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\left\{\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\rho}_f}{\rho_f}Q_f^2-\frac{\dot{\rho}_i}{\rho_i}Q_i^2\right]\right]\right\}\int\mathcal{D}\mathcal{Q}\,\mathcal{D}\mathcal{P}\exp\left[\frac{i}{\hbar}\int_{s_i}^{s_f}[\mathcal{P}\dot{\mathcal{Q}}-\mathcal{H}'(\mathcal{Q},\mathcal{P},s)]ds\right]. \tag{14}$$

Relation (14) needs to be commented upon. A naive description of the transformation $p\to P$ would be, according to (1), $p_j\propto P_j/\rho(t_j/t_0)$; in this case, the extremity $j$ of the time interval $\{j-1,j\}$ would have a preferential part. This post-point discretization would then give us a new measure

$$\mathcal{D}x\,\mathcal{D}p=\frac{1}{\rho_f}\mathcal{D}Q\,\mathcal{D}P.$$

A second choice, not much less naive would be to take $p_j\propto[P_j/\rho(t_{j-1}/t_0)]$, since $p_j$ and $P_j$ are constants on $]j-1,j[$, the extremity $(j-1)$ now having a preferential part. This prepoint discretization would lead to the measure

$$\mathcal{D}x\,\mathcal{D}p=\frac{1}{\rho_i}\mathcal{D}Q\,\mathcal{D}P.$$

In both cases $t_i$ and $t_f$ do not have the same part in the propagator. More precisely, we would then obtain through symmetrization of the measure, two different propagators:

$$K_{\pm}=\frac{1}{(\rho_i\rho_f)^{1/2}}\int \mathcal{D}Q\,\mathcal{D}P\exp\left[\frac{i}{\hbar}S_{\pm}\right], \tag{15a}$$

which correspond to two complex actions in the $(Q,P,t)$ coordinate system

$$S_{\pm}=A\pm\int_{t_i}^{t_f}\frac{\hbar}{2i}\frac{d\ln\rho}{dt}dt=A\pm\frac{\hbar}{2it_0}\int_{t_i}^{t_f}\frac{\dot\rho}{\rho}dt\ .$$

Performing the time transformation (2), these complex actions will be written in the $(Q,P,s)$ coordinate system as

$$\begin{aligned}S_{\pm}=\int_{s_i}^{s_f}ds\left[P\dot Q-\frac{P^2}{2m}+\frac{PQ\dot{\bar\rho}}{t_0\bar\rho}\right.&\\ -\bar\rho^2V\left[\bar\rho Q,\int^{s}\bar\rho^2(\sigma/t_0)d\sigma\right]&\\ \left.\pm\frac{\hbar}{2i}\frac{\dot{\bar\rho}}{t_0\bar\rho}\right]&=\int_{s_i}^{s_f}ds[P\dot Q-H_{\pm}]\ .\end{aligned} \tag{15b}$$

The elementary actions belonging to the $[j-1,j]$ interval,

$$S_{+}(j,j-1)=P_j\Delta Q_j-H_{+}(P_j,Q_j;s_j)\Delta s_j\ , \tag{15c}$$

$$S_{-}(j,j-1)=P_j\Delta Q_j-H_{-}(P_j,Q_{j-1};s_{j-1})\Delta s_j\ , \tag{15d}$$

correspond to the $QP$ and $PQ$ ordering, respectively.

