Quantum-Electrodynamical Theory of Atoms Interacting with High-Intensity Radiation Fields*

C. S. Chang† and P. Stehle
Physics Department, University of Pittsburgh, Pittsburgh, Pennsylvania 15213
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The interaction of a bound electron with external radiation fields of finite intensity is treated with the standard quantum-electrodynamical (QED) formalism of Feynman and Dyson. We first construct an equation for a bound electron in finite-intensity radiation fields, such as those encountered in early molecular-beam experiments and in recent masers and lasers. The Green's function for the equation so obtained enables us to compute the induced transition probabilities for various systems of interest. In this paper, we carry out the calculations on the two-level and three-level systems explicitly, where we find that, to order e^4, the QED method based on forward scattering and the semiclassical treatment differ. As we consider only the interaction of electrons with low-energy photons, we may ignore the virtual photon processes, in accordance with the low-energy theorem. As a result, this treatment contains only finite calculations. We demonstrate explicitly that our results are in qualitative agreement with the molecular-beam experiments of Kusch, for which the semiclassical treatment of Salwen fails to predict the results. Consequently, we expect an intensity-dependent effect which can be properly explained only by QED and not by the semiclassical treatment. We also include the effect of nonelectromagnetic relaxation on the induced transition probability of a two-level system, for which we obtain an expression slightly different from that used by Ramsey for the hydrogen maser. Finally, we derive some expressions which will be of interest in experiments related to lasers and masers.

I. INTRODUCTION

During the past decade, there has been much theoretical work on the interaction of atoms with radiation fields. The typical method used is to solve the Schrödinger equation by treating the radiation fields as classical electromagnetic fields satisfying Maxwell's equations. Such an approach is usually referred to as the semiclassical treatment. It has become apparent that the semiclassical treatment is inadequate to explain the experimental results on the Lamb shift and the anomalous magnetic moment of the electron. In this connection, quantum electrodynamics (QED) is needed. The success of QED in explaining the experiments on the electromagnetic interactions leads us to trust its formalism. Nevertheless, the QED formalism is not without difficulties; in particular, the question of renormalization has not yet been treated quite satisfactorily. However, if we can overlook this difficulty, the QED formalism should be excellent with regard to the electromagnetic interactions. Guided by the experimental evidence, we shall apply the formalism to some problems which have been treated in the past by the semiclassical method.

In the following, we shall apply the QED formalism to a bound electron interacting with the radiation field. One reason for doing this is the fact that, in recent years, there has been considerable controversy over the problem of a free electron interacting with high-intensity fields.[1,2] The QED method and the semiclassical treatment give different results. It is naturally of great theoretical interest to look at the situation of a bound electron. We shall demonstrate under what kind of conditions the two methods give rise to different results, and to what extent they agree with each other.

Since experiments on a free electron interacting with high-intensity fields are more difficult to perform, a conclusive test of both theories is not yet available. However, the experimental situation for bound electrons is quite different. In this respect, one may mention molecular beams.[3] More recently, work on lasers and masers has also involved high-intensity fields. The theoretical survey in these areas is almost restricted to the semiclassical treatments. Moreover, due to the rapid advancements of the experimental technique, the need for a more satisfactory treatment is apparent. We expect that the QED method will give better results.

It may be worthwhile to point out some important techniques commonly used to solve the problem of a bound electron interacting with radiation fields. There are (a) the method of Weisskopf and Wigner,[4] (b) the time-dependent perturbation method of Heitler and Ma,[5] and (c) the QED method of Low.[6] All three methods have been used for the calculation of natural line shapes. The first two methods have been generalized to other types of problems involving the electromagnetic interactions. For example, the method of Weisskopf and Wigner has been used by Ernst and Stehle[7] and by Scully and Lamb[8]; the

method of Heitler and Ma, has been used by Hutchinson and Hameka[9] and many others. However, the QED method of Low is not yet widely used. We briefly outline the approach in the next paragraph.

Consider the one-electron problem. The interaction of an electron and the radiation field can be analyzed in terms of the photon scattering process concerned. According to the general formulation of QED, one can write the equation for the electron Green's function in the following form:

$$G(x_1, x_2) = S_F(x_1, x_2) - i \int S_F(x_1, x_3) M(x_3, x_4) \times G(x_4, x_2) d^4x_3 d^4x_4 ,$$

where $S_F(x_1, x_2)$ is the electron propagator in the presence of the static fields, and $M(x_3, x_4)$ is the mass operator taking into account the effect of the time-dependent fields.[10] The first step is to construct the mass operator $M(x, x')$ from the S matrix by analyzing scattering processes. After $M(x, x')$ has been obtained, one can solve the above integral equation by means of Fourier-series techniques. Once $G(x, x')$ is determined, one can use the properties of Green's functions to compute the physical quantities of interest, such as the transition probabilities and scattering cross sections.

Following the method outlined above, we shall consider the effect of high-intensity photon scattering. Speaking in the language of the semiclassical treatments, it amounts to the electron interacting with a time-dependent electromagnetic field. Our major purpose is to compare our results with those of the semiclassical treatment. To this end, we compute the induced transition probabilities. We first consider a two-level system which, under certain reasonable assumptions, enables us to obtain simple analytical expressions for the induced transition probabilities. We then consider a more complicated three-level system. After some algebraic developments, we find that the above QED equation to order e^2 with respect to the mass operator $M(x, y)$ reproduces the expression of the induced transition probabilities obtained by the semiclassical methods. However, to order e^4 or higher, both approaches do not yield the same results. Finally, we demonstrate the theoretical consequences of such differences. It is hoped that our results for the multiple-quantum transitions can be used to explain the experiments of Kusch[11] and some other related phenomena.

All the expressions derived for molecular beams assume that the atomic levels are sharp. When the levels concerned acquire widths due to some relaxation mechanism, it is necessary to average the induced transition probabilities in certain ways. A detailed account of this situation is given in Sec. III where we obtain an expression, Eq. (29), which is different from that which has been used for the hydrogen maser.[12]

We believe that an extensive study of this type of problem is appropriate for the present experimental situation, in particular for masers and lasers, which can be used to test the validity of high-intesity QED to a very high accuracy.

II. BASIC FORMALISM

As was mentioned in the Introduction, the investigation of high intensity is very important both theoretically and experimentally. In the case of a free electron interacting with high-intensity radiation fields, there arose a considerable controversy over the merits of the QED method and the semiclassical treatment.[1,2] Various arguments have since been devised to resolve the discrepancy,[13–15] and possible experimental resolutions have been suggested. However, the problem is not accessible to experiment as yet. A more hopeful system from the experimental point of view is provided by bound systems where energy-level separations can be accurately measured. The high accuracy in molecular-beam experiments[3] makes it worthwhile to calculate the effect of high-intensity fields on the basis of QED and to compare the results with the semiclassical calculations.

In the following discussions, the interaction of a bound electron with the radiation field will be described as essentially due to forward scattering. This is true in the case of free electrons as was demonstrated by Ehlotzky.[15] Since we will restrict ourselves to the interaction of a bound electron with low-energy photons and the major purpose is to compare our results with the semiclassical treatment, the interaction of electrons with virtual photons will be neglected. This is well justified, since in the interaction of low-energy photons with the electrons, the low-energy theorem states that the only effect of virtual processes is to renormalize the charge and the mass of the electron, so that these effects are fully accounted for if the observed charge and mass are used.[16,17] Moreover, there is a close resemblance between the self-energy of an electron in a radiative correction and forward scattering. This can be easily seen by drawing the Feynman diagrams for the self-field (emission and absorption of virtual photons) and the external-field electron self-energy (emission and absorption of real photons at the same four-momentum k_μ, or forward scattering) to each corresponding order in e^2. For example, to order e^2, one can obtain the forward-scattering diagrams by simply breaking apart the appropriate self-energy loop at momentum k_μ. In the same way, in any order all the external-field self-energy diagrams may be obtained by breaking open the closed photon loops which occur in QED in the same order.[13] This

picture will be very helpful in the discussions to follow.

