Order-Parameter Fluctuations in a Weakly Interacting Bose Gas near the Superfluid Transition*

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(Received 8 May 1970)

The contribution of order-parameter fluctuations to the static and dynamic properties of a weakly interacting Bose gas near the Bose-Einstein condensation temperature T_c is investigated via a novel approach. Self-consistent approximations for the propagator of order-parameter fluctuations G are generated in successive temperature ranges closer and closer to T_c. Mean-field theory corresponds to assuming that the correlation of fluctuations G_2 is given by $G_2^{(1)}=GG$. This first approximation is valid in the range $\tau_1<\tau=(T-T_c)/T_c<1$, where at $\tau=\tau_1$ the neglected terms are comparable to those included. In this paper we consider the second approximation $G_2^{(2)}=GGVGG$, which involves the first-order interaction between the fluctuations. Two independent criteria for the range of validity $\tau_2<\tau<\tau_1$ of the second approximation are established. It is conjectured that the qualitative results of the present model calculation are applicable to liquid helium in the presently accessible temperature range. The predicted temperature dependences of the specific heat, superfluid density, and fourth sound are in agreement with the scaling hypothesis and available experiments in helium. The condensate density is predicted to vanish linearly with τ. It is shown that the contribution of order-parameter fluctuations to the thermal conductivity diverges as $\tau^{-1/3}$ above T_c, and it is suggested that the absence of logarithmic factors is due to the inconsistent treatment of entropy fluctuations.

I. INTRODUCTION

We consider in this paper a calculation of the critical behavior of a weakly interacting Bose gas near the superfluid transition. A study of this simple model should be helpful in understanding the nature of the λ transition in helium, even though liquid helium is a strongly interacting system.

In the absence of a microscopic theory that is valid in the immediate vicinity of the superfluid transition, we analyze the behavior of the weakly interacting Bose gas close to, but not in, the immediate vicinity of the transition. Mean-field theory is a notable example of a theory that in general is valid near the transition but not in its immediate vicinity. The temperature range in which mean-field theory is valid is referred to as the classical range; the temperature range inside the classical range is known as the critical range. In the present approach we consider the critical range to be divided into critical subranges and investigate the first critical subrange neighboring the classical range. It is expected that it might be easier to construct a consistent theory in the first critical subrange than in the immediate vicinity of the transition.

We introduce a simple physical picture of the superfluid transition that will help motivate the more mathematical discussion to follow. Let us consider temperatures T above the Bose-Einstein condensation temperature T_c, and define the dimensionless quantity $\tau=(T-T_c)/T_c$. The order parameter appropriate to the superfluid transition is the Bose field operator $\psi(r,t)$, and hence order-parameter fluctuations are described by the fluctuation propagator G, the one-particle Green's function.[1] Similarly the correlation of fluctuations in the system is described by G_2, the two-particle Green's function. A determination of G yields the thermodynamic properties of the system; G_2 and higher-order correlation functions are related to the dynamical properties of the system.

Mean-field theory assumes that the fluctuations are small, and corresponds in the present context to the assumption that the fluctuations propagate independently,[2] i.e., $G_2^{(1)}=GG$ (in symbolic notation). This first approximation will be valid as long as the terms neglected are small compared to those included. The classical range is then given by $\tau_1<\tau<1$, where at $\tau=\tau_1$ the neglected diagrams are comparable to those included. We now seek a second approximation that will be valid in the critical range $\tau<\tau_1$. Let us assume that there exists a temperature range where the fluctuations begin to interact, but the interaction is not divergent. We are then led to the second approximation

$$G_2^{(2)}=GGVGG, \tag{1.1}$$

which represents the first-order interaction between the fluctuations. As T approaches T_c still closer, it will be necessary to include higher-order interactions among the fluctuations, and the second approximation will also break down for some $\tau=\tau_2$.

The second approximation, valid in the first critical subrange $\tau_2<\tau<\tau_1$, is formally similar in certain respects to the work of Patashinskii and Pokrovskii[3] (PP) in the *immediate vicinity* of the λ transition in helium. Although the work of PP accounts for the observed specific-heat behavior, it suffers from a number of internal inconsistencies and misinterpretations[4,5] which are avoided in the second approximation.

Our approach is to generate self-consistent approximations for the fluctuation propagator in successive temperature ranges closer and closer to T_c. The critical range, where it is necessary to include the interaction of the fluctuations, is then divided into several critical subranges.[6] It is important to note that although τ has been assumed to be small, the present approach is not an expansion about $\tau = 0$.

In Sec. II the mathematical formulation of the second approximation is introduced and extended to temperatures below T_c. The nonmathematically inclined reader can skip to Sec. III, where the spectrum of the order-parameter fluctuations is calculated self-consistently within the second approximation. The equilibrium properties of the system are also calculated in Sec. III, and it is found that the specific heat at constant pressure diverges logarithmically and that the superfluid density vanishes as $\tau^{2/3}$ below T_c. In Sec. IV the contribution of the order-parameter fluctuations to the thermal conductivity[7] above T_c and to the macroscopic modes of the system is found. Two independent criteria for the range of validity $\tau_2 < \tau < \tau_1$ of the second approximation are established in Sec. V. It is shown that the diagrams neglected in the second approximation become important for $\tau \approx \tau_2$, and it is suggested that the breakdown of the second approximation is related to the critical behavior of the density fluctuations. The application of the present approach to liquid helium near the λ transition is discussed in Sec. VI. It is conjectured that the singular behavior of various static quantities for a weakly interacting Bose gas in the first critical subrange is the same as the singular behavior of the corresponding quantities for liquid helium in the presently accessible temperature range near the λ transition.

II. FORMAL STRUCTURE OF THEORY

Although the formal development of the theory of interacting bosons near the Bose-Einstein condensation T_c can be found in numerous papers,[3,8-10] it is convenient to summarize the microscopic theory both above and below T_c. We use the usual Hamiltonian for a system of bosons of mass m interacting via an instantaneous two-body potential $V(r)$,

$$H = -(2m)^{-1}\int d\vec{r}_1\, \psi^\dagger(1)\nabla_1^2\psi(1) + \tfrac{1}{2}\int d\vec{r}_1 d\vec{r}_2 dt_2\, \psi^\dagger(1)\psi^\dagger(2)V(12)\psi(2)\psi(1), \quad (2.1)$$

where $\psi^\dagger(1)$ creates a boson at space-time point $1 = \vec{r}_1, t_1$ and $V(12) = V(r_1 - r_2)\delta(t_1 - t_2)$. We adopt the standard notation of Ref. 1 and set $\hbar = c = k_B = 1$, where k_B is the Boltzmann constant, unless otherwise stated.

A. Above Transition

Above T_c where the order parameter $\langle\psi\rangle$ is zero, the fluctuation propagator can be defined as

$$G(11') = -i\langle T\{\psi(1)\psi^\dagger(1')\}\rangle \quad (2.2)$$

and the correlation of the fluctuations is contained in

$$G_2(121'2') = (-i)^2\langle T\{\psi(1)\psi(2)\psi^\dagger(2')\psi^\dagger(1')\}\rangle, \quad (2.3)$$

where T is the time-ordering operator and the brackets denote a statistical average over the grand canonical ensemble. The functions G and G_2 can also be given the usual interpretation in terms of particle propagation and correlation. However, near T_c it is more useful to interpret G and G_2 in terms of fluctuations, since order-parameter fluctuations dominate the behavior of the system. The nature of the fluctuations will be discussed in Sec. III.

The equation of motion for G is

$$G_0^{-1}(1\bar{2})G(\bar{2}1') = \delta(11') + \Sigma(1\bar{2})G(\bar{2}1'), \quad (2.4)$$

$$G_0^{-1}(12) = \left(i\frac{\partial}{\partial t_1} + \frac{1}{2m}\nabla_1^2 + \mu\right)\delta(12), \quad (2.5)$$

where the self-energy Σ is given by

$$\Sigma(11') = iV(1\bar{2})G_2(1\bar{2}\bar{3}\bar{2}^+)G^{-1}(\bar{3}1'). \quad (2.6)$$

All barred indices are integrated over. We define a four-point vertex function[8] $C(3456)$ by

$$G_2(121'2') = G(11')G(22') + G(12')G(21') + iG(1\bar{3})G(2\bar{4})C(\bar{3}\bar{4}\bar{5}\bar{6})G(\bar{5}1')G(\bar{6}2'). \quad (2.7)$$

The vertex function C can be interpreted as the effective interaction between the fluctuations, and thus the last term in (2.7) describes the correlation of the fluctuations. Substituting (2.7) into (2.6) we write the self-energy in the form

$$\Sigma(11') = iV(1\bar{2})G(\bar{2}\bar{2}^+)\delta(11') + iV(11')G(11') + i^2V(1\bar{2})G(1\bar{3})G(\bar{2}\bar{4})C(\bar{3}\bar{4}1'\bar{5})G(\bar{5}\bar{2}), \quad (2.8)$$

which is represented diagrammatically by Fig. 1.

As discussed in Sec. I, the first approximation for the fluctuation propagator G sufficiently far from T_c is obtained by neglecting the effective interaction between the fluctuations, i.e., $C^{(1)} = 0$ or

$$G_2^{(1)}(121'2') = G(11')G(22') + G(12')G(21'), \quad (2.9)$$

$$\Sigma^{(1)}(11') = iV(1\bar{2})G(\bar{2}\bar{2}^+)\delta(11') + iV(11')G(11'). \quad (2.10)$$

The first approximation, (2.9) and (2.10), is usu-

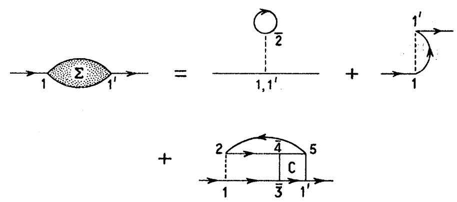

FIG. 1. Self-energy $\Sigma(11')$ above T_c as given by (2.8). The dashed lines represent the potential V, the solid lines represent the fluctuation propagator G, and the square indicates the vertex part C.

ally called the Hartree-Fock approximation.[1] Closer to T_c, the second approximation is obtained by approximating the effective interaction C by the first-order interaction V:

$$C^{(2)}(1234) = V(12)[\delta(24)\delta(13) + \delta(23)\delta(14)]. \qquad (2.11)$$

Substituting (2.11) into (2.8), we obtain the self-energy

$$\Sigma^{(2)}(11') = \Sigma^{(1)}(11') + i^2 V(1\bar{2})V(\bar{3}1') \times [G(11')G(\bar{2}\bar{3})G(\bar{3}\bar{2}) + G(1\bar{3})G(\bar{3}\bar{2})G(\bar{2}1')]. \qquad (2.12)$$

The second approximation, (2.11) and (2.12), is usually called the Born-collision approximation.[1]

In order to go beyond the second approximation, we examine the structure of the effective interaction $C(1234)$, which can be interpreted diagrammatically as the sum of all connected diagrams with four external points.[8] We can group the four external points of $C(1234)$ into two pairs, e.g., (12) and (34). A diagram is irreducible with respect to (12) or (34) if the pairs (12) and (34) cannot be disconnected by cutting one internal pair of G lines. Let $S(12; 34)$ be the contribution to C of all four-point connected diagrams that are irreducible with respect to (12), and let $T(12; 34)$ be the contribution from reducible diagrams with respect to (12). It follows that

$$C(1234) = S(12; 34) + T(12; 34), \qquad (2.13)$$

and C satisfies the Bethe-Salpeter integral equation

$$C(1234) = S(12; 34) + S(12; \bar{5}\bar{6})G(\bar{5}\bar{7})G(\bar{6}\bar{8})C(\bar{7}\bar{8}34). \qquad (2.14)$$

Clearly, we can repeat the same analysis for the other pairs, i.e.,

$$C(1234) = S(13; 24) + T(13; 24) = S(14; 23) + T(14; 23), \qquad (2.13')$$

and write Bethe-Salpeter equations analogous to (2.14).