Using again the definitions of the derivatives in (7), we get the following discretized version of the canonical transformation (12b): For the $QP$ ordering,

$$\begin{aligned}P_j&\simeq\frac{\Delta F_2'(j)}{\Delta Q_j}\\&=\frac{F_2'(Q_j,\mathcal{P}_j;s_j)-F_2'(Q_{j-1},\mathcal{P}_j;s_j)}{Q_j-Q_{j-1}}\\&=\mathcal{P}_j+\frac{m}{t_0}\tilde Q_j\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_j}=\mathcal{P}_j+\frac{m}{t_0}\left[Q_j-\frac{\Delta Q_j}{2}\right]\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_j}\ ,\\ \mathcal{Q}_j&=\frac{\Delta F_2'(j)}{\Delta\mathcal{P}_j}=\frac{F_2'(Q_j,\mathcal{P}_{j+1};s_j)-F_2'(Q_j,\mathcal{P}_j;s_j)}{\mathcal{P}_{j+1}-\mathcal{P}_j}=Q_j\ .\end{aligned} \tag{16a}$$

For the $PQ$ ordering,

$$\begin{aligned}P_j&\simeq\frac{\Delta F_2'(j)}{\Delta Q_j}=\frac{F_2'(Q_j,\mathcal{P}_j;s_{j-1})-F_2'(Q_{j-1},\mathcal{P}_j;s_{j-1})}{Q_j-Q_{j-1}}\\&=\mathcal{P}_j+\frac{m}{t_0}\left[Q_{j-1}+\frac{\Delta Q_j}{2}\right]\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_{j-1}}\\ \mathcal{Q}_j&=\frac{\Delta F_2'(j)}{\Delta\mathcal{P}_j}=\frac{F_2'(Q_j,\mathcal{P}_{j+1};s_j)-F_2'(Q_j,\mathcal{P}_j;s_j)}{\mathcal{P}_{j+1}-\mathcal{P}_j}=Q_j\ .\end{aligned} \tag{16b}$$

Exchanging $P_j$ for

$$P_j+\frac{m}{2t_0}\Delta Q_j\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_j}$$

in (15c) and $P_j$ for

$$P_j-\frac{m}{2t_0}\Delta Q_j\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_{j-1}}$$

in (15d), in order to get the expressions (16a) and (16b), and using (12b), yields in the $(Q,\mathcal{P},s)$ coordinate system (the measure being invariant),

$$\begin{aligned}K_{\pm}=&\frac{1}{(\rho_f\rho_i)^{1/2}}\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar\rho}_f}{\rho_f}Q_f^2-\frac{\dot{\bar\rho}_i}{\rho_i}Q_i^2\right]\right]\\&\times\lim_{N\to\infty}\int\prod_{j=1}^{N-1}dQ_j\prod_{j=1}^{N}d\mathcal{P}_j\frac{1}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}(\mathcal{P}_j\Delta Q_j-\mathcal{H}'_{\pm}\Delta s_j)\right],\end{aligned} \tag{16c}$$

where

$$\begin{aligned}\mathcal{H}'_{+}\Delta s_j&=\mathcal{H}'(\mathcal{P}_j,Q_j;s_j)\Delta s_j+\frac{m}{2t_0}\left[(\Delta Q_j)^2-\frac{i\hbar}{m}\Delta s_j\right]\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_j}\ ,\\ \mathcal{H}'_{-}\Delta s_j&=\mathcal{H}'(\mathcal{P}_j,Q_{j-1};s_{j-1})\Delta s_j-\frac{m}{2t_0}\left[(\Delta Q_j)^2-\frac{i\hbar}{m}\Delta s_j\right]\frac{\dot{\bar\rho}}{\bar\rho}\bigg|_{s=s_{j-1}}\ ,\end{aligned} \tag{16d}$$

$\mathcal{H}'$ being given by (13a).

When $\Delta s_j\to 0$, then

$$\langle(\Delta Q)^2\rangle=\int\int d\mathcal{P}d(\Delta Q)\frac{1}{2\pi\hbar}(\Delta Q)^2\exp\left[\frac{i}{\hbar}\left[\mathcal{P}\Delta Q-\frac{\mathcal{P}^2}{2m}\Delta s\right]\right]=\frac{i\hbar}{m}\Delta s\ .$$

Clearly, the expression (13a) is the limit of the discretized quantity (16d) as $\Delta s_j$ approaches zero. Thus $K_{+}=K_{-}=K$ [see Eq. (14)], and the expression (14) is independent of the discretization (4) (equivalence of discretizations).