According to the above discussion, we have the following QED equation,

$$G(x_1, x_2) = S_F(x_1, x_2) - i \int S_F(x_1, x_3) M(x_3, x_4) \times G(x_4, x_2) d^4x_3 d^4x_4 , \quad (1)$$

which has been given in the Introduction. $S_F(x_1, x_2)$ is the electron propagator in the absence of incoming radiation fields, and takes the form[18]

$$S_F(x_1, x_2) = \sum_n \psi_n(\vec{r}_1)\bar{\psi}_n(\vec{r}_2) \frac{1}{2\pi i} \times \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t_1 - t_2)}}{E_n(1 - i0) + \omega} , \quad (2)$$

where $\psi_n(\vec{r})$ is the eigenfunction of the Dirac equation in the presence of the static electromagnetic fields.

In Eq. (1), there are two unknown functions to be determined, namely, $G(x_1, x_2)$ and $M(x_1, x_2)$. Our first step is to determine the mass operator $M(x_1, x_2)$; this may be obtained from the general expansion of the S matrix, [19]

$$S = \sum_{n,\nu} K^{(n,\nu)}(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n; \xi_1, \xi_2, \cdots, \xi_\nu) \times : [\bar{\psi}(x_1) \cdots \bar{\psi}(x_n); \psi(y_1) \cdots \psi(y_n); A(\xi_1) \cdots A(\xi_\nu)] : , \quad (3)$$

where n is the number of electrons and ν is the number of photons participating the process under consideration, $\psi(x)$ is the electron-positron field operator, and $A(\xi)$ is the photon field operator, and the symbol $: \cdots :$ denotes the normal-ordered product. (Units are such that $\hbar = c = 1$ and the metric is $ab = a \cdot b = a_\mu b_\mu = a_0 b_0 - \vec{a} \cdot \vec{b}$ if a and b are four-vectors.) The functions $K^{(n,\nu)}$ may be obtained from the analysis of certain QED processes by using the general Feynman rules.

Before writing down the Feynman rules explicitly, we demonstrate the meaning of the expansion (3). First, let us consider the case $(n, \nu) = (1, 0)$; it corresponds to the emission and absorption of a virtual photon, and yields a self-energy correction to the electron. The case $(n, \nu) = (2, 0)$ corresponds to the emission and absorption of virtual photons from different electrons. For $(n, \nu) = (1, 2)$, one may have the well-known Compton scattering and the forward scattering. The latter process corresponds to the emission and absorption of real photons with the same energy and polarization for the incoming and outgoing photons and thereby leaves the whole system unchanged. Similarly, $(n, \nu) = (1, 4)$ includes forward scattering with two photons in and two photons out.

Since in the subsequent applications we shall consider the effect of the forward scattering only, we may write the S matrix in the form,

$$S = \sum_\nu K^{(1,\nu)}(x, y; \xi_1, \xi_2, \ldots \xi_\nu) \times : [\bar{\psi}(x)\psi(y); A(\xi_1)A(\xi_2) \cdots A(\xi_\nu)] : , \quad (4)$$

where $\nu = 2p$, p being a positive integer. The corresponding Feynman diagrams for the lower-lying values of ν or p are illustrated in Figs. 1 and 2. The matrix element for the total number ν of photons in the initial and final states can be written as

$$S_{i \to f} = \int K^{(1,\nu)}(x, y; \xi_1, \xi_2, \ldots \xi_\nu) \bar{\psi}(x)\psi(y) \times \Phi(\xi_1, \xi_2, \ldots \xi_\nu) d^4x d^4y d^4\xi_1 \cdots d^4\xi_\nu , \quad (5)$$

where Φ is the symmetrized wave function for the photons in both the initial and final states,

$$\Phi = \sum a(k_1) \cdots a(k_\nu) e^{i \sum_j k_j \xi_j} ,$$

where $\sum$ denotes the symmetrized sum for the product of the photon wave functions $a(k_1)$, $a(k_2)$, ..., $a(k_\nu)$.

From the theory of a free electron interacting with its self-field, one obtains the following relation between the mass operator M in Eq. (1) and the function $K^{(1,0)}$ defined in Eq. (3) [19]:

$$M(p) = i K^{(1,0)}(p) ,$$

and in the case of forward scattering, we may write[1,15]

$$M(p) = i \sum_{\nu = 1, 2, \ldots} K^{(1,2\nu)}(p) .$$

This connection between M and the expansion-coefficient functions $K^{(1,\nu)}$ can be easily generalized to the corresponding bound-electron case,

$$M(x, y) = i \sum_{\nu = 1, 2, 3, \ldots} K^{(1,2\nu)}(x, y) . \quad (6)$$

The following rules[20] are to be used in the construction of the S matrix in order to obtain $K^{(1,\nu)}$ by means of the photon-number-state representation:

(a) For each external vertex, we have a factor

$$-ie\sqrt{N}\gamma_\mu \int d^4x , \quad (7)$$

where N is the number of photons in the beam.

(b) For a photon of linear polarization which is emitted from or absorbed by the electron at an external vertex, we have

$$e_\mu (2\omega V)^{-1/2} e^{-i\vec{k} \cdot \vec{r} + i\omega t} \quad \text{and} \quad e_\mu (2\omega V)^{-1/2} e^{i\vec{k} \cdot \vec{r} - i\omega t} , \quad (8)$$

respectively, where V is the volume of the system, $(\omega, \vec{k}) = k_\mu$ is the photon energy-momentum four-

vector, and e_μ is the polarization unit four-vector.

(c) For each internal electron line, we insert a factor connecting two space-time points x_1 and x_2,

$$-iS_F(x_1, x_2) . \tag{9}$$

For example, on using the rules (7)–(9), the S matrices corresponding to Figs. 1(a) and 2(a) are

$$S_a^{(2)} = -i\left(\frac{e^2N}{2\omega V}\right)\int d^4x_1 d^4x_2\, \bar{\psi}(x_1)\hat{e}\, S_F(x_1, x_2)\hat{e} \times \psi(x_2)e^{ik(x_1-x_2)} ,$$

$$S_a^{(4)} = -i^3\left(\frac{e^2N}{2\omega V}\right)^2\int d^4x_1 d^4x_2 d^4x_3 d^4x_4\, \bar{\psi}(x_1)\hat{e} \times S_F(x_1, x_3)\hat{e}\, S_F(x_3, x_4)\, \hat{e}\, S_F(x_4, x_2)\hat{e}\, \psi(x_2) \times e^{ik(x_1+x_3)}e^{-ik(x_2+x_4)} ,$$

respectively. According to Eqs. (3) and (6), we have

$$M_a^{(2)}(x_1, x_2) = -\frac{e^2N}{2\omega V}\sum_l \hat{e}\,\psi_l(\vec{r}_1)\bar{\psi}_l(\vec{r}_2)\hat{e}\, e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} \times \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\, e^{i\omega'(t_1-t_2)}(E_l+\omega'-\omega)^{-1} , \tag{10}$$

$$M_a^{(4)}(x_1, x_2) = \left(\frac{e^2N}{2\omega V}\right)^2 \sum_{l,m,n} b_{lm}b_{mn}\hat{e}\,\psi_l(\vec{r}_1)\bar{\psi}_n(\vec{r}_2)\hat{e} \times e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\, e^{i\omega'(t_1-t_2)} \times [(E_l+\omega'-\omega)(E_n+\omega'-\omega)(E_m+\omega'-2\omega)]^{-1} , \tag{11}$$

where

$$b_{mn} = \int d^3r\, \bar{\psi}_m(\vec{r})\hat{e}\, e^{-i\vec{k}\cdot\vec{r}}\psi_n(\vec{r}) .$$

A generalization of Eqs. (10) and (11) to more than two emissions and absorptions is apparent.