An important subclass of $C(1234)$ is the set of four-point connected diagrams that are irreducible with respect to all three pairings of the external points. This subclass is called the absolutely irreducible vertex function $J(1234)$. A four-point connected diagram can only be reducible with respect to one and only one of the pairings and thus contributes unambiguously to the corresponding T. Therefore it follows that

$$S(12; 34) = T(13; 24) + T(14; 23) + J(1234). \qquad (2.15)$$

Analogous equations to (2.15) can be written for the other two pairs. From (2.13) and (2.15) we see that

$$C(1234) = \tfrac{1}{2}[S(12; 34) + S(13; 24) + S(14; 23) - J(1234)]. \qquad (2.16)$$

If the first-order contribution to J,

$$J^{(1)}(1234) = V(12)[\delta(24)\delta(13) + \delta(23)\delta(14)], \qquad (2.17)$$

is substituted in (2.16), then the diagrams generated for C by iterating (2.14) are known as the parquet diagrams.[11] If we take only the first term in (2.14) with (2.17) for J, then C reduces to the value in the second approximation (2.11). Thus (2.14) and (2.16) provide a systematic way of generating terms beyond the second approximation.

B. Below Transition

Below T_c, where the order parameter $\langle\psi\rangle$ is nonzero, it is convenient to define the fluctuation propagator $\tilde{G}_1$ in terms of the spinor $\Psi = (\psi, \psi^\dagger)$:

$$\tilde{G}_1(11') = G_1(11') - G_{1/2}(1)G^{+}_{1/2}(1'), \qquad (2.18)$$

$$G_{1/2}(1) = \sqrt{-i}\,\langle T\{\Psi(1)\}\rangle, \qquad (2.19)$$

$$G_1(11') = -i\langle T\{\Psi(1)\Psi^\dagger(1')\}\rangle, \qquad (2.20)$$

where the index 1 includes the spinor index, and the statistical average is taken over a restricted ensemble[9] in which the order parameter has well-defined phase and amplitude. Although the order parameter $\langle\psi\rangle$ is related to the first component of $G_{1/2}$, for brevity we refer to $G_{1/2}$ as the order parameter.

The equations of motion for $G_{1/2}$ and G_1 are, respectively,

$$G_0^{-1}(1\bar{2})G_{1/2}(\bar{2}) = \tfrac{1}{2}\,i\,V(1\bar{2})G_{3/2}(1\bar{2}\bar{2}^{+}) \equiv \Sigma_{1/2}(1), \qquad (2.21)$$

$$G_0^{-1}(1\bar{2})\tilde{G}_1(\bar{2}1') = \delta(11') + \Sigma(1\bar{2})\tilde{G}_1(\bar{2}1'), \qquad (2.22)$$

$$G_0^{-1}(12) = \left[\tau^{(3)}\, i\frac{\partial}{\partial t_1} + \left(\frac{1}{2m}\nabla_1^2 + \mu\right)\right]\delta(\vec{r}_1 - \vec{r}_2)\,\delta(t_1 - t_2), \qquad (2.23)$$

where repeated indices are summed over the spinor index, the potential $V(12)$ has no spinor indices, and the Pauli matrices $\tau^{(i)}$ $(i = 1, 2, 3)$ act in the spinor space. The fluctuation self-energy Σ can be generated by

$$\Sigma(11') = [\delta/\delta G_{1/2}(1')]\Sigma_{1/2}(1) , \qquad (2.24)$$

where $\Sigma_{1/2}$ is defined by the second equality in (2.21). Equation (2.24) yields self-energies that satisfy the Hugenholtz-Pines[12] relation and lead to a gapless $\tilde{G}_1$ spectrum.[9] The correlation function

$$G_{3/2}(122^+) = (-i)^{3/2} \langle T\{\Psi(1)\Psi(2)\Psi^\dagger(2^+)\}\rangle , \qquad (2.25)$$

which is a spinor, can be written in terms of the three-point vertex function $C_{3/2}$ as

$$G_{3/2}(122^+) = G_{1/2}(1)G_{1/2}(2)G^*_{1/2}(2) + G_{1/2}(1)\tilde{G}_1(22^+) + 2\tilde{G}_1(12)G_{1/2}(2) + i\tilde{G}_1(1\,\bar{3})\tilde{G}_1(2\,\bar{4}) \times C_{3/2}(\bar{3}\,\bar{4}\,\bar{5})\,\tilde{G}_1(\bar{5}\,2) . \qquad (2.26)$$

It is convenient to separate the diagonal terms $\tilde{\Sigma}$ and the off-diagonal terms $\hat{\Sigma}$ in the fluctuation self-energy:

$$\Sigma(11') = \tilde{\Sigma}(11') + \hat{\Sigma}(11') . \qquad (2.27)$$

We can now define a diagonal fluctuation propagator $\tilde{G}_d$ by the equation

$$G_0^{-1}(1\,\bar{2})\tilde{G}_d(\bar{2}\,1') = \delta(11') + \tilde{\Sigma}(1\,\bar{2})\tilde{G}_d(\bar{2}\,1') , \qquad (2.28)$$

which can be interpreted as the continuation below T_c of the normal fluctuation propagator G. Using (2.28) and (2.22) we obtain

$$\tilde{G}_1(11') = \tilde{G}_d(11') + \tilde{G}_d(1\,\bar{2})\hat{\Sigma}(\bar{2}\,\bar{3})\tilde{G}_1(\bar{3}\,1') . \qquad (2.29)$$

Near T_c the order parameter $G_{1/2}$ is small and the above equations can be simplified. The three-point vertex function $C_{3/2}$ in (2.26) has been expanded to lowest order in $G_{1/2}$ by Byckling,[10] who found

$$C_{3/2}(123) = C_2(123\bar{4})G_{1/2}(\bar{4}) , \qquad (2.30)$$

where C_2 can be interpreted as the effective interaction between fluctuations. In analogy to the treatment above T_c, we want to expand the effective interaction C_2 in powers of the first-order interaction V. It has been shown[9] that to low orders in V it is not possible to construct an approximation that leads to both a gapless $\tilde{G}_1$ spectrum (gapless approximation[9,12]) and consistent thermodynamics (conserving approximation[13]). One advantage of a gapless approximation is that the sound speed can be simply determined from the $\tilde{G}_1$ spectrum, whereas in a conserving approximation a calculation of the density-density correlation function is necessary. A gapless approximation by itself is not a good candidate for a model near T_c, since above T_c it reduces to an approximation one order lower in V. In order to get a sensible model, we add to the gapless approximation the continuation below T_c of a normal conserving approximation of at least the same order in V. Such a continuation corresponds to ignoring in (2.29) the factor containing $\hat{\Sigma}$ which vanishes at T_c, and replacing $\tilde{G}_1$ by $\tilde{G}_d$. Using (2.30), we can write (2.26) as

$$G_{3/2}(122^+) = G_{1/2}(1)G_{1/2}(2)G^*_{1/2}(2) + G_{1/2}(1)\tilde{G}_d(22^+) + 2\tilde{G}_d(12)G_{1/2}(2) + i\tilde{G}_d(1\bar{3})\tilde{G}_d(2\bar{4})\tilde{C}_2(\bar{3}\,\bar{4}\,\bar{5}\,\bar{6})G_{1/2}(\bar{6})\tilde{G}_d(\bar{5}2) + O(VG^3_{1/2}\,\tilde{G}_d^2) , \qquad (2.31)$$

where the first term in (2.31) gives rise to the lowest-order gapless approximation first used by Bogoliubov.[14] The effective interaction $\tilde{C}_2$ is obtained from C_2 by replacing $\tilde{G}_1$ by $\tilde{G}_d$, and can be interpreted as the continuation below T_c of the normal effective interaction C. Using (2.21), (2.24), and (2.31), we obtain

$$\Sigma(11') = \tfrac{1}{2}iV(1\bar{2})[G_{1/2}(\bar{2})G^*_{1/2}(\bar{2}) + \tilde{G}_d(\bar{2}\,\bar{2}^+)]\delta(11') + iV(11')[G_{1/2}(1)G^*_{1/2}(1') + \tilde{G}_d(11')] + \tfrac{1}{2}i^2\,V(1\,\bar{2})\tilde{G}_d(1\,\bar{3})\tilde{G}_d(\bar{2}\,\bar{4})\tilde{C}_2(\bar{3}\,\bar{4}\,\bar{5}\,1')\tilde{G}_d(\bar{5}\,\bar{2}) + O(V^2G^2_{1/2}\tilde{G}_d^2) . \qquad (2.32)$$

In contrast to (2.8) above T_c, (2.32) is not an exact equation and is valid only near T_c.

The first approximation is obtained by ignoring in (2.32) the effective interaction C_2 between the fluctuations and can be called the Bogoliubov–Hartree-Fock approximation,[15] since the approximation is a sum of the Bogoliubov approximation and the continuation below T_c of the normal Hartree-Fock approximation.

The second approximation is obtained by including the first-order interaction

$$\tilde{C}_2^{(2)}(1234) = V(12)[\delta(24)\delta(13) + \delta(23)\delta(14)] , \qquad (2.33)$$

which can be referred to as the Bogoliubov–Born-collision approximation. The second approximation (2.33) will be used in this paper. Discussion of the terms omitted in the second approximation below T_c parallels that above T_c and will not be reproduced here.

