Brownian Motion of a Quantum Oscillator*

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The theory of Brownian motion of a quantum oscillator is developed. The Brownian motion is described by a model Hamiltonian which is taken to be the one describing the interaction between this oscillator and a reservoir. Use is made of the master equation recently derived by the author, to obtain the equation of motion for the various reduced phase-space distribution functions that are obtained by mapping the density operator onto c-number functions. The equations of motion for the reduced phase-space distribution functions are found to be of the Fokker-Planck type. On transforming the Fokker-Planck equation to real variables, it is found to have the same form as the Fokker-Planck equation obtained by Wang and Uhlenbeck to describe the Brownian motion of a classical oscillator. The Fokker-Planck equation is solved for the conditional probability (Green's function) which is found to be in the form of a two-dimensional Gaussian distribution. This solution is then used to obtain various time-dependent quantum statistical properties of the oscillator. Next, the entropy for a quantum oscillator undergoing Brownian motion is calculated and we show that this system approaches equilibrium as $t \to \infty$. Finally we show that in the weak-coupling limit the Fokker-Planck equation reduces to the one obtained by making the usual rotating-wave approximation.

I. INTRODUCTION

Brownian motion of a classical oscillator has been studied in great detail by several workers. Most notable are the contributions made by Kramers,[1] by Uhlenbeck and Ornstein,[2] by Chandrasekhar,[3] and by Wang and Uhlenbeck.[4] The displacement $q(t)$ of the oscillator satisfies the equation of motion

$$\frac{d^2q}{dt^2} + 2\kappa \frac{dq}{dt} + \omega^2 q = F(t) , \tag{1.1}$$

where 2κ is the phenomenological damping coefficient and ω is the natural frequency of the oscillator. $mF(t)$ is a random force which is assumed to be a δ-correlated Gaussian process with zero mean, i.e.,

$$\langle F(t) \rangle = 0, \quad \langle F(t_1)F(t_2) \rangle = 2(D/m^2)\delta(t_1 - t_2) , \tag{1.2}$$

where D is the diffusion coefficient and m is the mass of the particle. Wang and Uhlenbeck replaced Eq. (1.1) by the following two first-order differential equations:

$$\dot{q} = p/m , \quad \dot{p} = -2\kappa p - m\omega^2 q + f(t) , \tag{1.3}$$

where p is the momentum of the particle. These are the Langevin equations which describe the Brownian motion of a classical oscillator. Moreover Eqs. (1.3) describe a two-dimensional Gaussian-Markoff process.[5] Wang and Uhlenbeck solved the Fokker-Planck equation equivalent to (1.3) for the conditional probability of the process and they also obtained the time dependence of the mean values of q and p and of the covariance matrix.

In the present paper, the Brownian motion of a quantum oscillator is studied.[6] The Brownian motion is described by a model Hamiltonian which is assumed to be the one that characterizes the interaction between the oscillator and a reservoir at temperature T. Throughout this paper, we employ the phase-space distribution functions obtained from the density operator via certain rules of mapping.[7,8] We use the master-equation approach, which we discussed elsewhere,[9] to derive an equation of motion for the *reduced* phase-space distribution function characterizing the oscillator system alone. The resulting equation for the reduced phase-space distribution function is found to be of the Fokker-Planck type. We find that when we specialize to the case of normal, antinormal, and Weyl rules of mapping,[8] the Langevin equations corresponding to the Fokker-Planck equation (which is the equation of motion for the reduced phase-space distribution function) are of the form (1.3) found previously for the Brownian motion of a classical oscillator. The Fokker-Planck equation that we derived is solved for the conditional probability (Green's function of the equation), which is then used to calculate the time-dependent quantum statistical properties of the oscillator executing Brownian motion. We then calculate the entropy of this system and show that it reaches equilibrium as $t \to \infty$. We also briefly consider the case when the reservoir is at zero temperature and show that if initially the state of the oscillator is a coherent state,[10] then it will remain in a coherent state for later times. Finally we show that in the weak-coupling limit ($\kappa \ll \omega$) our Langevin equations (or equivalently, the Fokker-Planck equation) reduce to the ones obtained by making the usual rotating-wave approximation.[11]

II. FOKKER-PLANCK EQUATION AND LANGEVIN EQUATIONS FOR BROWNIAN MOTION OF A QUANTUM OSCILLATOR

We take the following as the model Hamiltonian for the oscillator executing Brownian motion:

$$H=\omega\, a^\dagger a+\sum_j \omega_j\, a_j^\dagger a_j+\sum_j \{g_j\, a_j^\dagger (a+a^\dagger)+\text{H. c.}\}\,. \tag{2.1}$$

Here a and $a^\dagger$ are the annihilation and the creation operators for the oscillator (the system of interest), a_j and $a_j^\dagger$ are the annihilation and the creation operators of the jth oscillator of the reservoir, [12] and the g_j's are the coupling constants. The operators a, $a^\dagger$, a_j, and $a_j^\dagger$ satisfy the commutation relations

$$[a,\ a^\dagger]=1,\quad [a_j,\ a_k^\dagger]=\delta_{j_k}\,, \tag{2.2}$$

and all other commutators vanish.

The reduced density operator $\rho_S(t)$ corresponding to the oscillator system alone is obtained from the total density operator $\rho_{R+S}(t)$ by taking the trace over the reservoir variables, i.e.,

$$\rho_S(t)=\mathrm{Tr}_R[\rho_{R+S}(t)]\,. \tag{2.3}$$

We assume that at time $t=0$, the reservoir is in thermal equilibrium at temperature T, i.e., its density operator $\rho_R(0)$ is given by

$$\rho_R(0)=\exp(-\beta\textstyle\sum_j \omega_j\, a_j^\dagger a_j)/\mathrm{Tr}[\exp(-\beta\sum_j \omega_j\, a_j^\dagger a_j)]\,, \tag{2.4}$$

where $\beta=1/K_BT$ and K_B is the Boltzmann constant. We also assume that at time $t=0$, the reservoir and the oscillator system are statistically independent, i.e.,

$$\rho_{R+S}(0)=\rho_R(0)\rho_S(0)\,. \tag{2.5}$$

Let $\Phi_S^{(N)}(z, z^*, t)$ be the reduced phase-space distribution function, which is obtained from $\rho_S(t)$ by mapping it according to the normal rule of mapping. [8(a),8(b)] The equation of motion for $\Phi_S^{(N)}$ can be found by using standard techniques. [9] In fact it has been derived previously in Ref. 9. The master equation for $\Phi_S^{(N)}$, obtained from Eq. (5.26) of Ref. 9 by letting $q=1$ and $F_S^{(N)}=\Phi_S^{(N)}$, is

$$\frac{\partial\Phi_S^{(N)}}{\partial t}=\kappa\left(2[1+\langle n(\omega)\rangle]\,\frac{\partial^2\Phi_S^{(N)}}{\partial z\partial z^*}+\frac{\partial}{\partial z}(z\,\Phi_S^{(N)})+\frac{\partial}{\partial z^*}(z^*\,\Phi_S^{(N)})\right)-\kappa\left(\frac{\partial}{\partial z}(z^*\,\Phi_S^{(N)})\,e^{2i\omega t}+\text{c. c.}\right)-\kappa[1+\langle n(\omega)\rangle]\left(e^{2i\omega t}\,\frac{\partial^2\Phi_S^{(N)}}{\partial z^2}+\text{c. c.}\right), \tag{2.6}$$

where

$$\kappa=\pi h(\omega)|g(\omega)|^2,\qquad \langle n(\omega)\rangle=(e^{\beta\omega}-1)^{-1}\,. \tag{2.7}$$