Note that in the above, one has used the photon number state to obtain the mass operator. Alternatively, one may also use a coherent state to represent the external radiation field. To this end, consider the second-order S matrix

$$S^{(2)} = \tfrac{1}{2}\int d^4x_1 d^4x_2\, T[U(x_1)U(x_2)] ,$$

where T denotes the time-ordered product, and

$$U(x) = -j_\mu(x)A_\mu(x) = -ie{:}\,[\bar{\psi}(x)\hat{A}(x)\psi(x)]{:}$$

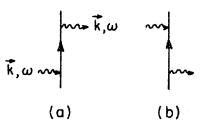

FIG. 1. Lowest-order forward scattering.

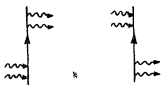

FIG. 2. Four-photon process.

We have, for the process shown in Fig. 1(a), [21]

$$\begin{aligned} S_a^{(2)} &= -\tfrac{1}{2}e^2\int d^4x_1 d^4x_2\, T\{{:}\,[\bar{\psi}(x_1)\hat{A}(x_1)\psi(x_1)]{:} \\ &\quad {:}\,[\bar{\psi}(x_2)\hat{A}(x_2)\psi(x_2)]{:}\} \\ &= -e^2\int d^4x_1 d^4x_2\hat{A}(x_1)S_F(x_1, x_2) \\ &\quad \times \hat{A}(x_2){:}\,[\bar{\psi}(x_1)\psi(x_2)]{:} . \end{aligned} \tag{12}$$

If one writes

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x) ,$$

where $A_\mu^{(+)}(x)$ is the absorption operator, then the eigenstate $|a\rangle$ of $A_\mu^{(+)}(x)$ with eigenvalue $a_\mu(x)$ is the coherent state in the usual definition

$$A_\mu^{(+)}(x)|a\rangle = a_\mu(x)|a\rangle .$$

We may also have

$$\langle a|A_\mu^{(-)}(x) = a_\mu^*(x)\langle a| .$$

For the free fields, we write[22]

$$\begin{aligned} &\vec{a}(x) = (2\omega V)^{-1/2}\vec{e}_\lambda(\vec{k})z_\lambda(\vec{k})\, e^{-i\omega t+i\vec{k}\cdot\vec{r}} , \\ &a_0(x) = 0 , \end{aligned} \tag{13}$$

where $\vec{e}_\lambda(\vec{k})$ are the transverse polarization vectors, and summation over λ is implied.

By taking the matrix element of Eq. (12) between two coherent states defined above, and on using Eqs. (5) and (6), we find

$$M_a^{(2)}(x_1, x_2) = -ie^2\hat{a}^*(x_1)S_F(x_1, x_2)\hat{a}(x_2) . \tag{14}$$

Similarly, one may also construct $M^{(4)}$ of order e^4 by using the coherent-state representation. On taking the matrix elements of Eqs. (10) and (14) between two electron states, one can demonstrate the equivalence of the two representations of radiation fields for our purpose. [23]

After the mass operator has been obtained in accordance with the above procedure, we find that $M(x_1, x_2) = M(\vec{r}_1, \vec{r}_2, t_1 - t_2)$. Using this property and Eq. (2), the solution of Eq. (1) can be found from the expansion

$$\begin{aligned} G(x_1, x_2) &= \frac{1}{2\pi i}\sum_n \psi_n(\vec{r}_1)\bar{\psi}_n(\vec{r}_2)\int_{-\infty}^{\infty} d\omega f_{n,n}(\omega)\, e^{i\omega(t_1-t_2)} \\ &\quad + \frac{1}{2\pi i}\sum_{m\neq n}\psi_m(\vec{r}_1)\bar{\psi}_n(\vec{r}_2)\int_{-\infty}^{\infty} d\omega f_{n,m}(\omega)\, e^{i\omega(t_1-t_2)} . \end{aligned} \tag{15}$$

From Eqs. (1) and (15), we have

$$f_{n,n}(\omega) = \frac{1}{E_n(1-i0)+\omega}\left(1 - H_{nn}(-\omega)f_{n,n}(\omega)\right.$$

$$- \sum_{m\neq n} H_{nm}(-\omega) f_{m,n}(\omega)\Big) ,$$

$$f_{n,m}(\omega) = \frac{1}{E_n(1-i0)+\omega}\Big(-H_{nm}(-\omega) f_{m,m}(\omega) - \sum_{p\neq m} H_{np}(-\omega) f_{p,m}(\omega)\Big) , \quad (16)$$

where

$$H_{nm}(\omega) = \int_{-\infty}^{\infty} \tilde{H}_{nm}(t)\, e^{i\omega t}\, dt ,$$
$$\tilde{H}_{nm}(t) = \int d^3r_1 d^3r_2 \bar{\psi}_n(\vec{r}_1) M(\vec{r}_1, \vec{r}_2, t) \psi_m(\vec{r}_2) . \quad (17)$$

Equations (10), (11), and (15)–(17) will be used for the determination of the electron Green's function $G(x_1, x_2)$ and to deduce some physical quantities of interest in the subsequent discussions.

It may be noted that the form of Eq. (4) ignores the interaction of electrons with virtual photons. This may lead one to believe that the present results should be the same as those of the semiclassical treatments. However, by explicit calculations which we will do very soon, this equivalence can only extend to the order e^2. It will be seen that the higher-order effects will give some interesting results.

We note that the above determination of the mass operator $M(x, y)$ depends on perturbation theory. The validity of such a procedure certainly depends on the convergence of the perturbation series. We shall not investigate the question of convergence here. Nevertheless, it may be pointed out[24] that if the photon density is less than $m^2\omega e^2$ or $(r_0 \lambda\!\!\!^{-}_c \lambda\!\!\!^{-})^{-1}$, where r_0 is the classical radius of the electron, $\lambda\!\!\!^{-}_c$ is its Compton wavelength, and $\lambda\!\!\!^{-}$ is the wavelength of the photon, then the perturbation procedure of constructing the mass operator is plausible. We shall assume that it is justified in the following applications.

III. APPLICATIONS TO THE TWO-LEVEL SYSTEM

Following the general discussion of Sec. II, it is not difficult to solve the system of algebraic Eqs. (16) when the atomic levels are finite in number. In particular, if there are only two levels, the solution is easy to obtain. We consider this case first.

Our purpose is to compute the probability amplitudes for the system concerned. To this end, we have to obtain the Green's function $G(x_1, x_2)$ for the generalized Dirac equation (1). Before doing this, we make some general remarks.

We depict our two-level system as shown in Fig. 3. The matrix elements for the mass operator defined in Eq. (17) will simplify in the nonrelativistic limit and the dipole approximation, which are valid for the atomic system and long-wavelength photons. With these approximations, we write

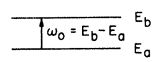

FIG. 3. Two-level system.

$$\xi = \left(\frac{e^2 N}{2\omega V}\right)^{1/2} |\langle a|\hat{e}\, e^{i\vec{k}\cdot\vec{r}}|b\rangle| = \left(\frac{e^2 N\omega}{2V}\right)^{1/2} \langle a|\hat{d}|b\rangle , \quad (18)$$

where $\hat{d}$ is the dipole operator, and $\langle a|\hat{d}|b\rangle$ is taken to be real.

Since we are considering a two-level system in the present section, it is sufficient to take the effect of forward scattering to the order e^2, because by starting with any level at time $t=0$, an emission or absorption of one photon connects the initial level with the other level.