By changing $\mathcal{P}$ into $P$ in Eqs. (16), we finally obtain the propagator

$$K(x_f,t_f;x_i,t_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\left\{\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar\rho}_f}{\rho_f}Q_f^2-\frac{\dot{\bar\rho}_i}{\rho_i}Q_i^2\right]\right]\right\}\int \mathcal{D}Q\,\mathcal{D}P\exp\left[\frac{i}{\hbar}\int_{s_i}^{s_f}[P\dot Q-H'(Q,P;s)]ds\right], \tag{17}$$

where

$$H'(Q,P;s)=\frac{P^2}{2m}+\frac{m\Omega^2Q^2}{2}+\bar\rho^2V\left[\bar\rho Q,\int^s\bar\rho^2(\sigma/t_0)d\sigma\right]$$

is the final Hamiltonian governing the dynamics of the system with variables $(Q,P,s)$. A comparison between the propagators (3) and (17) shows that the space-time transformations (1) and (2), has resulted in the appearance of a phase and a quadratic term $\frac{1}{2}m\Omega^2Q^2$, which are both potential independent but depend solely on the GCT.

The result of Eq. (17) is independent of the discretization. It describes the evolution of a physical system of which the space-time coordinates were submitted to a generalized canonical transformation. This result (17) seems important to us and also really interesting through its relative simplicity. Let us illustrate the usage one can make of this equation (17), with two applications of real epistemological interest: the variable-frequency harmonic oscillator and the rectangular well with moving wall (nonquadratical potential).

## III. APPLICATION TO THE VARIABLE FREQUENCY HARMONIC OSCILLATOR

The variable-frequency harmonic oscillator $V(x,t)=m\omega^2(t)x^2/2$ is a continued subject of interest because of its simplicity. The propagator (17) is written, in this particular case,

$$K(x_f,t_f;x_i,t_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\left\{\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar\rho}_f}{\rho_f}Q_f^2-\frac{\dot{\bar\rho}_i}{\rho_i}Q_i^2\right]\right]\right\}$$
$$\times\int \mathcal{D}Q\,\mathcal{D}P\exp\left\{\frac{i}{\hbar}\int_{s_i}^{s_f}\left[P\dot Q-\left[\frac{P^2}{2m}+\frac{m}{2}[\Omega^2+\omega^2(s)\bar\rho^4]Q^2\right]\right]ds\right\}. \tag{18}$$

In order to be able to utilize the well-known result of Feynman and Hibbs,[8] with respect to the constant-frequency harmonic oscillator, let us set the global time-dependent frequency appearing in Eq. (18) equal to a constant:

$$\Omega^2+\omega^2(s)\bar\rho^4=\frac{\omega_0^2}{t_0^2}=\text{const}\,, \tag{19}$$

which amounts to imposing a constraint on $\rho$. As

$$\Omega^2=\frac{\rho^3\ddot\rho}{t_0^2}\,,$$

Eq. (19) then becomes

$$\ddot\rho+\omega'^2(t)\rho=\frac{\omega_0^2}{\rho^3}\,, \tag{20}$$

where $\omega'(t)=t_0\omega(t)$, which is the well-known auxiliary equation.[3]

It is also easy to obtain the invariant. The Hamiltonian,

$$H'(Q,P;s)=\frac{P^2}{2m}+\frac{m}{2}\frac{\omega_0^2}{t_0^2}Q^2=\frac{m}{2}\left[\left[\frac{dQ}{ds}\right]^2+\frac{\omega_0^2}{t_0^2}Q^2\right], \tag{21}$$

expressed as a function of $x$ and $t$, is then exactly the invariant[3]

$$I=\frac{m}{2t_0^2}\left[(t_0\dot x\rho-x\dot\rho)^2+\omega_0^2\left[\frac{x}{\rho}\right]^2\right]. \tag{22}$$