III. STATIC PROPERTIES

We present a brief discussion of the criterion for the superfluid transition. The spectrum of the fluctuations is calculated within the second approximation and then used to evaluate explicitly the isobaric specific heat, the superfluid density, and the correlation length.

A. Bose-Einstein Condensation

The Bose-Einstein (BE) condensation is characterized by the onset of a macroscopic occupation of the zero-momentum state. The number of particles with momentum p is given by

$$N_p = \int (d\omega/2\pi) A(p, \omega) f(\omega) \tag{3.1}$$

above the transition, where in the usual notation $A(p, \omega)$ is the spectral function of G, $f(\omega) = (e^{\omega/T} - 1)^{-1}$ is the Bose statistical factor, and the energy ω is measured from the chemical potential μ. The BE condensation occurs when $N_0 \to \infty$ in the thermodynamic limit. It is easy to see that the integrand in (3.1) is well behaved for large ω, and any divergence in N_0 must come from small ω. It is known that for bosons $A(p, \omega)$ is well behaved for small ω and thus the divergence of N_0 must come from the pole of $f(\omega)$, that is, the divergence of N_0 arises from the Bose statistics. Therefore when $N_p \gg 1$ we expand $f(\omega) \approx T/\omega$ and write

$$N_p = T \int \frac{d\omega}{2\pi} \frac{A(p, \omega)}{\omega} = -TG(p, z_\nu = 0)$$

$$= T/W(p) \gg 1 \,, \tag{3.2}$$

where

$$W(p) = p^2/2m + \Sigma(p, 0) - \Sigma(0, 0) + \eta \,, \tag{3.3}$$

$$\eta = -\mu + \Sigma(0, 0) \,. \tag{3.4}$$

Equation (3.2) will be referred to as the dense-state limit. We see from (3.2) that the criterion for the BE condensation is $W(0) = 0$, or

$$\eta(\mu, T) = 0 \,. \tag{3.5}$$

The criterion (3.5) is sometimes referred to as the Landau criterion[3,4] and will be adopted as the criterion for the transition. Above the transition we have $\eta > 0$.

We now want to establish a criterion below T_c for the BE condensation. In Sec. II the second approximation was constructed to be conserving above T_c, and below T_c to be the sum of a gapless approximation and the continuation of the normal conserving approximation. The continuation of the normal conserving approximation is physically motivated by the continuous nature of the condensation and gives consistent thermodynamics for the fluctuations on both sides of the transition. The gapless approximation is necessary to ensure that the criterion for the condensation be the same above and below the transition. In general the gapless approximations satisfy the Hugenholtz-Pines relation[12]

$$\mu = \Sigma_{11}(0, 0) - \Sigma_{12}(0, 0) \,, \tag{3.6a}$$

where $\Sigma_{11}(0, 0)$ and $\Sigma_{12}(0, 0)$ are, respectively, the $p = z_\nu = 0$ values of the diagonal and off-diagonal elements of the matrix self energy Σ. The BE condensation is given by the vanishing of the condensate density $n_0 = N_0/V$, so that (3.6a) reduces to

$$\mu = \Sigma_{11}(0, 0; n_0 = 0) \tag{3.6b}$$

since Σ_{12} is of order n_0. $\Sigma_{11}(0, 0; n_0 = 0)$ reduces to the normal $\Sigma(0, 0)$ at the transition for a gapless-conserving approximation, and (3.6b) is the same as (3.5). Below the transition it is convenient to define η by

$$\eta = -\mu + \Sigma_{11}(0, 0) = \Sigma_{12}(0, 0) \,, \tag{3.7}$$

which is positive. Hence η defined by (3.4) above and (3.7) below provides a measure of the distance from the transition (3.5) in the μT plane.

The criterion (3.5) defines a λ curve in the μT plane. The thermodynamic transformation of the λ curve in the μT plane to the pressure-temperature (PT) plane has been discussed by Lee and Puff[16] and will not be reproduced here. It is convenient to refer to the normal phase above the transition in either the μT or PT plane as phase I, and the condensed phase as phase II.

B. Fluctuation Spectrum

The fluctuation self-energy in phase I is given in the second approximation by (2.12) and can be written as

$$\Sigma^{(2)}(p, z_\nu) = \Sigma^{(1)}(p) + \Sigma(p, z_\nu) \,, \tag{3.8}$$

$$\Sigma(p, z_\nu) = \frac{2V_0^2}{(2\pi)^9} \int \{d^3p_i\} \delta(\vec{p} + \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \int \frac{d\omega}{z_\nu - \omega} \times \int \{d\omega_i\} \delta(\omega + \omega_1 - \omega_2 - \omega_3) A(1) A(2) A(3) \times [f_1 f_2^+ f_3^+ - f_1^+ f_2 f_3] \,, \tag{3.9}$$

where $A(1) = A(p_1, \omega_1)$, $f_1 = f(\omega_1)$, and $f_1^+ = 1 + f_1$. $\Sigma^{(1)}$ is given by (2.10) and for the simple choice of the interparticle potential reduces to

$$\Sigma^{(1)}(p) = \Sigma^{(1)}(0) = 2nV_0 \,, \tag{3.10}$$

where n is the total number density. It is convenient to introduce the real and imaginary parts

$$\Sigma(p, z = \omega \pm i0^+) = \Delta(p, \omega) \mp \tfrac{1}{2} i \Gamma(p, \omega) \,, \tag{3.11}$$

where the real part Δ is related to the imaginary part Γ by the dispersion relation

$$\Delta(p, \omega) = P \int \frac{d\omega'}{2\pi} \frac{\Gamma(p, \omega')}{\omega - \omega'} \,. \tag{3.12}$$

Near the transition we expect that the significant contribution to the self-energy comes from the states which are densely occupied, the low-momentum states, and we apply the dense-state limit (3.2) to all the intermediate states in (3.9). The angular integrals in (3.9) can be performed by introducing the Fourier transform of the momentum δ function. We thus write the real part of the self-energy as

$$\Delta'(p) = -\pi^{-5} V_0^2 T^2 \int_0^\infty dr\, r^2 [j_0(rp) - 1] D^3(r) , \qquad (3.13)$$

$$D(r) = \int_0^\infty dq \frac{q^2 j_0(rq)}{W(q)} , \qquad (3.14)$$

where $\Delta(p) = \Delta(p, 0)$, $\Delta'(p) = \Delta(p) - \Delta(0)$, and $j_0(x) = x^{-1} \sin x$. Because of the application of the dense-state limit, the momentum range should be restricted to $p \ll p_T$, where p_T is defined by $T = W(p_T) \approx p_T^2/2m$. However, because the integral $D(r)$ converges for large q, the upper limit has been extended to infinity.

In the momentum range $p \ll p_0$ where p_0 satisfies $p_0^2/2m = \Delta'(p_0)$, the self-energy $\Delta'(p)$ dominates the kinetic energy $p^2/2m$, and the system is said to be characterized by strong coupling. For $p \ll p_0$, (3.14) can be written as

$$D(r) = \int_0^\infty dq \frac{q^2 j_0(rq)}{\Delta'(q) + \eta} . \qquad (3.15)$$

At the transition $\eta = 0$ and a solution of the nonlinear integral equation (3.13), (3.15) can be found by substituting the form $\Delta'(p) = A_0 p^l$. We obtain for $p \ll p_0$

$$\Delta'(p) = A_0 p^{3/2} , \qquad (3.16a)$$

$$A_0^4 = (2/15\pi^3)(V_0 T_c)^2 . \qquad (3.16b)$$

Above the transition we define a temperature-dependent momentum p_η by $\Delta'(p_\eta) = \eta$. In the range $p_\eta \ll p \ll p_0$ we find the solution (3.16). In the range $p \ll p_\eta$, $D(r)$ decays exponentially for large r and the major contribution to the (3.13) comes from small r. We expand j_0 in (3.13) for small r and find for $p \ll p_\eta$

$$\Delta'(p) = A_1 p^2 , \qquad (3.17a)$$

$$A_1^3 = (432\pi^2)^{-1} (V_0 T)^2/\eta . \qquad (3.17b)$$

Outside the strong-coupling range $p > p_0$, we set $\Delta'(p) = 0$.

To perform calculations, we extrapolate the above results for the fluctuation spectrum to a continuous function $W(p)$ given by

$$W(p) = \begin{cases} A_1 p^2 + \eta , & 0 < p < p_c \\ A_0 p^{3/2} + \eta , & p_c < p < p_0 \\ p^2/2m + \eta , & p_0 < p \end{cases} \qquad (3.18)$$

where the momentum p_c satisfies the relation $A_1 p_c^2 = A_0 p_c^{3/2}$ or $p_c = (A_0/A_1)^2$. The momentum p_c is related to p_η by a numerical factor: $(p_c/p_\eta)^2 = 288/5\pi$. The momentum p_0 is given by $A_0 p_0^{3/2} = p_0^2/2m$ or $p_0 = (2mA_0)^2$. Note that p_c^{-1} can be interpreted as the correlation length that divides the (p, p_c) plane into a critical region $p_c < p < p_0$ and a hydrodynamic region $p < p_c$, which is a characteristic feature of the scaling hypothesis. A schematic summary of the various regions is shown in Fig. 2.

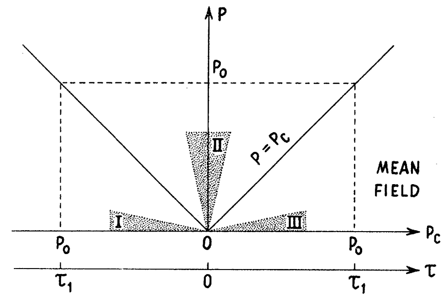

FIG. 2. Schematic plot of the various momentum regions near T_c. In the hydrodynamic regions (I and III) we have $W(p) = A_1 p^2 + \eta$. In the critical region (II) we have $W(p) = A_0 p^{3/2} + \eta$. The rectangular region bounded by p_0 is the strong-coupling region where the scaling hypothesis is valid.

Using the spectrum (3.18) we can now go back and calculate $\Delta(0)$ [see (3.13)] at the transition from

$$\Delta(0) = -\pi^{-5} V_0^2 T_c^2 \int_0^\infty dr\, r^2 D^3(r) . \qquad (3.19)$$

In the evaluation of $D(r)$ we consider only the contribution due to fluctuations in the strong-coupling region $p < p_0$. We obtain (see Appendix)

$$\Delta(0) = -(64/3\pi) T_c \epsilon^2 , \qquad (3.20)$$

where the small parameter ϵ is given by

$$\epsilon = V_0 / T_c \lambda_{T_c}^3 \qquad (3.21)$$

and the thermal wavelength λ_T is given by $\lambda_T^2 = 2\pi/mT$. Note that $p_0 \sim O(\epsilon)$, $p_c \tau^{-2/3} \sim O(\epsilon^{1/3})$, $A_1 \tau^{1/3} \sim O(\epsilon^{1/3})$, $A_0 \sim O(\epsilon^{1/2})$.