According to the discussion of the last two paragraphs, we may write

$$H_{aa}(-\Omega) = -\frac{\xi^2}{E_b+\omega+\Omega} , \quad H_{bb}(-\Omega) = -\frac{\xi^2}{E_a-\omega+\Omega} ,$$
$$H_{ab}(-\Omega) = H_{ba}(-\Omega) = 0 , \quad (19)$$

from Eqs. (10) and (17) in the rotating wave approximations. Using Eqs. (16) and (19), we have

$$f_{a,a}(\Omega) = \frac{1}{E_a(1-i0)+\Omega+H_{aa}(-\Omega)} ,$$
$$f_{b,b}(\Omega) = \frac{1}{E_b(1-i0)+\Omega+H_{bb}(-\Omega)} , \quad (20)$$
$$f_{a,b}(\Omega) = f_{b,a}(\Omega) = 0 .$$

Suppose that at $t=0$ the system is in the level a, described by the wave function $\psi_a(\vec{r})$, then the probability amplitude for the system to be in level a at time t is given by[25]

$$C_a(t) = (1/2\pi i)\int_{-\infty}^{\infty} f_{a,a}(\Omega)\, e^{i\Omega t}\, d\Omega . \quad (21)$$

Using Eqs. (19) and (20), and evaluating the integral of Eq. (21) by the theorem of residues by closing the contour in the upper half-plane, we have

$$C_a(t) = \left(\cos\eta t + i\frac{\omega_0-\omega}{2\eta}\sin\eta t\right) e^{-i(E_a+E_b-\omega)t/2}, \quad (22)$$

where

$$\omega_0 = E_b - E_a , \quad \eta = \tfrac{1}{2}[(\omega-\omega_0)^2 + 4\xi^2]^{1/2} .$$

The induced transition probability out of the level a is then

$$P_{a\to b}(t) = 1 - |C_a(t)|^2 = (\xi^2/\eta^2)\sin^2\eta t . \quad (23)$$

It is seen that the phase factor of the induced tran-

sition amplitude is not determined. This is due to the fact that the Green's function obtained in this way involves only the correction to the electron part, which does not include the external radiation field as a whole. It is of great interest both theoretically and experimentally[26] to determine this phase factor. In this connection, consider the photon to be absorbed at the space-time point $(\vec{r}, t)$ as shown in Fig. 4. The transition matrix element $T_{i\to f}$ is defined by

$$T_{i\to f} = -i\langle N-1, b|S-1|N, a\rangle$$
$$= -i\langle N-1, b|S|N, a\rangle \ .$$

Taking the effect of high-intensity forward scattering into account as shown in Fig. 4, and using the rules of Eqs. (7)–(9), the amplitude for this transition is found to be

$$C_b(t) = ie\int d^3r\, d^3r'\, \theta(t')\bar{\psi}_b(\vec{r})G(\vec{r}, \vec{r}', t-t')\hat{e}$$
$$\times (N/2\omega V)^{1/2}\, e^{-i\omega t' + i\vec{k}\cdot\vec{r}'}\,\psi_a(\vec{r}')\, e^{-iE_a t'}, \quad (24)$$

where $\theta(t)$ is the unit-step function. Carrying out the integration, we have

$$C_b(t) = i(\xi/\eta)\sin\eta t\, e^{-i(E_a+E_b+\omega)t/2} \ . \quad (25)$$

Comparing Eq. (25) with Eq. (23), it is easily seen that

$$P_{a\to b}(t) = |C_b(t)|^2 \ ,$$

as it should. Equations (22) and (25) are in agreement with the semiclassical treatments.[3]

Note that Eq. (23) assumes that the energy levels are sharp. If this is not the case, then an average must be taken, depending on the nature of relaxation mechanism. For example, in the hydrogen maser,[12] besides the level width there also exists mode damping inside the maser cavity. Generally speaking, there are two different types of relaxation mechanism; one is due to the electromagnetic interaction, and the other arises from nonelectromagnetic interaction. In the former case, it can be included in the fundamental equation (1). In the latter case, one cannot do so. In order to deal with the latter type of relaxation in the hydrogen maser, we assume that the level spreading and mode damping are Lorentzian. This assumption reflects the fact that the probability that an atom ceases to radiate by leaving the field region or by having its radiation state relaxed is described by a simple exponential distribution function of the type $\Gamma e^{-\Gamma t}$,[12] and its power spectrum is a Lorentzian distribution. Let the full widths of the atomic levels and the photon mode be δ and γ, respectively. The effect can be taken into account by smearing out each level and mode by some relevant Lorentzian distributions in Eq. (23). This amounts to replacing E_b, E_a, and ω by $E_b + x_1$, $E_a + x_2$, and $\omega + x_3$, respectively, and then integrating over x_1, x_2, and x_3 with the following Lorentzian distributions as weighting functions:

$$\frac{\delta}{2\pi}\frac{1}{x_1^2+\frac{1}{4}\delta^2}\ , \quad \frac{\delta}{2\pi}\frac{1}{x_2^2+\frac{1}{4}\delta^2}\ , \quad \frac{\gamma}{2\pi}\frac{1}{x_3^2+\frac{1}{4}\gamma^2}\ ,$$

in each integration variable. We end up with the following expression:

$$\bar{P}_{a\to b}(t) = \frac{\xi(2\xi+\Gamma)}{(\omega-\omega_0)^2+(2\xi+\Gamma)^2}$$
$$-\frac{2\xi^2\Gamma}{\pi}\int_{-\infty}^{\infty}\frac{\cos[t(x^2+4\xi^2)^{1/2}]dx}{(x^2+4\xi^2)[(x+\omega-\omega_0)^2+\Gamma^2]} \ , \quad (26)$$

where

$$\Gamma = \delta + \tfrac{1}{2}\gamma \ .$$

Using the method of stationary phase,[27] it can be shown[28] that if

$$t \gg \frac{\xi\Gamma^2}{\pi[\Gamma^2+(\omega-\omega_0)^2]^2} \ , \quad (27)$$

then

$$\frac{2\xi^2\Gamma}{\pi}\int_{-\infty}^{\infty}\frac{\cos[t(x^2+4\xi^2)^{1/2}]dx}{(x^2+4\xi^2)[(x+\omega-\omega_0)^2+\Gamma^2]} \ll 1 \ ,$$

and thus

$$\bar{P}_{a\to b} = \frac{\xi(2\xi+\Gamma)}{(\omega-\omega_0)^2+(2\xi+\Gamma)^2} \ . \quad (28)$$

Note that Eq. (28) represents the fraction of atoms in the lower level if initially they are in the upper level and vice versa. Since the relaxation processes involving Γ do not involve the emission of radiation, in order to get the power delivered by the atomic beam to the cavity, one must multiply Eq. (28) by the factor $2\xi/(2\xi+\Gamma)$, which is the branching ratio for the radiative transtion; the result is

$$P = I\hbar\omega\,\bar{P}_{a\to b}\frac{2\xi}{2\xi+\Gamma} = \frac{I\hbar\omega 2\xi^2}{(\omega-\omega_0)^2+(2\xi+\Gamma)^2} \ , \quad (29)$$

where I is the beam intensity. From Eq. (29), we see that the power spectrum is Lorentzian with the half-width $2\xi+\Gamma$, and is proportional to the field intensity in agreement with the correspondence principle. Our result in Eq. (29) indicates that the broadenings due to the radiation field and to nonelectromagnetic relaxations are independent of each

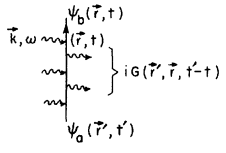

FIG. 4. Graphical illustration for the calculations of the induced transition amplitude.

other as they should be. However, instead of smearing the levels in this way, Kleppner *et al.*[12] averages the transition probability including the electromagnetic width over the time with a weighting factor depending on the nonelectromagnetic relaxtion, $e^{-\Gamma t}$, obtaining in this way an expression similar to Eq. (29) but with a width

$$(4\xi^2+\Gamma^2)^{1/2}$$

instead of the sum appearing in Eq. (29). This nonadditivity of the width indicates that the relaxation processes involved are not independent, but a dynamical basis for their mutual dependence is not at all obvious.