In deriving Eq. (2.6) we took the infinite-volume limit for the reservoir and made the Born approximation in conjunction with a short-memory approximation. The quantity κ plays the role of the damping coefficient. $\Phi_S^{(N)}$ in Eq. (2.6) is associated with the density operator in the interaction picture. On transforming to the Schrödinger picture, we obtain[13]

$$\frac{\partial}{\partial t}\Phi_S^{(N)}=i\omega\left(\frac{\partial}{\partial z}(z\,\Phi_S^{(N)})-\frac{\partial}{\partial z^*}(z^*\,\Phi_S^{(N)})\right)+\kappa\left(\frac{\partial}{\partial z}[(z-z^*)\,\Phi_S^{(N)}]+\text{c. c.}\right)-\kappa[1+\langle n(\omega)\rangle]\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right)^2\Phi_S^{(N)}\,. \tag{2.8}$$

One may similarly derive the equation of motion for the Sudarshan-Glauber distribution function[14] $\Phi_S^{(A)}$ and the Wigner distribution function[15,16] $\Phi_S^{(W)}$. We write these equations collectively as follows:

$$\frac{\partial\Phi_S}{\partial t}=i\omega\left(\frac{\partial}{\partial z}(z\,\Phi_S)-\frac{\partial}{\partial z^*}(z^*\,\Phi_S)\right)+\kappa\left(\frac{\partial}{\partial z}[z-z^*)\,\Phi_S]+\text{c. c.}\right)-\kappa\left[\lambda+\tfrac{1}{2}+\langle n(\omega)\rangle\right]\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right)^2\Phi_S\,. \tag{2.9}$$

Here the parameter λ takes values $-\frac{1}{2}$, 0, $\frac{1}{2}$ for $\Phi_S^{(A)}$, $\Phi_S^{(W)}$, and $\Phi_S^{(N)}$, respectively. [17] Equation (2.9) is an equation of the Fokker-Planck type and its solution makes it possible to calculate[18] all the time-dependent quantum statistical properties of the oscillator which is undergoing Brownian motion.

We now make the transformation to the real variables q and p defined by

$$z=\sqrt{\tfrac{1}{2}}\,[(m\omega)^{1/2}q+ip/(m\omega)^{1/2}]\,,\qquad z^*=\sqrt{\tfrac{1}{2}}\,[(m\omega)^{1/2}q-ip/(m\omega)^{1/2}]\,. \tag{2.10}$$

Then the Fokker-Planck equation (2.9) transforms into the following equation:

$$\frac{\partial\Phi_S}{\partial t}=-\frac{\partial}{\partial q}\left(\frac{p}{m}\,\Phi_S\right)+\frac{\partial}{\partial p}[(m\omega^2q+2\kappa p)\,\Phi_S]+2m\omega\kappa\left[\lambda+\tfrac{1}{2}+\langle n(\omega)\rangle\right]\frac{\partial^2\Phi_S}{\partial p^2}\,. \tag{2.11}$$

We will now for a while consider only the equation of motion for $\Phi_S^{(A)}$. Substituting $\lambda=-\frac{1}{2}$ in Eq. (2.11), we find that

$$\frac{\partial\Phi_S^{(A)}}{\partial t}=-\frac{\partial}{\partial q}\left(\frac{p}{m}\,\Phi_S^{(A)}\right)+\frac{\partial}{\partial p}[(m\omega^2q+2\kappa p)\,\Phi_S^{(A)}]+2m\omega\kappa\eta\,\frac{\partial^2\Phi_S^{(A)}}{2p^2}\,, \tag{2.12}$$

where

$$\eta=\langle n(\omega)\rangle\equiv(e^{\beta\omega}-1)^{-1}\,. \tag{2.13}$$

The Fokker-Planck equation (2.12) is equivalent[5]

to the Langevin equations given by

$$\dot{q} = p/m\,,\quad \dot{p} = -2\kappa p - m\omega^2 q + f(t)\,, \tag{2.14}$$

where $f(t)$ is a real Gaussian random process with zero mean and the correlation function for $f(t)$ is given by

$$\langle f(t_1) f(t_2)\rangle = 2D\,\delta(t_1 - t_2)\,. \tag{2.15}$$

Here D is the diffusion coefficient and is given by

$$D = 2m\omega\kappa\langle n(\omega)\rangle\,. \tag{2.16}$$

The Langevin equations (2.14) are of the same form as the Wang-Uhlenbeck equations (1.3). Equations (2.14)[19] may be taken to be the quantum analog of the Wang-Uhlenbeck equations.[4] We recall that in the classical case q and p are the position and the momentum variables, respectively, whereas in our case q and p are the c numbers onto which the position and the momentum operators are mapped by the normal rule of mapping. Moreover, in the classical case the diffusion coefficient for the random force $f(t)$ is given by

$$D = 2\kappa m K_B T\,, \tag{2.17}$$

which can be obtained from (2.16) by taking the high-temperature limit.

Let $K^{(A)}(q, p, t \mid q_0, p_0, 0)$ be the Green's function (conditional probability) associated with (2.12). $K^{(A)}$ is, by definition, the solution of (2.12) subject to the initial condition

$$K^{(A)}(q, p, 0 \mid q_0, p_0 0) = \delta(q - q_0)\,\delta(p - p_0)\,. \tag{2.18}$$

It is shown in Appendix B that $K^{(A)}$ is given by

$$K^{(A)}(q, p, t \mid q_0, p_0, 0) = [(2\pi^2\Delta]^{-1/2} \times \exp\{-(1/2\Delta)[\beta(q - \langle q(t)\rangle)^2 + \alpha(p - \langle p(t)\rangle)^2 - 2\gamma(q - \langle q(t)\rangle)(p - \langle p(t)\rangle)]\}\,, \tag{2.19}$$

where

$$\langle q(t)\rangle = \left[\left(\cos\omega_0 t + \frac{\kappa}{\omega_0}\sin\omega_0 t\right) q_0 + \frac{\sin\omega_0 t}{m\omega_0} p_0\right] e^{-\kappa t}\,, \tag{2.20a}$$

$$\langle p(t)\rangle = \left[\left(\cos\omega_0 t - \frac{\kappa}{\omega_0}\sin\omega_0 t\right) p_0 - \frac{m\omega^2 \sin\omega_0 t}{\omega_0} q_0\right] e^{-\kappa t}\,, \tag{2.20b}$$

$$\alpha = \frac{\eta}{m\omega}\left[1 - \left(\frac{\omega^2}{\omega_0^2} - \frac{\kappa^2}{\omega_0^2}\cos 2\omega_0 t + \frac{\kappa}{\omega_0}\sin 2\omega_0 t\right) e^{-2\kappa t}\right], \tag{2.21a}$$

$$\beta = \eta m\omega\left[1 - \left(\frac{\omega^2}{\omega_0^2} - \frac{\kappa^2}{\omega_0^2}\cos 2\omega_0 t - \frac{\kappa}{\omega_0}\sin 2\omega_0 t\right) e^{-2\kappa t}\right], \tag{2.21b}$$

$$\gamma = 2\kappa\omega\eta\,\frac{\sin^2\omega_0 t}{\omega_0^2}\, e^{-2\kappa t}\,, \tag{2.21c}$$

and

$$\omega_0 = (\omega^2 - \kappa^2)^{1/2},\quad \Delta = (\alpha\beta - \gamma^2)\,. \tag{2.22}$$