We now calculate the imaginary part of the spectrum by using the fact that for bosons $A(p, \omega) \propto \omega$ for small ω. It then follows from the general relation between $A(p, \omega)$ and $\Gamma(p, \omega)$ that $\Gamma(p, \omega)$ can be written in the form

$$\Gamma(p, \omega) = \gamma(p)\omega \sim O(\epsilon^2) , \qquad (3.22)$$

where $\gamma(p)$ is independent of ω and $\omega \ll W(p)$. The linear relation (3.22) on ω should be compared with quadratic relation $\Gamma(p, \omega) \propto \omega^2$ for normal Fermi systems near $T = 0$. If we substitute the form (3.22) into (3.12), we can relate $\gamma(p)$ to $\Delta(p)$ in the limit of strong coupling, $p \ll p_0$. We assume that the the major contribution to the integral[17] in (3.12) comes from $\omega \ll W(p_0) \approx p_0^2/2m \sim O(\epsilon^2)$ and find

$$\Delta(p) = -p_0^2 \gamma(p)/2\pi m . \qquad (3.23)$$

In particular we find that $\gamma_0 \equiv \gamma(0) \sim O(\epsilon^0)$:

$$\gamma_0 = \tfrac{5}{2} . \tag{3.24}$$

The spectral function has the form

$$A(p, \omega) = \frac{\gamma(p)\omega}{[\omega - W(p)]^2 + \frac{1}{4}[\gamma(p)\omega]^2} \quad \text{for } p \ll p_0 \text{ and } \omega \ll W(p_0). \tag{3.25}$$

We see that contrary to PP and in analogy to the superconducting transition, the order-parameter fluctuations cannot be interpreted as quasiparticles since the width of the spectral function (3.25) is comparable to the positition of the peak.

It is easy to see that the small-ω form (3.25) satisfies the identity

$$\int \frac{d\omega}{2\pi} \frac{A(p, \omega)}{\omega} = \frac{1}{W(p)} , \tag{3.26}$$

which demonstrates the consistency of the dense-state limit. Integrating (3.26) from $p = 0$ to $p = p_0$, we see that the total number of particles involved in strong coupling is of the order $O(\epsilon)$.

In phase II, the second approximation is given by the Bogoliubov–Born-collision approximation (2.32) and (2.33). For a short-ranged potential, the zero-frequency matrix self-energy can be written

$$\Sigma(p, 0) = \begin{pmatrix} 2nV_0 + \Delta(p) & n_0 V_0 \\ n_0 V_0 & 2nV_0 + \Delta(p) \end{pmatrix} , \tag{3.27}$$

where $\Delta(p)$ satisfies the same equations [(3.13) and (3.14)] as in phase I with $\eta(\mu, T)$ defined by (3.7). For small ω we have

$$\Sigma'_{11}(p, z_\nu = \omega \pm i0^+) = \Delta'(p) \mp \tfrac{1}{2}\gamma(p)\omega , \tag{3.28}$$

where $\Sigma'_{11}(p, z_\nu) = \Sigma_{11}(p, z_\nu) - \Sigma_{11}(0, 0)$.

The first element of the diagonal fluctuation propagator in (p, z_ν) space is given by

$$\tilde{G}_{11}(p, z_\nu) = \frac{z_\nu + p^2/2m + \Sigma'_{11}(p, z_\nu) + \eta}{D(p, z_\nu)} , \tag{3.29}$$

$$D(p, z_\nu) = z_\nu^2 - [p^2/2m + \Sigma'_{11}(p, z_\nu) + \eta]^2 + \Sigma_{12}(p, z_\nu)^2 . \tag{3.30}$$

In the strong-coupling region $p \ll p_0$, $z_\nu = 0$, we can express the fluctuation propagator in terms of $W(p)$ [(3.18)];

$$\tilde{G}_{11}(p, 0) = -W(p)/[W^2(p) - \eta^2] . \tag{3.31}$$

It is easy to see that the spectral function $A_{11}(p, \omega)$ at low frequency and momentum is given by two peaks of the form (3.25) centered at $\pm[W^2(p) - \eta^2]^{1/2}$.

C. Macroscopic Properties

We now use the fluctuation spectrum (3.18) to calculate the critical behavior of the isobaric specific heat C_P, the superfluid density ρ_s, and the correlation length ξ. The consistency of the various thermodynamic derivatives near the superfluid transition has been discussed phenomenologically by several authors[16,18] and will not be pursued here.

Near a point (μ_c, T_c) on the transition curve $\eta = 0$, the function $\eta(\mu, T)$ is expanded in a power series in $\mu - \mu_c$ and $T - T_c$:

$$\eta(\mu, T) = (\mu - \mu_c)\left[\left(\frac{\partial \eta}{\partial \mu}\right)_T\right]_{\mu \to \mu_c} + (T - T_c)\left[\left(\frac{\partial \eta}{\partial T}\right)_\mu\right]_{T \to T_c} + \ldots , \tag{3.32}$$

where the derivatives are given by

$$\left(\frac{\partial \eta}{\partial \mu}\right)_T = \begin{cases} -1 + [\partial \Sigma(0,0)/\partial\mu]_T , & \text{phase I} \\ [\partial \Sigma_{12}(0,0)/\partial\mu]_T , & \text{phase II} \end{cases} \tag{3.33}$$

and

$$\left(\frac{\partial \eta}{\partial T}\right)_\mu = \begin{cases} [\partial \Sigma(0,0)/\partial T]_\mu , & \text{phase I} \\ [\partial \Sigma_{12}(0,0)/\partial T]_\mu , & \text{phase II} . \end{cases} \tag{3.34}$$

In Sec. V these derivatives are calculated in the second approximation, and it is shown that in the first critical subrange these derivatives are well behaved. Thus in the first critical subrange, the expansion (3.32) for $\eta(\mu, T)$ is valid and we write

$$\left[\left(\frac{\partial \eta}{\partial \mu}\right)_T\right]_{\mu \to \mu_c} = a , \quad \left[\left(\frac{\partial \eta}{\partial T}\right)_\mu\right]_{T \to T_c} = b , \tag{3.35}$$

where a and b are constants. We also find in Sec. V that $a \sim O(\epsilon^0)$ and $b \sim O(\epsilon)$.

The isobaric specific heat (per unit volume) C_P can be expanded in terms of the grand potential $\Omega = -T \ln \mathrm{Tr}\,(e^{-\beta(H - \mu N)})$:

$$C_P = TV^{-1}[\Omega_{\mu T}^2/\Omega_{\mu\mu} - \Omega_{TT} - (S/N - \Omega_{\mu T}/\Omega_{\mu\mu})^2 \Omega_{\mu\mu}] , \tag{3.36}$$

where S is the entropy, V is the volume, and $\Omega_{\mu\mu} = (\partial^2\Omega/\partial_\mu^2)_{T,V}$, etc. The grand potential can be written in the form $\Omega(\mu, T) = \Omega'(\eta(\mu, T)) + \Omega_{\text{reg}}(\mu, T)$, where Ω_{reg} is well behaved as $\eta \to 0$. As emphasized in Refs. 16 and 18, Ω_{reg} plays an important role in the understanding of the pseudoasymptotic behavior, e.g., behavior in the first critical subrange, of the thermodynamic derivatives near the λ transition in helium. However, we will ignore Ω_{reg} here, concentrate only on the singular part Ω', and write

$$\Omega'_{\mu\mu} = a^2 \Omega'_{\eta\eta} , \quad \Omega'_{\mu T} = ab\, \Omega'_{\eta\eta} , \quad \Omega'_{TT} = b^2 \Omega'_{\eta\eta} . \tag{3.37}$$

If we substitute (3.37) into (3.32), the singular part of C_P can be reduced to the form

$$C_P = -(T/V)(S/N - b/a)^2 a^2 \Omega'_{\eta\eta} \quad . \tag{3.38}$$

Using (3.38) and the thermodynamic identity

$$-\frac{b}{a} = \left(\frac{\partial \mu}{\partial T}\right)_{\eta=0} = -\frac{S}{N} + \frac{1}{n}\left(\frac{\partial P}{\partial T}\right)_{\eta=0} \quad , \tag{3.39}$$

we can express the singular part of C_P as

$$C_P = -\frac{T}{V}\frac{1}{n^2}\left(\frac{\partial P}{\partial T}\right)^2_{\eta=0} a^2 \Omega'_{\eta\eta} \quad . \tag{3.40}$$

In the dense-state limit $\Omega_{\eta\eta}$ can be written in the form

$$-\frac{1}{V}\Omega_{\eta\eta} = T\int \frac{d^3p}{(2\pi)^3} G^2(p,0)\left(1 + \frac{\partial \Delta'(p)}{\partial \eta}\right) \quad . \tag{3.41}$$

Substituting the fluctuation spectrum (3.18) into (3.41) and keeping only the divergent contribution, we obtain

$$-(1/V)\Omega'_{\eta\eta} = (15/18\pi)^{1/2}(1/V_0)\ln(\beta\eta)^{-1} \quad . \tag{3.42}$$

The logarithmic singularity in (3.42) arises from the integration in (3.41) over the critical region $p_c < p < p_0$. In phase II, $\Omega'_{\eta\eta}$ is also given by (3.42). Combining (3.40) and (3.42) we find the singular part of the isobaric specific heat to be given by

$$C_P = \left(\frac{15}{18\pi}\right)^{1/2}(n\lambda_T^3)^{-1}\frac{1}{n}\left(\frac{\partial P}{\partial T}\right)^2_{\eta=0}\frac{a^2}{\epsilon}\ln\left|\frac{1}{\tau}\right| \quad . \tag{3.43}$$

Note that $C_P/\ln\tau^{-1} \sim O(\epsilon^{-1})$. The jump in C_P that is superimposed on the symmetric logarithmic singularity (3.43) can be obtained from the spectrum (3.18), but will not be considered here. Since the static properties are sensitive to only the real part of the fluctuation spectrum (3.18), it is not surprising that the above results coincide with those of PP.

The condensate density n_0 can be found from the relation (3.7), and to lowest order in ϵ in the first critical subrange we find

$$n_0 = \eta/V_0 = C_0\tau \quad , \tag{3.44}$$

where the constant $C_0 \sim O(\epsilon^0)$.