We may give a numerical example to demonstrate the validity of Eq. (29) for the hydrogen maser. Typical operating data are $2\xi = 10$ cps,[12] $\delta = 300$ cps,[29] $|\omega_0 - \omega| = 80$ cps,[30] $\gamma \approx 0$; thus we have

$$\frac{\xi\,\Gamma^2}{\pi[\Gamma^2+(\omega-\omega_0)^2]^2} \approx 10^{-4} \text{ sec} .$$

Since the storage time, which is the time available for the interaction between the hydrogen atoms and the radiation field, is of the order of one second,[12] it is seen that Eq. (27) is satisfied. This justifies dropping the second term in Eq. (26) for the hydrogen maser.

IV. EFFECTS OF MULTIPLE MODES ON TWO-LEVEL SYSTEM

In Sec. III, we considered a two-level system interacting with a single mode. It is easy to generalize to multiple modes. In this case, the matrix element of the mass operator becomes

$$H_{aa}(-\Omega) = -\sum_{j=1}^{n} \frac{\xi_j^2}{E_b - \omega_j + \Omega} ,$$

$$\xi_j = (e^2 N_j \omega_j / 2V)^{1/2} \langle a|\hat{d}|b\rangle ,$$

$$j = 1, 2, \ldots, n, \quad \xi_j \text{ real.} \tag{30}$$

Computing $C_a(t)$ as before, we have the following expression for the induced transition probability in the presence of n photon modes:

$$P_{a\to b}(t) = 1 - |C_a(t)|^2 , \tag{31}$$

where

$$C_a(t) = \frac{1}{2\pi i} e^{-iE_a t} \int_{-\infty}^{\infty} \left(z - \sum_{j=1}^{n} \frac{\xi_j^2}{z+\omega_{0j}} \right)^{-1} e^{izt} dz ,$$

$$\omega_{0j} = \omega_0 - \omega_j . \tag{32}$$

It can be shown[31] that the roots of the equation

$$f(z) = z - \sum_{j=1}^{n} \frac{\xi_j^2}{z+\omega_{0j}} = 0 \tag{33}$$

are all real and different. Let the $n+1$ different real roots be $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$. It can further be shown that [see Appendix B(i)]

$$\sum_{j=0}^{n} [f'(\alpha_j)]^{-1} = 1 , \tag{34}$$

where

$$f'(z) = \frac{df(z)}{dz} = 1 + \sum_{j=1}^{n} \frac{\xi_j^2}{(z+\omega_{0j})^2} .$$

Evaluating the integral of Eq. (32) by the theorem of residues and making use of Eq. (34), we have

$$P_{a\to b}(t) = 4 \sum_{j>l=0}^{n} \sum \frac{\sin^2[\frac{1}{2} t(\alpha_j - \alpha_l)]}{f'(\alpha_j) f'(\alpha_l)} . \tag{35}$$

It must be noted that Eq. (35) is not convenient in analytical discussion. The difficulty consists in obtaining the roots of Eq. (33). In the following paragraphs, we derive some analytical expressions suitable tor certain limiting cases. In this connection, we may simplify our presentation in the two-mode case. Then Eq. (30) may be written

$$H_{aa}(-\Omega) = -\frac{\xi^2}{E_b + \Omega - \omega} - \frac{\xi'^2}{E_b + \Omega - \omega'} ,$$

where the primed term is due to the presence of the extraneous mode. In the absence of the resonance mode ω, we have

$$C_a(t) = \left(\cos\eta' t + i \frac{\omega_0 - \omega'}{2\eta'} \sin\eta' t \right) e^{-i(E_a + E_b - \omega')t/2} ,$$

in accordance with Eq. (22), where

$$\eta' = \tfrac{1}{2}[(\omega_0 - \omega')^2 + 4\xi'^2]^{1/2} .$$

If $|\omega_0 - \omega'| \gg 2\xi'$, we have

$$C_a(t) \approx \exp\left[-i \left(E_a - \frac{\xi'^2}{\omega_0 - \omega'} \right) t \right] .$$

This indicates that in the presence of the extraneous mode ω', the level is modified to $\bar{E}_a$, where

$$\bar{E}_a = E_a - \xi'^2/(\omega_0 - \omega') .$$

A similar treatment can be applied to level B: E_b is modified to $\bar{E}_b$, where

$$\bar{E}_b = E_b + \xi'^2/(\omega_0 - \omega') ,$$

and the level separation becomes

$$\bar{\omega}_0 = \bar{E}_b - \bar{E}_a = \omega_0 + 2\xi'^2/(\omega_0 - \omega') , \tag{36}$$

because of the presence of the extraneous mode ω', whenever the condition

$$|\omega_0 - \omega'| \gg 2\xi'$$

is satisfied. Applying the results of Sec. III, it is

easy to establish the induced transition probability,

$$P_{a\to b}(t)=\frac{4\xi^2}{(\bar{\omega}-\omega)^2+4\xi^2}\sin^2\{\tfrac{1}{2}t[(\bar{\omega}_0-\omega)^2+4\xi^2]^{1/2}\}. \tag{37}$$

Therefore, one finds a shift of the central maximum by an amount

$$\delta\omega=\omega_r-\omega_0=2\xi'^2/(\omega_0-\omega')\ . \tag{38}$$

If $\omega'=-\omega_0$, one obtains the Bloch-Siegert shift.[32] Equation (38) is in agreement with the semiclassical treatments.[33]

We now consider the situation when $|\omega_0-\omega'|\gg 2\xi'$ is not true. In that case, the above approximation clearly breaks down. The following simple calculation will show that, instead of Eq. (38), we have $\delta\omega\to 0$ when $\omega'\to\omega_0$.

Consider the following integral for the $a\to a$ transition:

$$C_a(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(z-\frac{\xi^2}{z+a}-\frac{\xi'^2}{z+a'}\right)^{-1}e^{izt}dz\ , \tag{39}$$

where

$$a=\omega_0-\omega,\quad a'=\omega_0-\omega',\quad z=z-i0\ .$$

Looking at the function in the denominator, which is of the form

$$z^2+z\left(a-\frac{\xi'^2}{\alpha_\pm+a'}\right)-\left(\xi^2+\frac{\xi'^2}{\alpha_\pm+a'}\right), \tag{40}$$

we seek its zeros which approach $\alpha_\pm$ as $\xi'^2\to 0$. Here $\alpha_\pm$ are roots of

$$z^2+az-\xi^2=0\ .$$

The zeros are

$$\lambda_\pm=-\frac{1}{2}\left(a-\frac{\xi'^2}{\alpha_\pm+a'}\right)\pm\frac{1}{2}\left[\left(a+\frac{\xi'^2}{\alpha_\pm+a'}\right)^2+4\xi^2\right]^{1/2}. \tag{41}$$

Thus if in the integrand of (39) the denominator is replaced by (40), we get

$$|C_a(t)|^2=1+\frac{4(\lambda_++a)(\lambda_-+a)}{(\lambda_+-\lambda_-)^2}\sin^2[\tfrac{1}{2}(\lambda_+-\lambda_-)t]$$

and

$$|C_b(t)|^2=1-|C_a(t)|^2$$

$$=\frac{4(\lambda_++a)(\lambda_-+a)}{4(\lambda_++a)(\lambda_-+a)-(\lambda_++\lambda_-+2a)^2}$$

$$\times\sin^2[\tfrac{1}{2}(\lambda_+-\lambda_-)t]\ .$$

The central maximum for $|C_b(t)|^2$ is at

$$\lambda_++\lambda_-+2a=0\ . \tag{42}$$

Using Eq. (41), and assuming that the square roots in $\lambda_\pm$ nearly cancel, we have

$$2a(-\xi^2-aa'+a'^2)+\xi'^2(-a+2a')=0\ .$$

Neglecting cubic terms, we get

$$\delta\omega=a=[2\xi'^2/(2\xi^2+\xi'^2)]a'\ .$$

If $\xi=\xi'$, then the above expression reduces to

$$\delta\omega=\tfrac{2}{3}(\omega_0-\omega')\ . \tag{43}$$

Hence, we see that if $\omega'\to\omega_0$, then $\delta\omega\to 0$. The result is in qualitative agreement with the experiments of Lewis *et al.*[34]

V. INDUCED TRANSITION PROBABILITIES FOR THREE-LEVEL SYSTEM

We now investigate the QED formalism for the three-level system. What we are going to calculate is essentially the same as for the two-level system. The explicit computations made here are restricted to order e^2 only. Our purpose is to compare the QED method and the semiclassical treatments, and reinterpret the semiclassical results whenever possible.