It is seen from (2.19) that the solution for the Green's function $K^{(A)}$ is of the form of a two-dimensional Gaussian distribution with nonzero mean.[20] The coefficient ω_0 is real for the underdamped case ($\kappa < \omega$) and pure imaginary for the overdamped case ($\kappa > \omega$).

On transforming back to the complex variables via Eq. (2.10), we obtain the following Langevin equations:

$$\dot{z} = -i\omega z - \kappa(z - z^*) + \mathcal{F}(t)\,, \tag{2.23a}$$

$$\dot{z}^* = +i\omega z^* - \kappa(z^* - z) + \mathcal{F}^*(t)\,, \tag{2.23b}$$

where $\mathcal{F}(t)$ is a *complex* Gaussian random process[21] with zero mean and

$$\langle \mathcal{F}(t_1)\,\mathcal{F}(t_2)\rangle = -2\kappa\eta\delta(t_1 - t_2)\,, \tag{2.24a}$$

$$\langle \mathcal{F}^*(t_1)\,\mathcal{F}(t_2)\rangle = 2\kappa\eta\delta(t_1 - t_2)\,. \tag{2.24b}$$

The corresponding solution for the Green's function $K^{(A)}$ in terms of the complex variables z and z^* is

$$K^{(A)}(z, z^*, t \mid z_0, z_0^*, 0) = (\pi^2\Delta_0)^{-1/2} \times \exp\{\Delta_0^{-1}[\mu(z^* - \langle z^*(t)\rangle)^2 + \mu^*(z - \langle z(t)\rangle)^2 - \tau|z - \langle z(t)\rangle|^2]\}\,, \tag{2.25}$$

where

$$\langle z(t)\rangle = \left(\cos\omega_0 t - \frac{i\omega}{\omega_0}\sin\omega_0 t\right) z_0 e^{-\kappa t} + \frac{\kappa}{\omega_0}\sin\omega_0 t\, z_0^* e^{-\kappa t}\,, \tag{2.26}$$

$$\mu = -\frac{\kappa\eta}{\omega_0}\sin\omega_0 t\left(\cos\omega_0 t - \frac{i\omega}{\omega_0}\sin\omega_0 t\right) e^{-2\kappa t}, \tag{2.27a}$$

$$\tau = \eta\left[1 - \left(1 + \frac{2\kappa^2}{\omega_0^2}\sin^2\omega_0 t\right) e^{-2\kappa t}\right], \tag{2.27b}$$

and

$$\Delta_0 = (\tau^2 - 4|\mu|^2)\,. \tag{2.28}$$

In these equations z_0 is, of course, related to q_0 and p_0 in the same way as z to q and p [cf. Eq. (2.10)]. We will now calculate some of the statistical properties under two conditions of initial excitation.

A. Initial Coherent State Excitation

We first consider the case when the oscillator was initially excited to a coherent state[10] $|z_0\rangle$, i.e.,

$$\Phi_S^{(A)}(z, z^*, 0) = \delta^{(2)}(z - z_0) . \quad (2.29)$$

The phase-space distribution function at time t is then obviously identical with the Green's function $K^{(A)}$ given by Eq. (2.25). We thus conclude that the reduced phase-space distribution function is a nonstationary *complex* Gaussian distribution with nonzero mean. The time dependence of some of the lower-order moments is readily found to be

$$\langle a(t)\rangle = \langle z(t)\rangle , \quad (2.30a)$$

$$\langle [a^\dagger(t) - \langle a^\dagger(t)\rangle][a(t) - \langle a(t)\rangle]\rangle = \tau , \quad (2.30b)$$

$$\langle [a(t) - \langle a(t)\rangle]^2\rangle = +2\mu , \quad (2.30c)$$

$$\langle [a^\dagger(t) - \langle a^\dagger(t)\rangle]^2 [a(t) - \langle a(t)\rangle]\rangle^2 = 2\tau^2 + 4|\mu|^2 . \quad (2.30d)$$

The distribution function $\Phi_S^{(A)}$ may be used to calculate all the normally ordered moments. The multitime correlation functions may also be calculated[18]; for example, we have

$$\langle a^\dagger(t_1)a(t_2)\rangle = \int\int (z_1^* z_2) K^{(A)}(z_1, z_1^*, t_1 | z_2, z_2^*, t_2) \times K^{(A)}(z_2, z_2^*, t_2 | z_0, z_0^*, 0)\, d^2z_1 d^2z_2 , (t_1 > t_2) . \quad (2.31)$$

On substituting (2.25) into (2.31), we find that

$$\langle [a^\dagger(t_1) - \langle a^\dagger(t_1)\rangle][a(t_2) - \langle a(t_2)\rangle]\rangle = (-2\kappa^2\eta/\omega_0^2)\sin\omega_0 t_2 \sin\omega_0 t_1 \times e^{-\kappa(t_1+t_2)} + \eta(1 - e^{-2\kappa t_2}) e^{-\kappa(t_1 - t_2)} \times [\cos\omega_0(t_1 - t_2) + (i\omega/\omega_0)\sin\omega_0(t_1 - t_2)] . \quad (2.32)$$

We now calculate the entropy S associated with the quantum oscillator undergoing Brownian motion. We have shown in a recent paper[22] that the entropy for a system, which is in a state characterized by the Gaussian-Wigner distribution function of the form

$$\Phi^{(W)}(q, p) = [(2\pi)^2(\alpha_0\beta_0 - \gamma_0^2)]^{-1/2} \exp\{-\tfrac{1}{2}(\alpha_0\beta_0 - \gamma_0^2)^{-1} \times [\beta_0(q - \langle q\rangle)^2 + \alpha_0(p - \langle p\rangle)^2 - 2\gamma_0(q - \langle q\rangle)(p - \langle p\rangle)]\} , \quad (2.33)$$

is given by

$$S = K_B[(\sigma + 1)\ln(\sigma + 1) - \sigma\ln\sigma] , \quad (2.34)$$

where

$$\sigma = (\alpha_0\beta_0 - \gamma_0^2)^{1/2} - \tfrac{1}{2} . \quad (2.35)$$

We prove in Appendix C that the Wigner distribution function for the problem under consideration is also given by (2.25) but with τ replaced by $(\tau + \frac{1}{2})$. This result may be used to show that the Wigner distribution function is of the form (2.33) with the parameters given by

$$\alpha_0 = (\alpha + 1/2m\omega), \quad \beta_0 = (\beta + \tfrac{1}{2}m\omega), \quad \gamma_0 = \gamma , \quad (2.36)$$

and hence the entropy is given by (2.34) with

$$\sigma = \{[\eta(1 - e^{-2\kappa t}) + \tfrac{1}{2}]^2 - (4\kappa^2/\omega_0^2)\sin^2\omega_0 t\, \eta(\eta + \tfrac{1}{2}) e^{-2\kappa t}\}^{1/2} - \tfrac{1}{2} . \quad (2.37)$$