The superfluid density ρ_s can be obtained from the relation[9]

$$\rho_s = -n_0 m^2 \lim_{p\to 0}[p^2 \tilde{G}_{11}(p,0)]^{-1} \quad . \tag{3.45}$$

Substituting (3.31) into (3.45), we find[19]

$$\rho_s = 2n_0 m^2 A_1 \quad , \tag{3.46}$$

which is the microscopic analog of the phenomenological relation for ρ_s found by Josephson.[20] Using (3.17b) and $b \sim O(\epsilon)$, we see that to lowest order in ϵ, ρ_s is given by

$$\rho_s/\rho \propto \epsilon^{1/3}(-\tau)^{2/3} \quad , \tag{3.47}$$

where $\rho = mn$. Note that (3.47) vanishes as expected for the ideal Bose gas. The ϵ dependence in (3.47) can easily be obtained by matching ρ_s at τ_1, the temperature where mean-field theory breaks down. For $|\tau| > \tau_1$, we have $\rho_s = mn_0 \propto |\tau|$, independent of ϵ. Let us assume that in the critical range $\tau < \tau_1$, ρ_s has the form $\rho_s \propto \epsilon^\alpha |\tau|^{2/3}$. Then at $|\tau| = \tau_1$ we have $\epsilon^\alpha \tau_1^{2/3} \propto \tau_1$, which implies $\alpha = \frac{1}{3}$, since $\tau_1 \propto \epsilon$ [see (5.4)].

The correlation length ξ can be obtained from the asymptotic behavior of the fluctuation propagator for large r. In phase I we have from (3.2) and (3.17)

$$-TG(r, z_\nu = 0) = (T/4\pi A_1 r)\, e^{-r/\xi} \quad , \tag{3.48}$$

where ξ is given by

$$\xi = p_\eta^{-1} = (\eta/A_1)^{1/2} \propto \lambda_{T_c}\epsilon^{-1/3}\tau^{-2/3} \quad . \tag{3.49}$$

The ϵ dependence of ξ can also be obtained from a matching condition at τ_1. The asymptotic (large r) behavior of the fluctuation propagator in phase II can be found from (2.18) and (3.31) and is given by

$$-T\tilde{G}_{11}(r, z_\nu = 0) = [n_0 + (T/8\pi A_1 r)\, e^{-r/\xi_a}] \times (1 + T/8\pi A_1 n_0 r) \quad , \tag{3.50}$$

where the first factor on the right-hand side represents the amplitude (longitudinal) fluctuations and the second factor represents the phase (transverse) fluctuations. The amplitude correlation length ξ_a is given by

$$\xi_a = (\sqrt{2}\, p_\eta)^{-1} = (\eta/2A_1)^{1/2} \quad , \tag{3.51}$$

and phase correlation length can be defined as

$$\xi_p = T/8\pi n_0 A_1 = m^2 T/4\pi\rho_s \quad . \tag{3.52}$$

The Ginzburg criterion[21] has been used to determine the range of validity of phenomenological theories[22,23] that give the order-parameter correlation function in a form similar to (3.50). Applying the Ginzburg criterion to (3.50), we find the condition of validity $(\xi_p/e\xi_a) \ll 1$. Since $\xi_p \propto \xi_a$, the Ginzburg criterion would imply that if the second approximation is valid, it is valid right up to the superfluid transition. This result disagrees with a more detailed calculation (see Sec. V) which demonstrates that the fluctuation contribution from the critical region $p_c < p < p_0$ plays an important role in determining the range of validity. The Ginzburg criterion as applied to (3.50) overestimates the range of validity for the second approximation.

IV. DYNAMICAL PROPERTIES

An essential feature of the theory is the damping of the critical fluctuations, which is seen from Sec. III to be the same order of magnitude as the real part, namely, $O(\epsilon^2)$. It is misleading to refer to a Bose gas near the superfluid transition as behaving as an ideal gas of quasiparticles, even though the damping plays no role in the calculation of the thermodynamic properties.

A. Thermal Conductivity

We begin the discussion of dynamical properties by considering the calculation of the thermal conductivity of phase I of the Bose gas near T_c. The thermal conductivity κ can be related to the retarded correlation function of $\vec{j}' = \vec{j}^q + \lambda\vec{g}$ by the general expression

$$\kappa = \text{Re}\left(\frac{1}{3iT}\frac{d}{d\omega} K^R(p=0,\omega)\bigg|_{\omega=0}\right), \tag{4.1}$$

where

$$K^R(p,\omega) = i\int d\vec{r}\, dt\, e^{-i\vec{p}\cdot\vec{r}+i\omega t}\langle[j'(r,t),\, j'(0,0)]\rangle\,\theta(t) \tag{4.2}$$

and $\theta(t)$ is the usual step function. $\vec{j}^q$ can be interpreted as the heat-current density, $\vec{g}$ is the momentum-current density, and λ is an arbitrary constant, where the notation of Ref. 24 is used unless otherwise noted. The interpretation of $\vec{j}'$ depends on the choice of λ, but for convenience $\vec{j}'$ will be referred to as the heat-current density also. Thus it is necessary to calculate the heat-current fluctuation propagator $\langle j'\, j'\rangle$ self-consistently in the same spirit as the order-parameter fluctuation propagator was determined earlier. Since the order-parameter fluctuation propagator has the structure of a one-particle Green's function, the first-order interaction between the order-parameter fluctuations can be taken to be the interparticle potential. However, the nature of the first-order interaction between the heat-current fluctuations is not known.

Although the heat-current fluctuation propagator is not known self-consistently, it is possible to calculate the contribution of the order-parameter fluctuations to κ.[25] A similar calculation has been done by Aslamazov and Larkin[26] for the electrical conductivity of a normal metal above the superconducting transition in the classical range and has been extended by Tsuzuki[6] into the critical range. It is convenient to choose $\lambda = (P+Ts)/nm$, which gives

$$\vec{j}' = [Ts(1) - Tsn(1)/n + P(1) - P + Ts]\,\vec{v}\,, \tag{4.3}$$

where P is the pressure and s is the entropy per unit volume. The Euler relation for the energy density (at fixed N) has been used to obtain (4.3). If entropy and density fluctuations are ignored, (4.3) reduces to $\vec{j}' = Ts\vec{v}$, and κ can be written as

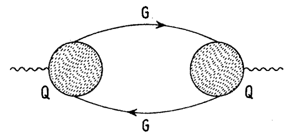

FIG. 3. Diagrammatic representation of the velocity-velocity correlation function with two fluctuation propagators.

$$\kappa = \tfrac{1}{3}Ts^2\,\text{Re}\left(-i\frac{d}{d\omega}\chi^R(p=0,\omega)\bigg|_{\omega=0}\right), \tag{4.4}$$

where χ^R is the Fourier transform of the retarded velocity correlation function defined as in (4.2). Equation (4.4) represents the contribution of order-parameter fluctuations to the thermal conductivity.

The practical effect of the above approximations is to reduce the correlation function $\langle j'\, j'\rangle$ to the form of a two-particle correlation function. The retarded correlation function $\chi^R(p,\omega)$ can be found in the usual way from the analytic continuation into the upper half-plane of the thermal function $\chi(p,z_\nu)$. By analogy to Refs. 6 and 26, χ can be factorized into a product of two fluctuation propagators as shown in Fig. 3. The corresponding integral for χ is

$$\chi(p=0,z_\nu) = -\frac{Q^2}{m^2}\int\frac{d^3p}{(2\pi)^3}\,p^2\,T \times\sum_n G(p,z_n+z_\nu)\,G(p,z_n)\quad, \tag{4.5}$$

where the vertex function Q is determined by the Ward identity

$$\vec{p}\,Q = -m\,\vec{\nabla}_p G^{-1}(p,0) = -2mA_1\vec{p}\quad. \tag{4.6}$$

The sum over frequencies in (4.5) can be converted into an integral in the standard way, and $\chi^R(\omega)$ can be written in the form

$$\chi^R(\omega) = -\frac{Q^2}{m^2}\int\frac{d^3p}{(2\pi)^3}\,p^2\int_{-\infty}^{\infty}\frac{dz}{2\pi i}\,f(z) \times[G^R(p,z)-G^A(p,z)] \times[G^R(p,z+\omega)-G^A(p,z-\omega)]\quad, \tag{4.7}$$

where $G^A(z) = G^R(z)^*$. $G^R(p,z)$ can be found by combining (3.18), (3.22), and (3.24) and has the form

$$G^R(p,z) = [z - W(p) + i\tfrac{1}{2}\gamma(0)z]^{-1}\quad, \tag{4.8}$$

where $W(p) = A_1p^2 + \eta$. The integral (4.7) is to be evaluated with the condition that the energy of the external disturbance ω be less than the energy of the fluctuations $W(p)$. Combining (4.4) and (4.6)–(4.8), we find that the singular contribution to the thermal conductivity is

$$\kappa' = \tfrac{1}{3} T_c s^2 \frac{Q^2}{m^2} \gamma(0) \int \frac{d^3p}{(2\pi)^3} p^2 W^{-3}(p) \qquad (4.9a)$$

$$= (8\pi)^{-1} T_c^2 s^2 (A_1\eta)^{-1/2} \gamma(0) \sim O(\epsilon^{-2/3}) \tau^{-1/3} . \qquad (4.9b)$$

A similar expression for κ' has been derived phenomenologically by Stauffer and Wong.[27]

B. Macroscopic Sound Mode

The macroscopic sound modes of the system should be determined from the poles of the density-density correlation function. As discussed earlier in this section, we do not know how to calculate such a correlation function self-consistently and are forced instead to consider the poles of the fluctuation propagator. It might be thought that the identification of the poles of $\tilde{G}_{11}$ with the macroscopic sound modes should be unambiguous, since it is known that the poles of $\tilde{G}_{11}$ are identical to the poles of the density-density correlation function for $T < T_c$ in the hydrodynamic limit.[9] The identity is based upon the assumption that the calculation of $\tilde{G}_{11}$ is consistent with the conservation laws (hydrodynamics) and the gapless condition (3.6). However, as mentioned in Sec. II, it is not possible to satisfy both conditions simultaneously to the first few orders in V_0.