Let $\vec{F}=\vec{I}+\vec{J}$ be the total angular momentum of the system, where $\vec{I}$ is the nuclear spin, and $\vec{J}$ is the total angular momentum for the electron. In the case of an atomic S state, we have $\vec{J}=\frac{1}{2}\vec{\sigma}$, where $\vec{\sigma}$ is the Pauli spin operator. We shall consider the transition with $\Delta F=0$ and the dipole selection rules $\Delta m=\pm 1$ as usual. Further, in order to facilitate the presentation, we consider the normal Zeeman splitting for the spin-1 system. The level scheme is shown in Fig. 5.

First, the matrix elements of the mass operator can be found easily in the nonrelativistic limit and the dipole approximation. They are

$$H_{13}=H_{31}=H_{11}=H_{33}=-b^2\left(\frac{1}{E_2+\omega+\Omega}+\frac{1}{E_2-\omega+\Omega}\right),$$

$$H_{22}=-b^2\left(\frac{1}{E_3+\omega+\Omega}+\frac{1}{E_3-\omega+\Omega}+\frac{1}{E_1+\omega+\Omega}\right.$$

$$\left.+\frac{1}{E_1-\omega+\Omega}\right), \tag{44}$$

$$H_{12}=H_{21}=H_{23}=H_{32}=0\ ,$$

to order e^2, where

$$\langle 1|\hat{d}|1\rangle=\langle 2|\hat{d}|2\rangle=\langle 3|\hat{d}|3\rangle=\langle 1|\hat{d}|3\rangle=\langle 3|\hat{d}|1\rangle=0$$

and (45)

$$b=(e^2N\omega/2V)^{1/2}\langle i|\hat{d}|j\rangle,\quad i,j=1,2,3,\quad |i-j|=1\ .$$

It has been shown that, in the normal Zeeman effect, the matrix elements of a dipole operator for general spin can be constructed from the elements of a spin-$\frac{1}{2}$ system.[35] This construction follows from the fact that a general n-level system

can be obtained from the symmetric state of n two-level systems. A simple calculation for the three-level system gives

$$b=\sqrt{2}\,\xi \quad , \tag{46}$$

where ξ is defined in (18). Using Eq. (46), a comparison of the following results with the Majorana formula,[36] which is obtained from the semiclassical theory, can be made.

Note that the expressions (44) are written in terms of the photon-number-state representation; one can also write them in terms of coherent states. Then the formulas corresponding to (44) are

$$H_{nn}=-\frac{e^2}{2\omega V}\sum_j |\langle n|\hat{e}_\lambda z_\lambda e^{i\vec{k}\cdot\vec{r}}|j\rangle|^2 \times\left(\frac{1}{E_j+\omega+\Omega}+\frac{1}{E_j-\omega+\Omega}\right) \quad ,$$

$$H_{nm}=-\frac{e^2}{2\omega V}\sum_j \langle n|\hat{e}_\lambda z_\lambda e^{i\vec{k}\cdot\vec{r}}|j\rangle\langle j|\hat{e}_\lambda z_\lambda e^{-i\vec{k}\cdot\vec{r}}|m\rangle \times\left(\frac{1}{E_j+\omega+\Omega}+\frac{1}{E_j-\omega+\Omega}\right) . \tag{47}$$

In these formulas, it is apparent that $|e_\lambda z_\lambda|^2$ takes the place of N, as was shown in Ref. 23.

We now compute the Green's function for the three-level system. It can be easily computed from Eq. (16). The result is

$$f_{1,1}(\Omega)=\frac{E_3+\Omega+H_{11}(-\Omega)}{(E_1+\Omega)(E_3+\Omega)+(E_1+E_3+2\Omega)H_{11}(-\Omega)} \quad ,$$

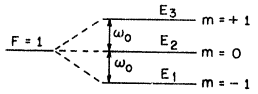

FIG. 5. Three-level system in the normal Zeeman effect.

$$f_{2,2}(\Omega)=\frac{1}{E_2+\Omega+H_{22}(-\Omega)} \quad ,$$

$$f_{3,3}(\Omega)=\frac{E_1+\Omega+H_{33}(-\Omega)}{(E_1+\Omega)(E_3+\Omega)+(E_1+E_3+2\Omega)H_{33}(-\Omega)} \quad , \tag{48}$$

$$f_{1,3}(\Omega)=f_{3,1}(\Omega) =\frac{-H_{11}(-\Omega)}{(E_1+\Omega)(E_3+\Omega)+(E_1+E_3+2\Omega)H_{11}(-\Omega)} \quad ,$$

$$f_{1,2}(\Omega)=f_{2,1}(\Omega)=f_{3,2}(\Omega)=f_{2,3}(\Omega)=0.$$

Computing the probability amplitude $C_{2\to2}(t)$ for the $2\to2$ transition from

$$C_{2\to2}(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} f_{2,2}(\Omega)\,e^{i\Omega t}d\Omega \quad , \tag{49}$$

and using (44) to order e^2 in the rotating wave approximation, we get

$$P_{2\to2}(t)=|C_{2\to2}(t)|^2 =\left(1-\frac{8\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\right)^2 . \tag{50}$$

Further, using the argument leading to Eq. (25), one can compute $C_{2\to1}(t)$ and $C_{2\to3}(t)$ for the $2\to1$ and $2\to3$ transitions, respectively; the results are

$$P_{2\to1}(t)=|C_{2\to1}(t)|^2=P_{2\to3}(t)=|C_{2\to3}(t)|^2 =\frac{8\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\{\cos^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]+\frac{a^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\} \quad , \tag{51}$$

where

$$a=\omega-\omega_0, \quad \omega_0=E_3-E_2=E_2-E_1 \quad .$$

Equations (50) and (51) are in agreement with the semiclassical treatments.[36]

Similarly, one may compute $C_{3\to2}(t)$ and $C_{3\to1}(t)$ for the $3\to2$ and $3\to1$ transitions, assuming that at $t=0$ the third level is occupied. Following the arguments leading to (24) and restricting ourselves here again to order e^2 in H_{nm}, we get the following expressions for $C_{3\to2}(t)$ and $C_{3\to1}(t)$:

$$C_{3\to2}(t)=ie\int d^3r\,d^3r'\,dt'\,\theta(t')\bar{\psi}_2(\vec{r})G(\vec{r},\vec{r}';t-t')\,\hat{e}\,(N/2\omega V)^{1/2}\,e^{i\omega t'-i\vec{k}\cdot\vec{r}'}\psi_3(\vec{r}')\,e^{-iE_3t'} \quad , \tag{52a}$$

$$C_{3\to1}(t)=(ie)^2\int d^3r\,d^3r'\,d^3r''\,dt'\,dt''\,\theta(t'')\bar{\psi}_1(r)G(\vec{r},\vec{r}';t-t')\,\hat{e}\,(N/2\omega V)^{1/2} \times e^{i\omega t'-i\vec{k}\cdot\vec{r}'}G(\vec{r}',\vec{r}'';t'-t'')\,\hat{e}\,(N/2\omega V)^{1/2}\,e^{i\omega t''-i\vec{k}\cdot\vec{r}''}\psi_3(\vec{r}'')\,e^{-iE_3t''} \quad . \tag{52b}$$