It is obvious from (2.37) that $\sigma \to \eta = \langle n(\omega)\rangle$ as $t \to \infty$ and the entropy S tends to

$$S \to K_B\{[\langle n(\omega)\rangle + 1]\ln[\langle n(\omega)\rangle + 1] - \langle n(\omega)\rangle \ln\langle n(\omega)\rangle\} \quad \text{as } t \to \infty . \quad (2.38)$$

The entropy given by the right-hand side of (2.38) will be recognized as the entropy for a quantum oscillator in thermal equilibrium. This result shows that the system approaches equilibrium as $t \to \infty$. In fact we show in Appendix B that the steady-state distribution functions given by

$$\Phi_S^{(A)}(z, z^*, t) \to \frac{1}{\pi\langle n(\omega)\rangle} \exp\left(-\frac{|z|^2}{\langle n(\omega)\rangle}\right) \quad \text{as } t \to \infty . \quad (2.39)$$

We will now briefly examine the zero-temperature case. For zero temperature $\langle n(\omega)\rangle = 0$ and the Langevin equations (2.23) reduce to

$$\dot z = -i\omega z - \kappa(z - z^*) , \quad (2.40a)$$

$$\dot z^* = +i\omega z^* - \kappa(z^* - z) . \quad (2.40b)$$

The solution to these equations is readily found to be

$$z(t) = [\cos\omega_0 t - (i\omega/\omega_0)\sin\omega_0 t]\, z(0)\, e^{-\kappa t} + (\kappa/\omega_0)\sin\omega t\, z^*(0) e^{-\kappa t} . \quad (2.41)$$

Since the initial distribution function is of the form (2.29) it is obvious from this solution that the phase-space distribution function for later times is given by

$$\Phi_S^{(A)}(z, z^*, t) = \delta^{(2)}(z - \langle z(t)\rangle), \quad (2.42a)$$

where

$$\langle z(t)\rangle = [\cos\omega_0 t - (i\omega/\omega_0)\sin\omega_0 t] z_0 e^{-\kappa t} + (\kappa/\omega_0)\sin\omega_0 t\, z_0^* e^{-\kappa t} . \quad (2.42b)$$

We therefore conclude that, for the case of zero temperature, if the system is initially in a coherent state $|z_0\rangle$, it will remain in the coherent state whose amplitude is given by Eq. (2.42b). The normally ordered moments will then be given by

$$\langle [a^\dagger(t)]^m [a(t)]^n\rangle = \langle z^*(t)\rangle^m \langle z(t)\rangle^n . \quad (2.43)$$

B. Initial Thermal Excitation

We next consider the case when the oscillator was initially excited to a state characterized by the phase-space distribution function

$$\Phi_S^{(A)}(z, z^*, 0) = \frac{1}{\pi\eta_0}\exp\left(-\frac{|z|^2}{\eta_0}\right),$$

$$\eta_0 = (e^{\omega/K_B T_0} - 1)^{-1}, \tag{2.44}$$

appropriate to thermal equilibrium at temperature T_0. The distribution function for later times is given by

$$\Phi_S^{(A)}(z, z^*, t) = \int K^{(A)}(z, z^*, t \mid z_0, z_0^*, 0)\Phi_S^{(A)}(z_0, z_0^*, 0)d^2 z_0. \tag{2.45}$$

On substituting (2.25) and (2.44) into (2.45), we find, after a straightforward but long calculation, that the reduced phase-space distribution function at time t is given by

$$\Phi_S^{(A)}(z, z^*, t) = [\pi^2(\tau_0^2 - 4\mid \mu_0\mid^2)]^{-1/2} \times \exp[-(\tau_0^2 - 4\mid \mu_0\mid^2)^{-1}(\tau_0 \mid z\mid^2 - \mu_0 z^{*2} - \mu_0^* z^2)], \tag{2.46}$$

where the parameters μ_0 and τ_0 are given by

$$\mu_0 = (\eta_0 - \eta)(\kappa/\omega_0)\sin\omega_0 t[\cos\omega_0 t - (i\omega/\omega_0)\sin\omega_0 t]e^{-2\kappa t}, \tag{2.47a}$$

$$\tau_0 = \eta + (\eta_0 - \eta)[1 + (2\kappa^2/\omega_0^2)\sin^2\omega_0 t]e^{-2\kappa t}. \tag{2.47b}$$

The reduced distribution function is in the form of a complex Gaussian distribution with zero mean. In the limit as $T_0 \to T$ (or $\eta_0 \to \eta$), we find that (2.46) reduces to (2.44). We conclude that if the oscillator executing Brownian motion was already in thermal equilibrium with the medium (reservoir) then, as one would expect, it will remain in equilibrium.

We conclude this section by giving the form of the corresponding equation of motion for the reduced density operator $\rho_S(t)$. It may be obtained from Eq. (2.9), for $\lambda = -\frac{1}{2}$, by applying the mapping operator[7,8(a),8(b)] $\Omega^{(A)}$ for the antinormal rule of association and by making use of the following identities[23]:

$$\Omega^{(A)}\left(\frac{\partial}{\partial z}(z\Phi_S^{(A)})\right) = -\frac{1}{\pi}[a^\dagger, a\rho_S], \tag{2.48a}$$

$$\Omega^{(A)}\left(\frac{\partial}{\partial z}(z^*\Phi_S^{(A)})\right) = -\frac{1}{\pi}[a^\dagger, \rho_S a^\dagger], \tag{2.48b}$$

$$\Omega^{(A)}\left(\frac{\partial^2}{\partial z\,\partial z^*}\Phi_S^{(A)}\right) = -\frac{1}{\pi}[a^\dagger, [a, \rho_S]], \tag{2.48c}$$

$$\Omega^{(A)}\left(\frac{\partial^2}{\partial z^2}\Phi_S^{(A)}\right) = \frac{1}{\pi}[a^\dagger, [a^\dagger, \rho_S]], \tag{2.48d}$$

and their adjoints. We find that the reduced density operator $\rho_S(t)$ satisfies the master equation

$$\frac{\partial \rho_S}{\partial t} = -i\omega[a^\dagger a, \rho_S] - \kappa(a^\dagger a\rho_S - 2a\rho_S a^\dagger + \rho_S a^\dagger a + a^2\rho_S - a\rho_S a - a^\dagger\rho_S a^\dagger + \rho_S a^{\dagger 2}) - \kappa\langle n(\omega)\rangle(2[a^\dagger, [a, \rho_S]] + [a^\dagger, [a^\dagger, \rho_S]] + [a, [a, \rho_S]]). \tag{2.49}$$