Finally, the propagator for the potential $V(x,t)=m\omega^2(t)x^2/2$ is given by

$$K(x_f,t_f;x_i,t_i)=\left[\frac{m\omega_0/t_0}{2\pi i\hbar\rho_f\rho_i\sin[(\omega_0/t_0)(s_f-s_i)]}\right]^{1/2}\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar\rho}_f}{\rho_f}Q_f^2-\frac{\dot{\bar\rho}_i}{\rho_i}Q_i^2\right]\right]$$
$$\times\exp\left[\frac{im\omega_0/t_0}{2\hbar\sin[(\omega_0/t_0)(s_f-s_i)]}\{(Q_f^2+Q_i^2)\cos[(\omega_0/t_0)(s_f-s_i)]-2Q_fQ_i\}\right], \tag{23}$$

with

$$\dot{\bar{\rho}}=\rho^2\dot{\rho},\quad Q_f=\frac{x_f}{\rho_f},\quad Q_i=\frac{x_i}{\rho_i},\quad s_f-s_i=\int_{t_i}^{t_f}\frac{d\sigma}{\rho^2(\sigma/t_0)}\ .$$

This equation (23) completely resolves the problem of the variable harmonic oscillator.

## IV. APPLICATION TO THE INFINITE POTENTIAL WELL WITH MOVING WALL

One wall is supposed fixed at the origin, the other is submitted to an arbitrary time-dependent movement. At any instant $t$, the position of the moving wall is $x=L\rho$, where $L$ represents the position of the moving wall at the initial instant.[4] We suppose that $\rho(0)=1$ and $s=0$ if $t=0$. This potential well, defined by

$$V(x,t)=\begin{cases}0, & \text{for } 0\le x\le L\rho\\ \infty, & \text{elsewhere}\end{cases}$$

thus has boundary conditions which are time dependent. Let us first notice that if $x_f$ or $x_i$ are outside of the interval $[0,L\rho]$, the propagator (3) is null, $V$ being infinite. Then, if $x_f$ and $x_i\in[0,L\rho]$, the propagator (17) is written as

$$K(x_f,t_f;x_i,t_i)=\frac{1}{(\rho_f\rho_i)^{1/2}}\left\{\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar{\rho}}_f}{\rho_f}Q_f^2-\frac{\dot{\bar{\rho}}_i}{\rho_i}Q_i^2\right]\right]\right\}\bar{K}(Q_f,s_f;Q_i,s_i)\ ,$$

with

$$\bar{K}(Q_f,s_f;Q_i,s_i)=\int\mathcal{D}Q\,\mathcal{D}P\exp\left\{\frac{i}{\hbar}\int_{s_i}^{s_f}\left[P\dot{Q}-\left[\frac{P^2}{2m}+\frac{m}{2}\Omega^2Q^2\right]\right]ds\right\}\ , \tag{24}$$

where $Q_f$ and $Q_i\in[0,L]$.

Thus the computation of the propagator (3) for this infinite well with variable width amounts to the determination of the propagator (24), with the quadratic term $\frac{1}{2}m\Omega^2Q^2$ added, but with the constant boundary conditions. In the general case, the expression (24) is not calculable, analytically. Let us thus take into consideration specific movements.

### A. The particular case with $\Omega=0$

#### 1. The propagator, for $\Omega=0$

If $\Omega=\ddot{\rho}=0$, the movement of the wall is uniform. Its position $x=L\rho=L[1+(t/t_0)]$ depends linearly on time, $t$ being related to $s$ through $s=t/(1+t/t_0)$, $t\in[0,\infty]$, $s\in[0,t_0]$. In order to calculate the propagator, we make use of the transformation

$$Q=\frac{L}{\pi}\theta,\quad P=\frac{L}{\pi}P_\theta\ ,$$

and of the following equation deduced from the rigid rotator,[9] submitted to the constraint $a=L/\pi$,