The pole of the fluctuation propagator $\tilde{G}_{11}$ can be obtained from (3.29) and (3.30). Using expressions (3.18), (3.27), and (3.28) for the self-energy, we can write for small p and ω

$$\tilde{G}_{11}^{-1}(p, \omega + i0^+) \propto \omega^2 - 2n_0 V_0 A_1 p^2 + i2\gamma_0 A_1 p^2 \omega + \cdots . \qquad (4.10)$$

Since the self-energy has been approximated for small p and ω, we do not get all the poles of $\tilde{G}_{11}$. The pole in (4.10) is a sound mode with speed c and damping D given, respectively, by

$$c^2 = 2n_0 V_0 A_1 = (\rho_s/\rho)(nV_0/m) \sim (-\tau)^{2/3} O(\epsilon^{4/3}), \qquad (4.11)$$

$$D = 2\gamma_0 A_1 \sim (-\tau)^{1/3} O(\epsilon^{1/3}). \qquad (4.12)$$

Lee and Yang[28] have shown that in the first approximation the pole in $\tilde{G}_{11}$ near T_c corresponds to oscillations in the superfluid component only, and thus it is appropriate to identify c and D of (4.11) and (4.12) with the fourth-sound mode. The speed of the fourth-sound mode near T_c is given in the hydrodynamic limit by

$$c_4^2 = (\rho_s/\rho) c_1^2, \qquad (4.13)$$

where c_1 is the first-sound speed. The microscopic result (4.11) agrees in form with (4.13) and thus is consistent with the assumption of dynamical scaling.

V. VALIDITY OF SECOND APPROXIMATION

It will be instructive to consider first the determination of the range of validity of the first approximation, since the range of validity of the second approximation can be established in a similar, although not as rigorous, manner.

Let us first investigate the implications of expansion (3.32) of the function $\eta(\mu, T)$ near the transition. In the first approximation, $\Sigma = 2nV_0$ and n is given by

$$n = \lambda_T^{-3} F_{3/2}((\mu - 2nV_0)/T), \qquad (5.1)$$

where $F_{3/2}$ is the familiar Bose-Einstein integral. Using (3.33) and (3.34), we find that in phase I

$$\left[\left(\frac{\partial\eta}{\partial\mu}\right)_T\right]_{\mu\to\mu_c} = -\{1 + 2\epsilon[\pi T/\eta(T)]^{1/2}\}^{-1}, \qquad (5.2a)$$

$$\left[\left(\frac{\partial\eta}{\partial T}\right)_\mu\right]_{T\to T_c} = 3n\lambda_T^{-3}\epsilon\{1 + 2\epsilon[\pi T_c/\eta(\mu)]^{1/2}\}^{-1}. \qquad (5.2b)$$

The derivatives (5.2a) and (5.2b) are singular as $\eta \to 0$, and thus expansion (3.32) appears to be unacceptable. However, the η-dependent term in (5.2a), which is proportional to $\partial n/\partial\mu$ and hence to density fluctuations, can be neglected if $\tau \gg \tau_1$, where τ_1 satisfies the relation $1 = 2\epsilon[\pi T_c/\eta(\tau_1)]^{1/2}$. Similarly the η-dependent term in (5.2b) can be neglected if $\sigma = |\mu - \mu_c|/\mu_c \gg \sigma_1$, where σ_1 satisfies the relation $1 = 2\epsilon[\pi T_c/\eta(\sigma_1)]^{1/2}$. Thus in the range $\tau > \tau_1$ and $\sigma > \sigma_1$, the derivatives (5.2a) and (5.2b) behave as finite constants (3.35) given by

$$a = -1 + O(\epsilon), \quad b = 3n\lambda_T^3\epsilon + O(\epsilon^2), \qquad (5.3)$$

where

$$\tau_1 = \tfrac{2}{3}\sigma_1 = [\tfrac{4}{3}\pi\zeta(\tfrac{3}{2})]\epsilon . \qquad (5.4)$$

This value of τ_1, determined by the internal consistency of the expansion of η [3.32)], is consistent with the value obtained by the Ginzburg criterion.[21] In phase II the constants a and b are just the negatives of the values (5.3) and η is symmetric about the transition.

The range of validity of the first approximation can also be determined by considering the magnitude of the self-energy diagrams neglected in the first approximation. The next term in Σ corresponds to (3.9), the term calculated self-consistently in the second approximation. Equation (3.9) is estimated within the first approximation by taking $W(p) = p^2/2m + \eta$ for the intermediate states. Using (3.13) and (3.14), we find that $W(p) - \eta$ has an additional contribution:

$$W(p) - \eta = (p^2/2m)[1 + (4\pi T_c/27\eta)\epsilon^2]. \qquad (5.5)$$

The second term in (5.5) diverges as $\eta \to 0$, and the first approximation breaks down when the second term becomes comparable to the first. We thus find that the first approximation is valid for $\tau > \tau_1'$ where

$$\tau_1' = [\tfrac{4}{81}\pi\zeta(\tfrac{3}{2})]\epsilon . \qquad (5.6)$$

This estimate of τ_1' implicitly assumes that the perturbation series for Σ converges for $\tau > \tau_1'$. The value τ_1' given in (5.6) differs from (5.4) only by a numerical factor, which should not be taken too seriously as the calculation is only an order-of-magnitude argument.

Finding the value of τ at which the second approximation reduces to the first approximation,

$$A_1(\tau)p^2 = p^2/2m , \qquad (5.7)$$

is our final and most direct determination of the range of validity of the first approximation. The previous criteria have involved the spectrum determined within the first approximation, whereas (5.7) can only be written down if we are able to calculate beyond the first approximation. The resulting value of τ_1 obtained from (5.7) is exactly the same as (5.6). We have now explicitly shown that three independent criteria for the range of validity of the first approximation are consistent with one another.

Let us now consider the range of validity of the second approximation in the same manner as was done for the first approximation. First we investigate the implications of the expansion (3.32) for the function $\eta(\mu, T)$ near the transition. In the second approximation we have $\Sigma(0,0) = 2nV_0 + \Delta(0)$, and from (3.33) and (3.34) we see that the derivatives of η to lowest order in ϵ are given by

$$\left(\frac{\partial\eta}{\partial\mu}\right)_T = \frac{-1}{D} , \qquad (5.8a)$$

$$\left(\frac{\partial\eta}{\partial T}\right)_\mu = 3n\lambda_T^{-3}\frac{\epsilon}{D} , \qquad (5.8b)$$

$$D = 1 - 2V_0\frac{\partial n}{\partial\eta} - \frac{\partial\Delta(0)}{\partial\eta} . \qquad (5.9)$$

The density n is given in the second approximation by

$$n = \frac{T}{2\pi^2}\left(\int_0^{p_c}\frac{p^2dp}{A_1p^2+\eta} + \int_{p_c}^{p_0}\frac{p^2dp}{A_0p^{3/2}+\eta} - \int_0^{p_0}\frac{p^2dp}{p^2/2m+\eta}\right) + \lambda_T^{-3}F_{3/2}(\beta\eta). \qquad (5.10)$$

The contribution to n of the free-particle spectrum in the momentum range $p > p_0$ has been written as the integral from $0 < p < p_0$ minus the integral from $p < p_0$. The dense-state limit [see (3.2)] has been applied only to states with momentum $p < p_0$. An inspection of (5.10) shows that the contribution to n from the momentum range $p < p_0$ is of the order $O(\epsilon)$. However, the range $p < p_0$ gives a singular contribution to the derivative $\partial n/\partial\eta$, and we find after a straightforward calculation that

$$2V_0\frac{\partial n}{\partial\eta} = -8.9 - 1.03\ln\left(\frac{p_0}{p_c}\right)^{3/2} , \qquad (5.11)$$

where $p_c^{3/2}$ is proportional to η and the numerical coefficients have been calculated to $O(\epsilon^0)$. Note that the term proportional to $\eta^{-1/2}$ in (5.2) does not appear in the second approximation and that the term proportional to $\ln\eta^{-1}$ in (5.11) comes from intergrating over the momentum range $p_c < p < p_0$. The form of (5.11), $A + B\ln\eta$, as well as the critical behavior of the quantities considered in Secs. III and IV, depends only on the limiting behavior (3.16) and (3.17) of the fluctuation spectrum $W(p)$; however, numerical coefficients such as A and B are sensitive to the extrapolation procedure used to obtain the spectrum (3.18).

It is shown in the Appendix that the derivative $\partial\Delta(0)/\partial\eta$ has the value

$$\frac{\partial\Delta(0)}{\partial\eta} = -2.97 - 0.11\ln\left(\frac{p_0}{p_c}\right)^{3/2} . \qquad (5.12)$$

Substituting (5.11) and (5.12) into (5.10), we find

$$D = 12.9 + 1.14\ln(p_0/p_c)^{3/2} . \qquad (5.13)$$

As $\eta \to 0$, we see from (5.8) and (5.13) that the derivatives of $\eta(\mu, T)$ are singular and the expansion (3.32) of $\eta(\mu, T)$ is suspect. However, in the range $\tau > \tau_2$ (and $\sigma > \sigma_2$), the derivatives behave as constants (3.35) given by

$$a = -1/12.9 + O(\epsilon), \quad b = 3n\lambda_T^3\epsilon/12.9 + O(\epsilon^2), \qquad (5.14)$$

with

$$\tau_2 = 12.9\, e^{-1/\lambda}\tau_1' , \quad 1/\lambda = 12.9/1.14 = 11.3 , \qquad (5.15)$$

where τ_1' is given by (5.6). Note that λ is of the order $O(\epsilon^0)$. The expansion (3.32) for $\eta(\mu, T)$ near the transition is thus verified in the second approximation for $\tau > \tau_2$.

We now show that the interactions between fluctuations that were ignored in the second approximation can be neglected in the temperature range $\tau > \tau_2'$, where τ_2' is estimated to be consistent with the value (5.15).

It is convenient to consider the diagrammatic expansion for the vertex part C discussed in Sec.

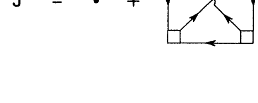

FIG. 4. First two diagrams for the irreducible vertex function J. The symmetrized potential (2.17) is represented by a dot.