III. ROTATING-WAVE APPROXIMATION AND BROWNIAN MOTION

In most of the problems in quantum optics and in other fields, one conventionally makes the rotating-wave approximation,[11] which amounts to ignoring the rapidly oscillating terms. On making the rotating-wave approximation our model Hamiltonian (2.1), describing the Brownian motion of a quantum oscillator, reduces to

$$H = \omega a^\dagger a + \sum_j \omega_j a_j^\dagger a_j + \sum_j (g_j a_j^\dagger a + \text{H.c.}). \tag{3.1}$$

We can again obtain the master equation for the reduced distribution function $\Phi_S^{(A)}$, as we have already done in [Ref. 9, Eq. (5.19)]. The equation of motion for $\Phi_S^{(A)}$ is

$$\frac{\partial \Phi_S^{(A)}}{\partial t} = \kappa\left(2\eta\frac{\partial^2\Phi_S^{(A)}}{\partial z\,\partial z^*} + \frac{\partial}{\partial z}(z\Phi_S^{(A)}) + \frac{\partial}{\partial z^*}(z^*\Phi_S^{(A)})\right). \tag{3.2}$$

On transforming back to the Schrödinger picture (see the discussion in Appendix A), we obtain the equation

$$\frac{\partial \Phi_S^{(A)}}{\partial t} = i\omega\left(z\frac{\partial \Phi_S^{(A)}}{\partial z} - z^*\frac{\partial \Phi_S^{(A)}}{\partial z^*}\right) + \kappa\left(2\eta\frac{\partial^2\Phi_S^{(A)}}{\partial z\,\partial z^*} + \frac{\partial}{\partial z}(z\Phi_S^{(A)}) + \frac{\partial}{\partial z^*}(z^*\Phi_S^{(A)})\right). \tag{3.3}$$

On introducing the real variables q and p defined by Eq. (2.10), we find that (3.3) reduces to

$$\frac{\partial \Phi_S^{(A)}}{\partial t} = \frac{\partial}{\partial q}[(\kappa q - p/m)\Phi_S^{(A)}] + \frac{\partial}{\partial p}[(\kappa p + m\omega^2 q)\Phi_S^{(A)}] + \kappa\eta\left(\frac{1}{m\omega}\frac{\partial^2\Phi_S^{(A)}}{\partial q^2} + m\omega\frac{\partial^2\Phi_S^{(A)}}{\partial p^2}\right). \tag{3.4}$$

This equation has the form of the Fokker-Planck equation. The corresponding Langevin equations are

$$\dot{q} = -\kappa q + p/m + F_q(t), \quad \dot{p} = -\kappa p - m\omega^2 q + F_p(t). \tag{3.5}$$

Here $F_q(t)$ and $F_p(t)$ are two independent real Gaussian random processes with zero means and

$$\langle F_q(t_1)F_q(t_2)\rangle = \frac{2\kappa\langle n(\omega)\rangle}{m\omega}\delta(t_1 - t_2), \tag{3.6a}$$

$$\langle F_p(t_1)F_p(t_2)\rangle = 2\kappa\langle n(\omega)\rangle m\omega\delta(t_1 - t_2). \tag{3.6b}$$

We find from Eq. (3.5) that the time derivative of q is no longer equal to p/m whereas the relation $\dot{q} = p/m$ is expected to hold good in general. We will now explain this anomaly.

In the weak-coupling limit ($\kappa \ll \omega$), we can replace ω_0 by ω and ignore all the terms of order $\kappa e^{-\kappa t}$ or higher. We then find that Eqs. (2.20)–(2.22) reduce to

$$\langle q(t)\rangle = [\cos\omega t\, q_0 + (\sin\omega t/m\omega)p_0]e^{-\kappa t}, \tag{3.7a}$$

$$\langle p(t)\rangle = (\cos\omega t\, p_0 - m\omega \sin\omega t q_0)e^{-\kappa t}, \quad (3.7b)$$

$$\alpha = (\eta/m\omega)(1 - e^{-2\kappa t}), \quad \beta = \eta m\omega(1 - e^{-2\kappa t}), \quad (3.8)$$

and

$$\gamma = 0. \quad (3.9)$$

Therefore in the weak-coupling limit, the solution (2.19) for the Green's function $K^{(A)}$ reduces to

$$K^{(A)}(q, p, t \mid q_0, p_0, 0) = [2\pi\eta(1 - e^{-2\kappa t})]^{-1} \times \exp\{-(1/2\alpha)[q - \langle q(t)\rangle]^2 - (1/2\beta)[p - \langle p(t)\rangle^2]\}, \quad (3.10)$$

where $\langle q(t)\rangle$ and $\langle p(t)\rangle$ are now given by (3.7a) and (3.7b), respectively. It is easily verified that (3.10) is the solution of the following Fokker-Planck equation:

$$\frac{\partial \Phi_S^{(A)}}{\partial t} = \frac{\partial}{\partial q}[(\kappa q - p/m)\Phi_S^{(A)}] + \frac{\partial}{\partial p}[(\kappa p + m\omega^2 q)\Phi_S^{(A)}] + \kappa\eta\left(\frac{1}{m\omega}\frac{\partial^2 \Phi_S^{(A)}}{\partial q^2} + m\omega\frac{\partial^2 \Phi_S^{(A)}}{\partial p^2}\right), \quad (3.11)$$

subject to the initial condition

$$\Phi_S^{(A)}(q, p, 0) = \delta(q - q_0)\delta(p - p_0). \quad (3.12)$$

The Fokker-Planck equation (3.11) is the same as (3.4), which was obtained by making the rotating-wave approximation. This then leads us to conclude that the Langevin equations (3.5) obtained by making the rotating-wave approximation are the weak-coupling limits[23a] ($\kappa \ll \omega$) of the Langevin equations (2.14), and this explains the anomaly that we noted above.

In the present paper, we considered only the Brownian of a single quantum oscillator. The results are easily generalized to the case of N quantum oscillators undergoing Brownian motion. In a future publication, we hope to discuss the non-Markoffian behavior of the quantum oscillator executing Brownian motion.