Equation (52a) can be evaluated very easily; we have

$$C_{3\to 2}(t)=\frac{\sqrt{2}\,\xi}{2\pi i}\int_{-\infty}^{\infty}\frac{(z+a)\,e^{izt}}{z\,[z^2-(a^2+4\xi^2)]}\,dz\ . \tag{53a}$$

In order to evaluate (52b), we first carry out the integration over $\vec{r}$, $\vec{r}'$, and $\vec{r}''$. In the nonrelativistic limit and the dipole approximation, and using the dipole selection rules, we find four terms of the following type contributing to the integral in (52b):

$$f_{1,1}f_{2,2},\quad f_{1,2}f_{3,2},\quad f_{1,3}f_{2,2},\quad f_{1,3}f_{2,3}\ ,$$

since the Green's function contains the diagonal and off-diagonal sums. According to (48), the terms involving $f_{1,2}$ and $f_{2,3}$ vanish. Upon using (48), we are left with the following sum of two terms:

$$f_{1,1}(\Omega)f_{2,2}(\Omega')+f_{1,3}(\Omega)f_{2,2}(\Omega')$$

$$=\frac{(E_1+\Omega)f_{2,2}(\Omega')}{(E_1+\Omega)(E_3+\Omega)+(E_1+E_3+2\Omega)H_{33}(-\Omega)}\ .$$

The above term is to be integrated over Ω and Ω' with other factors deliberately omitted. Including the omitted factors and carrying out the integration of (52b) over t' and t'', we get

$$C_{3\to 1}(t)=(2\xi^2/2\pi i)\int d\Omega\,d\Omega'\,\delta(\Omega-\Omega'+\omega)\,e^{i\Omega' t}$$

$$\times[E_2+\Omega+H_{22}(-\Omega)]^{-1}\{E_1+\Omega'+H_{11}(-\Omega')$$

$$+[H_{11}(-\Omega')/(E_3+\Omega')](E_1+\Omega')\}^{-1}(E_3-\omega+\Omega)^{-1}\ .$$

The integration over Ω' can be done immediately:

$$C_{3\to 1}(t)=\frac{2\xi^2}{2\pi i}\int_{-\infty}^{\infty}e^{itz}dz\,e^{-i\omega t}\left\{z\,[z^2-(a^2+4\xi^2)]\right.$$

$$\left.\times\left[1-2\xi^2\left(\frac{1}{z}+\frac{1}{z+2\omega}\right)\frac{1}{z+a}\right]\right\}^{-1}\ .$$

Expanding the factor

$$\left[1-2\xi^2\left(\frac{1}{z}+\frac{1}{z+2\omega}\right)\frac{1}{z+a}\right]^{-1}$$

as a power series in ξ^2, the second term is of order ξ^2, thus contributing a ξ^4 term to $C_{3\to 1}(t)$; we then get

$$C_{3\to 1}(t)=\frac{2\xi^2}{2\pi i}e^{-i\omega t}\int_{-\infty}^{\infty}\frac{e^{itz}dz}{z\,[z^2-(a^2+4\xi^2)]}+O(\xi^4)\ . \tag{53b}$$

Apart from the term $O(\xi^4)$, the results of (53a) and (53b) are in exact agreement with the semiclassical treatment,[36] as can be easily seen by comparing (53a) and (53b) with Eqs. (A16) of Appendix A.

In computing $C_{3\to 3}(t)$, the matter becomes complicated. This complication is closely related to the renormalization principle to be discussed briefly below. However, if we require that

$$|C_{3\to 3}(t)|^2+|C_{3\to 2}(t)|^2+|C_{3\to 1}(t)|^2=1\ ,$$

then it is easy to compute $P_{3\to 3}(t)$ from (53a) and (53b). The result is again the same as in the semiclassical theory.[37]

The results of the calculations made so far are the same as those of the semiclassical theory. This is not surprising because we have calculated only the effects which are proportional to the intensity in the low-frequency (dipole) limit and this is exactly where contact between quantum and classical electrodynamics is made.[1] It is in this limit that, for example, the Klein-Nishina formula including radiative corrections becomes the Thomson formula and thus provides a way of measuring e^2 directly. The semiclassical theory does not go beyond these effects, while in the quantum theory one can include in a systematic way the contribution of any number of forward scatterings and so can evaluate the dependence on intensity beyond the first linear term. This is done in Sec. VI for the special case of a two-level atom. The calculation on the multiple-level system and quantitative comparison with Kusch's experiments[11] are now in preparation.

In concluding this section, we discuss the problem of renormalization when higher-order effects are included. In the previous calculations, we have ignored such problems in order to get the results quickly. However, from a rigorous theoretical point of view, the problems must be carefully examined. First, let us consider the order e^2 effect. It is seen that the validity in the interpretation of (22) and (50) requires that $G(x_1, x_2)$ obtained must satisfy

$$G(\vec{r}_1,\vec{r}_2;0)=\delta(\vec{r}_1-\vec{r}_2)\ , \tag{54}$$

so that

$$\psi(\vec{r},t)=\int G(\vec{r},\vec{r}';t)\psi(\vec{r}',0)\,d^3r'\ . \tag{55}$$

To order e^2, condition (54) can be verified explicitly. Hence, no problem arises in the interpretation of (22) and (50). Alternatively, the validity of (54) to order e^2 can be seen by examining the equal-time anticommutation relation for the electron field operator to order e, and it is not difficult to demonstrate this fact. However, to higher order than e^2, one must introduce a wave function renormalization constant Z_2 and subtract a mass counter term δm in the QED terminology in order to preserve the form of the equal-time commutation relation. A general survey of the theory of forward scattering is deferred to a future publication. In Sec. VI, we are not interested in the interpretation of the expressions like (22) and (50),

because the experimental quantities are the induced transition probabilities which can be computed in a way similar to (24).

VI. FINITE INTENSITY EFFECTS IN MOLECULAR AND ATOMIC BEAM EXPERIMENTS

In Sec. V, we have shown how the semiclassical results arise. The formulas for the induced transition probabilities obtained there indicate that the semiclassical theory does not take the true fourth-order process into account. This appears to be unsatisfactory from the theoretical viewpoint of QED, though it is sometimes claimed by those who use the semiclassical theory that their results are "exact." From the experimental point of view, Kusch[11] has pointed out that the semiclassical results[38] show a clear discrepancy with experimental observations when the field intensity increases. This situation seems to indicate that the semiclassical theory cannot properly take care of the multiple quantum transition, which should be important in the region of high-intensity fields. In the following calculations, we shall explicitly demonstrate the anomalous behavior of the width of the induced transition probabilities in two-level atoms which is very like that observed by Kusch[11] in five-level atoms.