II, since the vertex part can be interpreted as representing the interaction between the order-parameter fluctuations. The parquet diagrams have been shown by PP to be convergent and to simply renormalize the "bare" vertex V_0. The first two contributions to the absolutely irreducible vertex part are shown in Fig. 4. The first diagram is simply the potential V_0 and is included in the second approximation. To evaluate the second diagram we approximate the vertex part by V_0, make the approximation $f(\omega) \approx T_c/\omega$ for all intermediate states, and use the self-consistent solution of G in the second approximation. It is easy to see that the dominant contribution comes from the $p^{3/2}$ part of the spectrum. The second diagram then corresponds to the integral

$$\frac{T_c^3 V_0^4}{A_0^6(2\pi)^9}\int \frac{d^3q_1\, d^3q_2\, d^3q_3}{q_1^{3/2}q_2^{3/2}q_3^{3/2}|\vec{p}_1-\vec{q}_1-\vec{q}_2|^{3/2}|\vec{p}_1+\vec{p}_2-\vec{q}_4|^{3/2}|\vec{p}_4-\vec{p}_1-\vec{q}_3|^{3/2}}\ , \tag{5.16}$$

where $\vec{q}_4=\vec{q}_1+\vec{q}_3-\vec{q}_2$ and the $\vec{p}_i$ are the external momenta. Since a logarithmic divergence arises for large values of the momenta q_i, it is convenient to transform to the new variables $Q^2=\sum_i q_i^2$ and $\nu_i = q_i/Q$, which are, respectively, the square of the radius and the directional cosines in a nine-dimensional momenta space. The integral in the new variables can be estimated by setting the external momenta equal to zero. The Q integration is over the momenta p_c to p_0 and can be done immediately to give the form

$$\lambda' V_0 \ln(p_0/p_c)^{3/2}. \tag{5.17}$$

The dimensionless number λ' is given by the integral

$$\lambda' = \frac{2T_c^3 V_0^3}{3A_0^6(2\pi)^9}\int \frac{d\Omega}{\nu_1^{3/2}\nu_2^{3/2}\nu_3^{3/2}\nu_4^{3/2}|\nu_1+\nu_2|^{3/2}|\nu_1+\nu_3|^{3/2}}\ , \tag{5.18}$$

where $d\Omega$ is the element of solid angle in the nine-dimensional momenta space. Since $(p_0/p_c)^{1/2} = 2mA_1$, the argument of the logarithm is proportional to τ^{-1}.

The appearance of the logarithmically divergent term (5.17) contradicts the earlier assumption that the interaction between fluctuations is well behaved for small momenta. Such a term would be expected to be important at temperatures very close to T_c, but it can be neglected in comparison with V_0 in the temperature range away from T_c. We define τ_2 by equating the two terms $V_0=\lambda' V_0 \ln(p_0/p_c)^{3/2}$. We find

$$\tau_2' = 12.9\ e^{-1/\lambda'}\tau_1'. \tag{5.19}$$

The value of τ_2' depends only weakly on the magnitude of the interparticle potential but is very sensitive to the value of λ'.

The integral (5.18) for λ' is very complicated, and we have been able to estimate λ' only very roughly. Let us first set $\overline{\nu}_i \approx 1$ so that the integral becomes equal to the area of a nine-dimensional sphere of radius 1. The corresponding value of $\lambda'^{-1} = 336\ (\frac{2}{15}\pi)^{1/2} \approx 218$. Such an approximation places a lower estimate on the value of τ_2'. A more realistic approximation might be to set $\overline{\nu}_i \approx \frac{2}{3}$, which yields $\lambda'^{-1} \approx 19.1$.

It is possible to show that the dominant behavior of the remaining diagrams for J have the same form as (5.17). Hence the remaining diagrams also contribute to λ' and a more accurate calculation of the integral (5.18) for λ' may be incomplete, without the additional assumption that the perturbation expansion for J converges in the range $\tau > \tau_2'$.

It is interesting to compare the estimate of τ_2 obtained by the above criteria to an estimate τ_n of the onset of critical density fluctuations. The contribution to the density fluctuations $\partial n/\partial\mu$ from the critical range $p_c < p < p_0$ can be obtained from (5.10), and we find

$$\frac{\partial n}{\partial \mu} \sim 1 + \lambda_n \ln\left(\frac{p_0}{p_c}\right)^{3/2}, \tag{5.20}$$

where

$$\lambda_n^{-1} = A_0^4/\eta A_1^3 = 288/5\pi = 18.3\ . \tag{5.21}$$

We see that the density fluctuations are not critical for $\tau > \tau_n$, where

$$\tau_n = 12.9\ e^{-1/\lambda_n}\tau_1'. \tag{5.22}$$

Comparing the form of (5.22) with (5.15) and (5.19), we conjecture that the breakdown of the second approximation at τ_2 might be related to the critical behavior of density fluctuations.

It would be desirable to extend the theory to the next approximation in $\tau < \tau_2$ and to show that the next approximation reduces to the second approximation for $\tau > \tau_2$. Unfortunately, the next approximation that includes both diagrams in Fig. 4 self-consistently does not appear to be tractable.[29]

We have seen that it is possible to establish criteria for the range of validity of the second approximation in the same manner and with the same assumptions as was done for the first approximation. Because of the greater complexity of the second approximation, it is not possible without much further investigation to estimate τ_2 with the same degree of confidence as it was possible to estimate τ_1 in the first approximation.

The assumptions described above are quite different from those made by PP. To maintain the consistency condition that the absolutely irreducible vertex part J is well behaved, PP assumed that the sum of the logarithmically divergent terms for J must give the result $\lambda' = 0$. However Migdal[4] has given general arguments to show that $\lambda' \neq 0$ and hence the assumption of PP is not correct.

For $\tau < \tau_2$ the perturbation series for J diverges and each term is of the same order of magnitude. It might be thought that the theory can be extended into the range $\tau < \tau_2$ by summing the leading powers of $\lambda' \ln(p_0/p_c)$, as has been done by Migdal.[4] However, the sum of a diverging series has been shown to lead to incorrect results near the transition of the two-dimensional Ising model.[30]

In the present approach it is not possible to make any statement about the critical behavior of the system for $\tau < \tau_2$ without including a larger set of diagrams self-consistently. The existence of at least two critical subranges has been indicated for the weakly interacting Bose gas, but it is still an open question whether the range $\tau < \tau_2$ can be still further subdivided. There exists the possibility that the different critical subranges will exhibit different critical behavior.

VI. DISCUSSION

It has been shown that it is possible to construct a simple self-consistent theory of the superfluid transition of a weakly interacting Bose gas, which takes into account the first-order interaction between the order-parameter fluctuations. A number of mathematical assumptions were not rigorously proven, but were shown to be consistent with a simple picture of the transition.

We now consider the possibility of regarding the second approximation for a weakly interacting Bose gas as a model calculation for helium near the λ transition. If we assume that the λ transition is mainly a consequence of Bose statistics, then the nature of the singularities near the λ transition should not be strongly dependent upon the quantitative details of the interaction. Let us investigate the consequences of this assumption and accept for the moment that the concept of critical subranges is applicable to liquid helium and that the first critical subrange includes the presently accessible temperature range. It is then expected that the static properties of helium have the following critical behavior in the first critical subrange:

$$n_0 \propto \tau, \quad \xi \propto \tau^{-2/3}, \quad \rho_s \propto \tau^{2/3}, \quad C_P \propto \ln \tau^{-1}.$$

The critical behavior can be expressed in terms of the standard notation[31] for the various critical exponents. There is, however, an ambiguity in choosing the order parameter for helium, e.g., we can choose either $\sqrt{n_0}$ or $\sqrt{\rho_s}$ apart from phase factors. Although the only quantity readily accessible to experiment is the superfluid density ρ_s, it is clear microscopically that $\sqrt{n_0}$ is the order parameter which is analogous to the magnetization in the ferromagnetic transition, to the density in the liquid-gas transition, and to the gap function in the superconducting transition. Thus the predicted critical exponents in the usual notation[31] are

$$\alpha = 0, \quad \beta = \tfrac{1}{2}, \quad \gamma = 1, \quad \eta = \tfrac{1}{2}, \quad \nu = \zeta = \tfrac{2}{3}\,. \tag{6.1}$$

(It is easy to see that if $\sqrt{\rho_s}$ were chosen as the order parameter then the set $\alpha = 0$, $\beta = \frac{1}{3}$, $\gamma = \frac{4}{3}$, $\eta = 0$, $\nu = \zeta = \frac{2}{3}$ would result.) The values of $\alpha = 0$ and $\nu = \zeta = \frac{2}{3}$ have been confirmed by experiments for helium, but there is no direct experimental evidence for the values of the remaining critical exponents. An extremely difficult but useful experiment would be to measure the condensate density near the λ transition, and that would be a direct check of our prediction $n_0 \propto \tau$ ($\beta = \frac{1}{2}$). Some authors have assumed that $\rho_s \propto n_0 \propto \tau^{2/3}$, but as Josephson[20] has emphasized, this assumption is equivalent to $\eta = 0$, which is the prediction of the classical Ornstein-Zernicke theory.

On the basis of an analogy[31] between the Bose lattice gas and the general Heisenberg-Ising magnet, it is generally assumed that the critical exponents for the λ transition in helium should be approximately the same as those for the magnetic transition. Since it is not the purpose of this paper to discuss magnetic models, we confine ourselves to one brief comment concerning the λ transition of helium and the Bose lattice gas analogs. The application of the present model calculation to the λ transition assumes that the λ transition is mainly a consequence of the Bose statistics. The existence of the Bose statistics can be expressed in terms of the commutation relations between the Bose field operators. However, the presence in the Bose lattice gas of an *infinitely* repulsive interaction profoundly modifies the structure and thus the symmetry of the underlying field description of the system. The change in symmetry might suggest that the Bose lattice gas is a poor model for helium near the λ transition. Near the critical point of helium where the interaction rather than the Bose statistics plays an essential role, it is expected that the lattice gas would be an acceptable model. We suggest that although the critical exponents for the critical point of helium and magnetic systems are similar, the critical exponents of the λ transition of helium may be different from magnetic systems because of the importance of the Bose statistics. Unfortunately the critical exponents that distinguish the various models and analogies, e.g., β and η, have not been measured for helium.

We now consider the applicability to helium of the dynamic calculation in Sec. IV. The contribution of order-parameter fluctuations to the thermal conductivity κ was found to be $\kappa' \propto \tau^{-1/3}$. The extended dynamical scaling[32] and the mode-mode coupling[33] theories give a critical temperature dependence for κ proportional to $\tau^{-1/3}C_P^+(C_P^-)^{-1/2}$, where $C_P^\pm$ are the specific heat at constant pressure above and below T_c, respectively. The predicted temperature dependence of κ from the above phenomenological theories is in excellent agreement with the bulk measurements[34] in helium in the temperature range $10^{-6} < \tau < 10^{-2}$. We see that the contribution of the order-parameter fluctuations to κ agrees with the phenomenological theories and with experiment except for the logarithmic factors. In the mode-mode coupling approach, the factors of C_P arise from the normalization of the entropy fluctuations. In the dynamical scaling approach, entropy fluctuations are built in though the use of hydrodynamics to obtain the poles of the various correlation functions. We thus are led to suspect that the absence of logarithmic factors in κ may be due to the inconsistent treatment of the entropy fluctuations. From this point of view, we do not expect logarithmic factors in the speed c_4 and damping D_4 of fourth sound in helium since the normal fluid, which carries the entropy, is constrained. It would be desirable to measure the properties of the fourth-sound mode near the λ transition in helium.