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APPENDIX A: RELATION BETWEEN EQUATIONS OF MOTION FOR PHASE-SPACE DISTRIBUTION FUNCTION IN SCHRÖDINGER PICTURE AND IN INTERACTION PICTURE

In this appendix, we discuss the relation between the equations of motion for the phase-space distribution function in the Schrödinger picture and in the interaction picture. We write the Hamiltonian of the system under consideration as

$$H = H_0 + H_1, \quad (A1)$$

where H_0 and H_1 are the unperturbed and perturbed Hamiltonians, respectively. We have shown elsewhere[7,8(b),9] that the phase-space distribution function Φ satisfies an equation of the form

$$\frac{\partial \Phi}{\partial t} = -i\mathcal{L}_0\Phi - i\mathcal{L}_1\Phi, \quad (A2)$$

where $\mathcal{L}_0$ and $\mathcal{L}_1$ are the Liouville operators (differential operators) corresponding to H_0 and H_1, respectively. We assume that $\mathcal{L}_0$ is explicitly time independent. Then the phase-space distribution function Φ_I in the interaction picture is defined by

$$\Phi_I = \exp(i\mathcal{L}_0 t)\Phi. \quad (A3)$$

It is then obvious from (A2) and (A3) that Φ_I satisfies the equation of motion

$$\frac{\partial \Phi_I}{\partial t} = -i\mathcal{L}_I(t)\Phi_I, \quad (A4)$$

where $\mathcal{L}_I(t)$ is the interaction Liouville operator in the interaction picture and is given by

$$\mathcal{L}_I(t) = \exp(i\mathcal{L}_0 t)\mathcal{L}_1 \exp(-i\mathcal{L}_0 t). \quad (A5)$$

On inverting (A5), we also obtain

$$\mathcal{L}_1 = \exp(-i\mathcal{L}_0 t)\mathcal{L}_I(t)\exp(i\mathcal{L}_0 t) = \mathcal{L}_I(t) + (-it)[\mathcal{L}_0, \mathcal{L}_I(t)] + [(-it)^2/2!][\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_I(t)]] + \cdots. \quad (A6)$$

Equations (A2)–(A6) are the desired relations.

For the problem considered in this paper, the unperturbed Hamiltonian for the oscillator system is

$$H_0 = \omega a^\dagger a. \quad (A7)$$

It can be easily shown[7] that for the normal rule of mapping, the unperturbed Liouville operator is given by

$$\mathcal{L}_0 = \omega\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) \quad (A8)$$

The equation of motion (2.6) is for the distribution function in the interaction picture and therefore $-i\mathcal{L}_I(t)$ is given by

$$-i\mathcal{L}_I(t) = \kappa\left(2 + z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*} - z^* e^{2i\omega t}\frac{\partial}{\partial z} - z e^{-2i\omega t}\frac{\partial}{\partial z^*}\right) + \kappa[\langle n(\omega)+1\rangle]\left(2\frac{\partial^2}{\partial z\,\partial z^*} - e^{2i\omega t}\frac{\partial^2}{\partial z^2} - e^{-2i\omega t}\frac{\partial^2}{\partial z^{*2}}\right). \quad (A9)$$

For $\mathcal{L}_0$ given by (A8), one may easily prove the following relations:

$$\exp(-i\mathcal{L}_0 t)\left(\frac{\partial}{\partial z}\right)\exp(i\mathcal{L}_0 t) = \exp(-i\omega t)\frac{\partial}{\partial z}, \quad (A10a)$$

$$\exp(-i\mathcal{L}_0 t)\left(\frac{\partial}{\partial z^*}\right)\exp(i\mathcal{L}_0 t) = \exp(i\omega t)\frac{\partial}{\partial z^*}, \quad (A10b)$$

$$\exp(-i\mathcal{L}_0 t)(z)\exp(i\mathcal{L}_0 t) = z\exp(i\omega t)\,, \tag{A10c}$$

$$\exp(-i\mathcal{L}_0 t)(z^*)\exp(i\mathcal{L}_0 t) = z^*\exp(-i\omega t)\,. \tag{A10d}$$

On making use of these relations, we find that

$$\begin{aligned} -i\mathcal{L}_1 &= \exp(-i\mathcal{L}_0 t)[-i\mathcal{L}_I(t)]\exp(i\mathcal{L}_0 t) \\ &= \kappa\left[2+(z-z^*)\frac{\partial}{\partial z}+(z^*-z)\frac{\partial}{\partial z^*}\right. \\ &\quad \left.-[1+\langle n(\omega)\rangle]\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right)^2\right]. \end{aligned} \tag{A11}$$

Finally on substituting (A11) and (A8) into (A2) we obtain the desired equation of motion (2.8) for the distribution function in the interaction picture.

APPENDIX B: DERIVATION OF SOLUTION (2.19) FOR LINEARIZED FOKKER-PLANCK EQUATION (2.12)

We first consider the following linearized Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\sum_{ij}\frac{\partial}{\partial x_i}(\beta_{ij}x_j P)+\sum_{ij}D_{ij}\frac{\partial^2 P}{\partial x_i \partial x_j}\,, \tag{B1}$$

where β_{ij} and D_{ij} are independent of the random variables x. The solution to (B1), subject to the initial condition

$$P(\{x_i\},0) = \prod_i \delta(x_i - x_i^0) \tag{B2}$$

is

$$P(\{x_i\},t|\{x_i^0\},0) = [(2\pi)^N|\det\sigma(t)|]^{-1/2}$$

$$\times\exp\{-\tfrac{1}{2}[X-b(t)X^0]^T\sigma^{-1}(t)[X-b(t)X^0]\}\,, \tag{B3}$$

where X is the column matrix

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

and the superscript T denotes the transpose of the matrix. The parameters $b(t)$ and $\sigma(t)$ are given by

$$b(t) = e^{\beta t}\,, \tag{B4}$$

$$\sigma(t) = \sigma(\infty) - b(t)\sigma(\infty)b^T(t)\,, \tag{B5}$$

and $\sigma(\infty)$ is the solution of

$$\beta\sigma(\infty)+\sigma(\infty)\beta^T = -2D\,. \tag{B6}$$

For the Fokker-Planck equation (2.12), we have

$$\beta = \begin{pmatrix} 0 & m^{-1} \\ -m\omega^2 & -2\kappa \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 2m\omega\kappa\eta \end{pmatrix}. \tag{B7}$$

The matrix $e^{\beta t}$ may be found by using the method of calculating the function of a matrix by the formula

$$e^{\beta t} = \frac{1}{2\pi i}\oint (z-\beta)^{-1}e^{zt}\,dz\,. \tag{B8}$$

It is shown by a straightforward calculation that

$$b(t) = \begin{pmatrix} \cos\omega_0 t+\frac{\kappa}{\omega_0}\sin\omega_0 t & \frac{\sin\omega_0 t}{m\omega_0} \\ -\frac{m\omega^2\sin\omega_0 t}{\omega_0} & \cos\omega_0 t-\frac{\kappa}{\omega_0}\sin\omega_0 t \end{pmatrix} e^{-\kappa t}\,, \tag{B9}$$

where

$$\omega_0 = (\omega^2-\kappa^2)^{1/2}\,. \tag{B10}$$

On using (B6) and (B7) it is easily found that

$$\sigma(\infty) = \frac{\eta}{m\omega}\begin{pmatrix} 1 & 0 \\ 0 & m^2\omega^2 \end{pmatrix}\,. \tag{B11}$$

On substituting (B9) and (B11) into (B5), we find that the matrix elements of $\sigma(t)$ are given by

$$\sigma_{11}(t) = \frac{\eta}{m\omega}\left[1-\left(\frac{\omega^2}{\omega_0^2}-\frac{\kappa^2}{\omega_0^2}\cos 2\omega_0 t + \frac{\kappa}{\omega_0}\sin 2\omega_0 t\right)e^{-2\kappa t}\right]\,, \tag{B12}$$

$$\sigma_{22}(t) = \eta m\omega\left[1-\left(\frac{\omega^2}{\omega_0^2}-\frac{\kappa^2}{\omega_0^2}\cos 2\omega_0 t - \frac{\kappa}{\omega_0}\sin 2\omega_0 t\right)e^{-2\kappa t}\right]\,, \tag{B13}$$

$$\sigma_{12}(t) = \sigma_{21}(t) = (2\kappa\eta\omega/\omega_0^2)\sin^2\omega_0 t\, e^{-2\kappa t}\,. \tag{B14}$$