$$\int\mathcal{D}\theta\,\mathcal{D}P_\theta\exp\left\{\frac{i}{\hbar}\int_{s_i}^{s_f}\left[P_\theta\dot{\theta}-\left[\frac{P_\theta^2}{2ma^2}+\frac{\hbar^2(n^2-\frac{1}{4})}{2ma^2\sin^2\theta}\right]\right]ds\right\}$$
$$=\sum_{l=0}^{\infty}\exp\left[-\frac{i}{\hbar}\frac{(l+n+\frac{1}{2})^2\hbar^2}{2ma^2}(s_f-s_i)\right](l+n+\tfrac{1}{2})\frac{(l+2n)!}{l!}(\sin\theta_f\sin\theta_i)^{1/2}P_{l+n}^{-n}(\cos\theta_f)P_{l+n}^{-n}(\cos\theta_i)\ .$$

Let us first set $n=\frac{1}{2}$. Then, the use of the formula[10]

$$P_{\nu-(1/2)}^{-1/2}(\cos\theta)=\left[\frac{2}{\pi\sin\theta}\right]^{1/2}\frac{\sin\nu\theta}{\nu},\quad\text{with } \nu=l+1$$

amounts to utilizing the image method[11] in the system $(Q,P,s)$ and leads to the result

$$K^0(x_f,t_f;x_i,t_i)$$
$$=\frac{1}{(\rho_f\rho_i)^{1/2}}\exp\left[\frac{im}{2\hbar t_0}\left[\frac{\dot{\bar{\rho}}_f}{\rho_f}Q_f^2-\frac{\dot{\bar{\rho}}_i}{\rho_i}Q_i^2\right]\right]$$
$$\times\bar{K}(Q_f,s_f;Q_i;s_i)\ , \tag{25}$$

where

$$\bar{K}(Q_f,s_f;Q_i,s_i)=\frac{2}{L}\sum_{l=1}^{\infty}\sin\left[\frac{l\pi}{L}Q_f\right]\sin\left[\frac{l\pi}{L}Q_i\right]$$
$$\times\exp\left[-\frac{i}{\hbar}E_l^0(s_f-s_i)\right]\ .$$

The wave functions and the energy spectrum are given by

$$\psi_l^{(0)}(Q)=\left[\frac{2}{L}\right]^{1/2}\sin\left[\frac{l\pi}{L}Q\right],\quad E_l^{(0)}=\frac{\hbar^2}{2m}\left[\frac{l\pi}{L}\right]^2,$$
$$l=1,2,3,\ldots,\infty\ .$$

### 2. *The adiabatic approximation*

In the case where the moving wall is moved very slowly (adiabatically) and linearly with respect to time, $x = L[1 + a(t/t_0)]$, with $a \ll 1$, the following expression applies to the propagator (25):

$$K^{\mathrm{ad}}(x_f,t_f;x_i,t_i) = \left[\frac{m}{2i\pi\hbar(t_f-t_i)}\right]^{1/2} \sum_{l=-\infty}^{+\infty} \left[\exp\left\{\frac{im}{2\hbar(t_f-t_i)}\left[(x_f-x_i)^2 + 4lL\rho_f\rho_i\left[lL + \frac{x_f}{\rho_f} - \frac{x_i}{\rho_i}\right]\right]\right\} - \exp\left\{\frac{im}{2\hbar(t_f-t_i)}\left[(x_f+x_i)^2 + 4lL\rho_f\rho_i\left[lL - \frac{x_f}{\rho_f} - \frac{x_i}{\rho_i}\right]\right]\right\}\right],$$

where the equation is obtained through the Poisson summation

$$\sum_{l=-\infty}^{+\infty} f(l) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy\, f(y)\, e^{2i\pi ny}.$$