The picture that arises from the present microscopic approach is somewhat analogous to the phenomenological mode-mode coupling approach. The divergence of the various properties of the system is tied to the existence of various critical fluctuations. A complete theory would consist of a closed set of self-consistent nonlinear coupled equations for the propagators of the critical fluctuations.[35] The critical fluctuations in helium are those in the order parameter and the entropy. The static entropy fluctuation is completely determined by the order-parameter fluctuations. On the other hand, in dynamic situations, entropy and order-parameter fluctuations should be considered as separate critical fluctuations. Since the entropy fluctuations are only weakly critical, the present model is useful for the calculation of the dominant behavior of dynamic properties, but it is still incomplete owing to the inconsistent treatment of the entropy fluctuations.

We now return to the basic question concerning the applicability of the concept of critical subranges to the λ transition in helium. Unfortunately, there is no definitive answer to this question and the remaining discussion is only meant to be suggestive. In Sec. V, it was seen that the criteria for the range of validity of the second approximation, $\tau_2 < \tau < \tau_1$, were related to the critical behavior of the density fluctuations. A possible conjecture for helium is that the existence of the first critical subrange is associated with the fact that the density fluctuations have not yet become critical. Estimates based upon phenomenological considerations[16] suggest that the density fluctuations in helium become critical only for extremely small $\tau \sim 10^{-60}$.

We have indicated the existence of the first critical subrange for a weakly interacting Bose gas. It would be interesting to see if the concept of critical subranges is useful in other systems. However, the applicability of the present approach might be model dependent.

ACKNOWLEDGMENT

One of us (V. K. W.) would like to thank Dr. A. M. Sessler for suggesting the thesis problem which led to this work.

APPENDIX

The derivative $\partial\Delta(0)/\partial\eta$ at T_c is needed in the evaluation of the constants a and b [see (3.35), (5.8), and (5.9)] and is given by

$$\frac{\partial\Delta(0)}{\partial\eta} = -\frac{3}{\pi^5} V_0^2 T_c^2 \int_0^\infty dr\, r^2 D^2(r) \frac{\partial D(r)}{\partial\eta} . \qquad \text{(A1)}$$

We shall take the limit $\eta \to 0$ in (A1) whenever divergences do not result. Using (3.15) and (3.16) we see that the contribution from the strong-coupling region $0 < p < p_0$ to $D(r)$ is given by

$$D(r) = \begin{cases} \frac{2}{3} A_0^{-1} p_0^{3/2}, & r \ll 1/p_0 \\ (\frac{1}{2}\pi)^{1/2} A_0^{-1} r^{-3/2}, & r \gg 1/p_0 . \end{cases} \qquad \text{(A2)}$$

The limiting forms (A2) for $D(r)$ can be extrapolated to join at $r = r_0 \equiv c/p_0$, where $c = (\frac{9}{8}\pi)^{1/3}$. The value of $\Delta(0)$ at T_c [see (3.20)] is obtained from (3.19) by the use of the above extrapolation for $D(r)$.

The derivative $\partial D(r)/\partial\eta$ can be obtained from (3.15) and (3.18),

$$-\frac{\partial D(r)}{\partial\eta} = \frac{1}{A_1^2 p_c} \int_0^1 dx\, j_0(rp_c x) \frac{x^2(1 - x^2/3\lambda_n)}{(x^2+\lambda_n)^2} + \frac{1}{A_0^2} \int_1^{p_0/p_c} dx\, j_0(rp_c x) \frac{x^2}{(x^{3/2}+\lambda_n)^2} \equiv I_1 + I_2 , \qquad \text{(A3)}$$

where $x = q/p_c$ and λ_n is given by (5.21). Since $j_0(rp_c) \to 0$ as $\eta \to 0$ for finite r, the integral I_1 reduces to a constant and can be evaluated analytically. In the limit $r \ll 1/p_0$, $j_0(rp_c x)$ in I_2 can be replaced by unity and I_2 is given by

$$I_2 = A_0^{-2} [\ln(p_0/p_c) - \tfrac{2}{3} - \tfrac{2}{3}\ln(1+\lambda_n)] , \quad r \ll 1/p_0 . \qquad \text{(A4)}$$

For $r \gg 1/p_0$ we neglect λ_n in the denominator of I_2 and obtain

$$I_2 = \frac{1}{A_0^2}\left(1 - \frac{\sin p_0 r}{p_0 r} - \ln(r p_c) + \mathrm{Ci}(r p_0) - \mathrm{Ci}(1)\right), \quad 1/p_0 \ll r \ll 1/p_c \qquad \text{(A5)}$$

where Ci(r) is the cosine integral

$$\mathrm{Ci}(r) = \gamma + \ln r + \int_0^r dx\, x^{-1}(\cos x - 1)$$

and $\gamma = 0.557$. The derivative $\partial\Delta(0)/\partial\eta$ can be evaluated by breaking the r integral in (A1) into two ranges, from 0 to c/p_0 and from c/p_0 to $1/p_c \rightarrow \infty$, using the appropriate forms for $D(r)$, I_1, and $I_2(r)$. We evaluated numerically the integral

$$\int_c^\infty dr\, r^{-1}\,\mathrm{Ci}(r) = -0.22 .$$

The remaining integrals are straightforward and we obtain the value (5.12) for $\partial\Delta(0)/\partial\eta$.

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*Work supported in part by the National Science Foundation and the Horace M. Rackham School of Graduate Studies.

[1]We use the standard definitions and notation, e.g., see L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

[2]Similar mean-field approximations have been made by L. P. Gorkov for the superconducting transition and N. D. Mermin for the liquid-gas transitions. See L. P. Gorkov Zh. Eksperim. i Teor. Fiz. 34, 735 (1958) [Sov. Phys. JETP 7, 505 (1958)]; and N. D. Mermin, Ann. Phys. (N. Y.) 18, 421 (1962). The mean-field theory for a superfluid transition has been investigated by V. K. Wong (unpublished).

[3]A. Z. Patashinskii and V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. 46, 994 (1964) [Sov. Phys. JETP 19, 677 (1964)].

[4]A. A. Migdal, Zh. Eksperim. i Teor. Fiz. 55, 1964 (1968) [Sov. Phys. JETP 28, 1036 (1969)].

[5]A. M. Polyakov, Zh. Eksperim. i Teor. Fiz. 55, 1026 (1968) [Sov. Phys. JETP 28, 533 (1969)].

[6]A similar approach has been suggested independently by T. Tsuzuki [Progr. Theoret. Phys. (Kyoto) 41, 296 (1969)] for the superconducting transition.

[7]A summary of the results for the thermal conductivity has been reported in Ref. 25.

[8]C. DeDominicis and P. C. Martin, J. Math. Phys. 5, 14 (1964); 5, 31 (1964).

[9]P. C. Hohenberg and P. C. Martin, Ann. Phys. (N. Y.) 34, 291 (1965).

[10]E. Byckling, Ann. Phys. (N. Y.) 47, 1 (1968).

[11]See, for example, A. A. Abrikosov, Physics 2, 5 (1965); and Yu. A. Buchkov, L. P. Gorkov, and I. M. Dzyaloshinskii, Zh. Eksperim. i Teor. Fiz. 50, 738 (1966) [Sov. Phys. JETP 23, 489 (1966)].

[12]N. M. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).

[13]G. Baym, Phys. Rev. 127, 1391 (1962).

[14]N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).

[15]The Bogoliubov–Hartree-Fock approximation is used in the last reference cited in Ref. 2 and was first used by N. Shohno, Progr. Theoret. Phys. (Kyoto) 31, 553 (1964). Further treatment of gapless-conserving approximations can be found in V. K. Wong, Ph. D. thesis, UCRL Report No. UCRL-17159 (University of California at Berkeley, 1966) (unpublished).

[16]K. C. Lee and R. D. Puff, Phys. Rev. 158, 170 (1967).

[17]This assumption for small ω is not the same nor as good as the dense-state limit.

[18]E. G. Batyev, A. Z. Patashinskii, and V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. 47, 598 (1964) [Sov. Phys. JETP 20, 398 (1965)].

[19]M. A. Moore and R. B. Stinchcombe, Phys. Letters 24A, 619 (1967).

[20]B. D. Josephson, Phys. Letters 21, 608 (1966).

[21]See, for example, P. C. Hohenberg, in *Conference on Fluctuations in Superconductors, Asilomar*, 1968, edited by W. S. Foree and F. Chilton (Stanford Reseach Institute, Menlo Park, Calif., 1968), p. 305; and R. A. Ferrell, *ibid.*, p. 239. The original suggestion is found in V. L. Ginzburg, Fiz. Tverd. Tela 2, 2031 (1960) [Sov. Phys. Solid State 2, 1824 (1961)].

[22]T. M. Rice, Phys. Rev. 140, A1889 (1965).

[23]V. K. Wong, Phys. Letters 27A, 269 (1968).

[24]L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) 24, 419 (1963).

[25]H. Gould and V. K. Wong, Phys. Letters 31A, 364 (1970). After the present manuscript was submitted for publication, we noticed the independent work of A. M. Polyakov, Zh. Eksperim. i Teor. Fiz. 57, 2144 (1969) [Sov. Phys. JETP 30, 1164 (1970)], in which a similar approach to dynamical properties is presented.

[26]L. G. Aslamazov and A. I. Larkin, Fiz. Tverd. Tela 10, 1104 (1968) [Sov. Phys. Solid State 10, 875 (1968)].

[27]D. Stauffer and V. K. Wong, J. Low Temp. Phys. 2, 599 (1970).

[28]T. D. Lee and C. N. Yang, Phys. Rev. 113, 1406 (1959). In the second approximation, there is no coupling between the superfluid and normal components and the conclusion of Lee and Yang is still valid.

[29]Note that the form of the estimate (5.17) implicitly assumes that the $p^{3/2}$ spectrum is not greatly modified for $\tau < \tau_2$. We would like to thank Professor C. N. Yang for pointing this out to us.

[30]T. T. Wu, Phys. Rev. 149, 380 (1966).

[31]See, for example, M. E. Fisher, Rept. Progr. Phys. 30, 615 (1967).

[32]B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[33]J. Swift and L. P. Kadanoff, Ann. Phys. (N. Y.) 50, 312 (1968).

[34]G. Ahlers, Phys. Rev. Letters 21, 1159 (1968).

[35]The kinetic equations proposed by K. Kawasaki [Ann. Phys. (N. Y.) 61, 1 (1970)] have the same structure as our equation of motion for G (or $\tilde{G}_1$) in the second approximation, with the exception that one of the G's is replaced by an entropy fluctuation propagator.