The desired solution to the Fokker-Planck equation (2.12) is obtained by substituting (B12)–(B14) and (B9) into (B3). One obviously has the relations

$$\alpha = \sigma_{11},\quad \beta = \sigma_{22},\quad \gamma = \sigma_{12}\,, \tag{B15}$$

$$\langle q(t)\rangle = b_{11}(t)q_0 + b_{12}(t)p_0\,, \tag{B16}$$

$$\langle p(t)\rangle = b_{21}(t)q_0 + b_{22}(t)p_0\,. \tag{B17}$$

On substituting the values of the matrix elements $b_{ij}(t)$ into (B16) and (B17), we find that $\langle q(t)\rangle$ and $\langle p(t)\rangle$ are given by Eqs. (2.20a) and (2.20b), respectively.

It is also obvious that the steady-state solution is given by

$$\Phi_S^{(A)}(q,p,t) \to \frac{1}{2\pi\eta}\exp\left[-\frac{1}{2\eta}\left(m\omega q^2+\frac{1}{m\omega}p^2\right)\right] \quad \text{as } t\to\infty, \tag{B18}$$

or equivalently, in terms of the variables z and z^*, one has

$$\Phi_S^{(A)}(z,z^*,t) \to (1/\pi\eta)\,e^{-|z|^2/\eta} \quad \text{as } t\to\infty\,. \tag{B19}$$

APPENDIX C: TIME DEPENDENCE OF WIGNER DISTRIBUTION FUNCTION FOR INITIAL COHERENT STATE EXCITATION

The Wigner distribution function $\Phi_S^{(W)}$ for the oscillator undergoing Brownian motion for an initial coherent state excitation can be obtained by solving (2.9), for $\lambda = 0$, under the initial condition

$$\Phi_S^{(W)}(z,z^*,0) = (2/\pi)\,e^{-2|z-z_0|^2}\,, \tag{C1}$$

which is the Wigner distribution function associated with the density operator $\rho_s(0) = |z_0\rangle\langle z_0|$. We will, however, use an alternative method based on the connecting relations[8(a)] between the distribution functions corresponding to two different rules of association. The Wigner distribution function $\Phi_S^{(W)}$ is related to $\Phi_S^{(A)}$ by

$$\Phi_S^{(W)} = \exp\left(\frac{1}{2}\frac{\partial^2}{\partial z\,\partial z^*}\right)\Phi_S^{(A)}. \tag{C2}$$

We have already obtained the distribution function $\Phi_S^{(A)}$ which is given by (2.25), i.e.,

$$\Phi_S^{(A)} = \frac{1}{\pi(\tau^2 - 4|\mu|^2)^{1/2}}\exp\{-(\tau^2-4|\mu|^2)^{-1} \times[-\mu^*(z-\langle z(t)\rangle)^2 - \mu(z^*-\langle z^*(t)\rangle)^2 + \tau|z-\langle z(t)\rangle|^2]\}, \tag{C3}$$

which can also be rewritten as[24]

$$\Phi_S^{(A)} = \frac{1}{\pi^2\tau}\int \exp\left(-|\alpha|^2 + \frac{\mu^*\alpha^2}{\tau} + \frac{\mu\alpha^{*2}}{\tau} - \frac{[z^*-\langle z^*(t)\rangle]\alpha}{\sqrt{\tau}} + \frac{[z-\langle z(t)\rangle]\alpha^*}{\sqrt{\tau}}\right) d^2\alpha. \tag{C4}$$

On combining (C2) and (C4), we obtain

$$\Phi_S^{(W)} = \frac{1}{\pi^2\tau}\int \exp\left[-|\alpha|^2\left(1+\frac{1}{2\tau}\right) + \frac{\mu^*\alpha^2}{\tau} + \frac{\mu\alpha^{*2}}{\tau} - \frac{[z^*-\langle z^*(t)\rangle]\alpha}{\sqrt{\tau}} + \frac{[z-\langle z(t)\rangle]\alpha^*}{\sqrt{\tau}}\right] d^2\alpha. \tag{C5}$$

On changing the variable of integration to $\beta = (1+1/2\tau)^{1/2}\alpha$, (C5) reduces to

$$\Phi_S^{(W)} = \frac{1}{\pi^2\tau_0}\int \exp\left(-|\beta|^2 + \frac{\mu^*\beta^2}{\tau_0} + \frac{\mu\beta^{*2}}{\tau_0} - \frac{[z^*-\langle z^*(t)\rangle]\beta}{\sqrt{\tau_0}} + \frac{[z-\langle z(t)\rangle]\beta^*}{\sqrt{\tau_0}}\right) d^2\beta, \qquad \tau_0 = \tau + \tfrac{1}{2}. \tag{C6}$$

This integral is easily evaluated[24] and we find that

$$\Phi_S^{(W)} = \frac{1}{\pi(\tau_0^2 - 4|\mu|^2)^{1/2}}\exp\{-(\tau_0^2-4|\mu|^2)^{-1} \times[-\mu^*(z-\langle z(t)\rangle)^2 - \mu(z^*-\langle z^*(t)\rangle)^2 + \tau_0|z-\langle z(t)\rangle|^2]\}. \tag{C7}$$

This result shows that the Wigner distribution function $\Phi_S^{(W)}$ is the same as $\Phi_S^{(A)}$ [given by (2.25)] but with τ replaced by $\tau + \frac{1}{2}$.

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[1]H. A. Kramers, Physica 7, 284 (1940).

[2]G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. 36, 823 (1930).

[3]S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).

[4]M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323 (1945).

[5]For a discussion of Gaussian Markoff processes, see, e.g., R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. I (Gordon and Breach, New York, 1963).

[6]The Brownian motion of a quantum oscillator was studied previously by several workers; see, e.g., (a) P. Ullersma, Physica 32, 27 (1966); 32, 56 (1966) and some of the references therein; (b) J. Schwinger, J. Math. Phys. 2, 407 (1961). Our approach to the problem differs appreciably from others. Schwinger, for example, uses the quantum action principle and obtains the *effective* equations of motion for the annihilation and the creation operators. These equations of motion involve the *random forces* which are themselved operators. Since we employ the phase-space distribution functions (Refs. 7 and 8) to study the Brownian motion of a quantum oscillator, the structure of our theory is much closer to that of classical theory. Moreover to the best of our knowledge, the *Fokker-Planck equation* for the Brownian motion of a *quantum* oscillator has not been obtained before.

[7]G. S. Agarwal and E. Wolf, Phys. Rev. Letters 21, 180 (1968).

[8]For a detailed discussion of the mapping of operators onto c numbers and vice versa, see (a) G. S. Agarwal and E. Wolf, Phys. Rev. D 2, 2161 (1970); (b) 2, 2187 (1970). For a viewpoint different than ours, see (c) M. Lax, Phys. Rev. 172, 350 (1968).