The quantity $a$ being small, $\rho_f \sim \rho_i \sim \rho$, the propagator in the adiabatic approximation will thus be given by

$$K^{\mathrm{ad}}(x_f,t_f;x_i,t_i) \simeq \left[\frac{m}{2i\pi\hbar(t_f-t_i)}\right]^{1/2} \sum_{l=-\infty}^{+\infty} \left[\exp\left[\frac{im}{2\hbar(t_f-t_i)}(x_f-x_i+2lL\rho)^2\right] - \exp\left[\frac{im}{2\hbar(t_f-t_i)}[x_f+x_i-2l\rho]^2\right]\right], \tag{26}$$

which is the sum on all the classical paths. We can write this equation (26) in the form

$$K^{\mathrm{ad}}(x_f,t_f;x_i,t_i) \approx \frac{2}{L\rho}\sum_{l=1}^{\infty} \sin\left[\frac{l\pi}{L\rho}x_f\right]\sin\left[\frac{l\pi}{L\rho}x_i\right] \times \exp\left[\frac{-i\hbar l^2\pi^2}{2mL^2\rho^2}(t_f-t_i)\right].$$

The wave functions, as well as the energies in the system $(x,p,t)$, are then given by

$$\psi_l^{\mathrm{ad}}(x) = \left[\frac{2}{L\rho}\right]^{1/2} \sin\left[\frac{l\pi}{L\rho}x\right], \quad E_l^{\mathrm{ad}} = \frac{\hbar^2}{2m}\left[\frac{l\pi}{L\rho}\right]^2.$$

Finally, for motions of the moving wall such as $\Omega = 0$, there is an auxiliary equation $\ddot{\rho} = 0$. The system admits thus an invariant

$$I = \frac{P^2}{2m} + V(Q) = \frac{m\dot{Q}^2}{2} + V(Q) = \begin{cases} \dfrac{m\dot{Q}^2}{2}, & 0 \le Q \le L \\ \infty, & \text{otherwise} \end{cases}$$

or

$$I = \frac{m}{2t_0^2}(t_0\dot{x}\rho - x\dot{\rho})^2 + V\left[\frac{x}{\rho}\right] = \begin{cases} \dfrac{m}{2t_0^2}(t_0\dot{x}\rho - x\dot{\rho})^2, & 0 \le x \le L\rho \\ \infty, & \text{otherwise}. \end{cases}$$

#### B. The general case, $\Omega \neq 0$

##### 1. *Case of small $\Omega$ frequency*

If the wall movements are such that the $\Omega$ frequency is weak, the quadratic term $m\Omega^2 Q^2/2$ can be treated as a perturbation. The standard perturbation theory[8] amounts to calculating a series $K = K^{(0)} + K^{(1)} + K^{(2)} + \cdots$, where $K^{(0)}$ is the propagator without perturbation given by Eq. (25).

##### 2. *Case where $\Omega$ is not small*

It is obvious that the analytical solution does not exist. One can only consider a numerical solution. It is easier to obtain this solution starting from Eq. (25) than from the Eq. (3). In both cases 1 and 2 the system does not admit any auxiliary equation or invariant.

##### 3. *Case of constant frequency, $\Omega = \mathrm{const}$*

In this case the system admits an auxiliary equation,

$$\ddot{\rho} = \frac{t_0^2}{\rho^3}\Omega^2,$$

and an invariant,

$$I = \frac{m}{2t_0^2}(t_0\dot{x}\rho - x\dot{\rho})^2 + \tfrac{1}{2}m\Omega^2\left[\frac{x}{\rho}\right]^2 + V\left[\frac{x}{\rho}\right].$$

For the propagator (23) relevant to a particle which is bound to move harmonically on a segment $(0,L)$, an analytical solution is no longer possible. One can take into consideration a numerical calculation.

## V. CONCLUSION

In this paper, we have calculated the propagator of a time-dependent physical system when the space-time variables describing this system were submitted to a GCT. The calculation, which is done in phase space, does not presuppose the existence of an invariant or the knowledge of an auxilary equation.

The method was illustrated, in the case of a quadratic potential, by the variable-frequency harmonic oscillator, and in the case of a nonquadratic potential, by the rectangular well with a moving wall, the resolution of which had never been done, as far as we know, in the framework of a path-integral approach. Of course, this method can be applied to other problems of physics of similar nature.

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