[9]G. S. Agarwal, Phys. Rev. 178, 2025 (1969).

[10]For a detailed discussion of coherent states see, e.g., J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); or R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[11]For a discussion of the *rotating-wave approximation*, see, e.g., H. Haken, *Handbuch der Physik*, Vol. XXV/2C (Springer, Berlin, 1970).

[12]We regard the medium in which the oscillator is undergoing Brownian motion as the reservoir and adopt the harmonic-oscillator model for the reservoir. The harmonic-oscillator model of the reservoir is *not* necessary and we have assumed it here only for the sake of simplicity.

[13]This transformation for the phase-space distribution functions is a bit involved and is carried out in Appendix A.

[14]E. C. G. Sudarshan, Phys. Rev. Letters 10, 277 (1963); and Refs. 8(a), 8(b), and 10.

[15]E. Wigner, Phys. Rev. 40, 749 (1932); J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).

[16]We recall [Ref. 8(b)] that in our terminology the Sudarshan-Glauber distribution function $\Phi_S^{(A)}$ (Wigner distribution function $\Phi_S^{(W)}$) is equal to π^{-1} times the antinormal (Weyl) equivalent of the density operator.

[17]The equation for $\Phi_S^{(A)}$ and $\Phi_S^{(W)}$ can also be obtained from (2.8) by making use of the relations

$$\Phi_S^{(A)} = \exp\left(\frac{\partial^2}{\partial z\partial z^*}\right)\Phi_S^{(W)}, \quad \Phi_S^{(W)} = \exp\left(-\frac{1}{2}\frac{\partial^2}{\partial z\partial z^*}\right)\Phi_S^{(W)}.$$

For a derivation of these relations, see Ref. 8(a).

[18]G. S. Agarwal, Phys. Rev. 177, 400 (1969); G. S. Agarwal and E. Wolf, Phys. Rev. D 2, 2206 (1970); see also Refs. 8(b) and 8(c).

[19]If the oscillator is undergoing Brownian motion in an

external field, represented by $f_{ext}(t)$, then the Langevin Equations (2.14) are modified to

$$\dot{q} = p/m, \quad \dot{p} = -2\kappa p - m\omega^2 q + f(t) + f_{ext}(t).$$

[20]This result also holds good in presence of an external field but with $\langle q(t)\rangle$ and $\langle p(t)\rangle$ replaced by

$$\langle q(t)\rangle + \int_0^t d\tau\, f_{ext}(t-\tau)\,(\sin\omega_0\tau/m\omega_0)\, e^{-\kappa\tau},$$

$$\langle p(t)\rangle + \int_0^t d\tau f_{ext}(t-\tau)[\cos\omega_0\tau - (\kappa/\omega_0)\sin\omega_0\tau]e^{-\kappa\tau},$$

respectively.

[21]Since $\mathfrak{F}(t)$ is a *complex* Gaussian random process with zero mean, its characteristic functional $\xi[\lambda(\cdot)]$ defined by

$$\xi[\lambda(\cdot)] \equiv \langle \exp\{i\int[\lambda^*(t)\mathfrak{F}(t) + \lambda(t)\mathfrak{F}^*(t)]dt\}\rangle$$

is given by

$$\xi[\lambda(\cdot)] = \exp\{-\tfrac{1}{2}\int\int[\lambda^*(t)\lambda^*(\tau)\langle\mathfrak{F}(t)\mathfrak{F}(\tau)\rangle + \lambda(t)\lambda(\tau)\langle\mathfrak{F}^*(t)\mathfrak{F}^*(\tau)\rangle + \lambda^*(t)\lambda(\tau)\langle\mathfrak{F}(t)\mathfrak{F}^*(\tau)\rangle + \lambda(t)\lambda^*(\tau)\langle\mathfrak{F}^*(t)\mathfrak{F}(\tau)\rangle]\,dt\,d\tau\}.$$

[22]G. S. Agarwal, Phys. Rev. A 3, 828 (1971).

[23]These identities are easily proved by making use of the general theory developed in Refs. 8(a) and 8(b).

[23a]We may also eliminate the rapidly oscillating terms directly from the stochastic equation (2.6) and then in the weak-coupling limit, we obtain the Fokker-Planck equation (3.2). For an excellent discussion of this elimination procedure, see, for example, R. L. Stratonovich, Ref. 5, Vol. II, p. 113.

[24]We have made use of the identity (1.18) given in V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961).

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Violation of Boltzmann's *H* Theorem in Real Gases*

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The well-known variational (maximum-entropy) property of the Maxwellian velocity distribution is used to shed some light on the range of validity of the Boltzmann transport equation. It permits a characterization of the initial states for which the Boltzmann *H* theorem is violated. In particular, it is shown that: (a) Any monatomic system for which the equilibrium potential energy exceeds the minimum possible value possesses a continuum of initial states for which the approach to equilibrium takes place through an increase, rather than a decrease, in Boltzmann's *H*. (b) If the initial distribution of particles is spatially homogeneous and Maxwellian, the approach to equilibrium will take place through an increase (decrease) in the Boltzmann *H*, according as the initial potential energy is less (greater) than the equilibrium value. (c) A necessary condition for the *H*-theorem-violating phenomenon is that the approach to equilibrium takes place through a conversion of kinetic energy into potential energy; a sufficient condition requires also that the initial velocity distribution be sufficiently close to Maxwellian. (d) These *H*-theorem-violating conditions are readily attained experimentally; for example, the free expansion of oxygen gas at 160 °K and 45-atm pressure produces an experimentally realizable violation of the Boltzmann *H* theorem.

I. INTRODUCTION

Ever since the famous *Umkehreinwand* and *Wiederkehreinwand* of Zermelo and Loschmidt, it has been clear that the Boltzmann *H* theorem, and therefore the Boltzmann transport equation, cannot be of universal validity, even for a dilute gas. Any system possesses certain initial states for which the *H* theorem is violated. In one sense, these *H*-theorem-violating states can be characterized at once, as those in which the particle positions and velocities are so correlated that *Stosszahlansatz* fails to hold. However, this is very abstract, and gives no hint as to how, or whether, such states could be produced experimentally.

It is often supposed that these *H*-theorem-violating states are in some way exceptional, so that they may be disregarded in practice. While this conclusion is undoubtedly correct in many cases, we show below that when the system has an appreciable potential energy, there is a class of initial conditions for which interparticle forces automatically produce and maintain *H*-theorem-violating states, with the result that $\dot{H}$ remains positive, on the average, throughout the approach to equilibrium. These conditions are, moreover, in no way exceptional; they can be (and undoubtedly have been) produced experimentally.

The existence of this *H*-theorem-violating phenomenon was pointed out briefly at the end of the writer's Brandeis lectures[1] on statistical mechanics; however, the class of states for which it occurs was described incompletely, in terms of the average force acting on a particle. We obtain below a simpler description, in terms of the kinetic and potential energy of the system.

II. DERIVATIONS

Consider a monatomic fluid consisting of N particles of mass m, confined to a box of volume V,