For the sake of simplicity, let us return to the two-level system treated in Sec. III. We shall use Eq. (24) to compute the induced transition probability, but, in contrast with Sec. III, the correction of the Green's function due to forward scattering is taken to all orders in e^2. Looking at (11) for the fourth-order process, it is not difficult to generalize it to the process of n incoming followed by n outgoing photons. By using (17), the matrix element corresponding to this process is

$$H_{pq}^{(2n)}(-\Omega) = \sum_{l_1,\ldots,l_{2n-1}} (-1)^n \left(\frac{e^2N}{2\omega V}\right)^n \hat{d}_{p,l_1}\hat{d}_{l_1,l_2}\cdots\hat{d}_{l_{2n-1},q} \times\{(E_{l_1}+\Omega-\omega)(E_{l_2}+\Omega-\omega)\cdots \times[E_{l_{2n-2}}+\Omega-(n-1)\omega](E_{l_{2n-1}}+\Omega-n\omega)\}^{-1}, \quad (56)$$

for $n \geq 2$, and $H_{pq}^{(2)}(-\Omega)$ is given by (19), where $\hat{d}$ is the dipole operator defined in (18). It is easily seen that for

$$p=a, \quad q=b, \quad \text{or} \quad p=b, \quad q=a,$$

we have, for all n,

$$H_{ab}^{(2n)} = H_{ba}^{(2n)} = 0,$$

due to the dipole selection rule. For

$$p=q=b,$$

we get

$$H_{bb}^{(2n)}(E_b - x) = (-1)^n \xi^{2n} \times [(x+a_1)\cdots(x+a_n)(x+b_1)\cdots(x+b_{n-1})]^{-1}, \quad (57)$$

where

$$x = E_b + \Omega, \quad \omega_0 = E_b - E_a > 0, \quad a_j = \omega_0 + j\omega, \quad b_j = j\omega, \quad j = 1, 2, \ldots, \quad (58)$$

and ξ is defined in (18). Thus the matrix element for the mass operator taking the higher-order forward scattering into account can be written in the form

$$H_{bb} = \sum_{n=1}^{\infty} H_{bb}^{(2n)} = -\frac{\xi^2}{x+a_1} + \sum_{n=2}^{\infty} (-1)^n \xi^{2n} \times [(x+a_1)\cdots(x+a_n)(x+b_1)\cdots(x+b_{n-1})]^{-1}. \quad (59)$$

Similarly, an expression for H_{aa} to all orders can be found. Now because

$$H_{ab} = H_{ba} = \sum_{n=1}^{\infty} H_{ab}^{(2n)} = \sum_{n=1}^{\infty} H_{ba}^{(2n)} = 0,$$

it is readily seen that the form (20) remains unchanged. Using (15), (20), (24), and (59), we get an expression for the induced transition probability taking the effect of one extra photon absorption into account,

$$P_{a\to b}(t) = |C_b(t)|^2 = \frac{\xi^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \frac{e^{itz}\,dz}{(z+a_1)[z-g(z)]} \right|^2, \quad (60)$$

where $g(z)$ is defined by

$$g(z) = -\frac{\xi^2}{z+a_1} + \sum_{n=2}^{\infty} (-1)^n \xi^{2n} \left(\prod_{j=1}^{n}(z+a_j) \prod_{l=1}^{n-1}(z+b_l) \right)^{-1}, \quad (61)$$

and the following notation has been introduced:

$$\prod_{j=1}^{n}(x+a_j) \equiv (x+a_1)(x+a_2)\cdots(x+a_n).$$

A partial-fraction decomposition of (57) can be achieved by putting

$$\prod_{j=1}^{n}\prod_{l=1}^{n-1}[(z+a_j)(z+b_l)]^{-1} = \sum_{j=1}^{n} \frac{A_j}{z+a_j} + \sum_{j=1}^{n-1} \frac{B_j}{z+b_j},$$

where A_j and B_j for each j are clearly dependent on n, and are given by

$$A_j = \lim_{z\to -a_j} \prod_{l=1}^{n}\prod_{k=1}^{n-1} \frac{z+a_j}{(z+a_l)(z+b_k)}, \qquad B_j = \lim_{z\to -b_j} \prod_{l=1}^{n}\prod_{k=1}^{n-1} \frac{z+b_j}{(z+a_l)(z+b_k)}. \quad (62)$$

Thus

$$(z+a_1)g(z) = -\left(\xi^2 - \sum_{n=2}^{\infty}(-1)^n \xi^{2n} A_1\right) + (z+a_1)$$

$$\times\left[\sum_{n=1}^{\infty}\xi^{2n}(-1)^n\left(\sum_{j=2}^{n}\frac{A_j}{z+a_j}+\sum_{j=1}^{n-1}\frac{B_j}{z+b_j}\right)\right], \quad (63)$$

where A_1 can be easily computed from (62) to get

$$A_1 = \left((n-1)!\ \omega^{n-1}\prod_{j=0}^{n-2}(\omega_0+j\omega)\right)^{-1}. \quad (64)$$

If the system is tuned near resonance for a single quantum transition where $\omega \approx \omega_0$, we may approximate (64) by

$$A_1 \approx \{[(n-1)!]^2\ \omega_0^{2n-2}\}^{-1}, \quad (65)$$

and ignore the last sum in (63) by writing

$$(z+a_1)g(z) \approx -\left(\xi^2 - \sum_{n=2}^{\infty}(-1)^n \xi^{2n} A_1\right) \equiv -\xi^2 J_0(s), \quad (66)$$

where

$$s \equiv 2\xi/\omega_0,$$

and $J_0(x)$ is the ordinary Bessel function of zeroth order.[39] With this approximation, Eq. (60) becomes

$$P_{a\to b}(t) = \frac{\xi^2}{4\pi^2}\left|\int_{-\infty}^{\infty}\frac{e^{itz}\,dz}{z(z+a)-\xi^2 J_0(s)}\right|^2, \quad (67)$$

where

$$a = a_1 = \omega_0 - \omega.$$

Let s_1 be the smallest root of $J_0(x)$, which is given by[37]

$$s_1 \approx 2.40.$$

It is known that

$$J_0(x) > 0 \quad \text{for } 0 < x < s_1.$$

Thus, if

$$0 < s = 2\xi/\omega_0 < 2.40, \quad (68)$$

we may evaluate (67) by theorem of residues to get

$$P_{a\to b}(t) = \frac{4\xi^2}{a^2+4\xi^2 J_0(s)}\sin^2\{\tfrac{1}{2}t[a^2+4\xi^2 J_0(s)]^{1/2}\}, \quad (69a)$$

and the amplitude is

$$C_b(t) = i\frac{2\xi}{[a^2+4\xi^2 J_0(s)]^{1/2}}\sin\{\tfrac{1}{2}t[a^2+4\xi^2 J_0(s)]^{1/2}\} \times e^{-i(E_a+E_b+\omega)t/2}. \quad (69b)$$

Note that Eq. (69a) attains a maximum for $a=0$ with $|\sin\{t\xi[J_0(s)]^{1/2}\}| = 1$ for some t. The maximum for $P_{a\to b}(t)$ is $[J_0(s)]^{-1}$, which is clearly greater than unity. Such a pathological situation arises from the fact that, to obtain $C_b(t)$, we only include one extra line for the absorption of an external photon. In fact, to compute the $a\to b$ transition, there are two, three, etc. extra absorptions of the external photon lines. Moreover, the contributions due to an even number of extra absorptions vanish as can be easily seen from (52b) in accordance with the dipole selection rule; this fact has been noted by Winter.[40] Thus one only needs to evaluate the contribution due to odd numbers of extra absorptions in order to get the correct formula for $P_{a\to b}(t)$. In this connection, we may calculate the amplitude due to the $(2n+1)$ extra absorptions of the external photons. It is given by

$$C_{a\to b}^{(2n+1)}(t) = (ie)^{2n+1}[(N/2\omega V)^{1/2}]^{2n+1} \times\int d^3r\, d^3r_1\cdots d^3r_{2n+1}\, dt_1\cdots dt_{2n+1}\,\theta(t_1) \times\bar{\psi}_b(\vec{r})\,G(\vec{r},\vec{r}_1;t-t_1)\hat{e}\,e^{i\omega t_1 - i\vec{k}\cdot\vec{r}_1} \times G(\vec{r}_1,\vec{r}_2;t_1-t_2)\cdots G(\vec{r}_{2n},\vec{r}_{2n+1};t_{2n}-t_{2n+1})\hat{e} \times e^{i\omega t_{2n+1} - i\vec{k}\cdot\vec{r}_{2n+1}}\psi_a(\vec{r}_{2n+1})\,e^{-iE_a t_{2n+1}}. \quad (70)$$

Carrying out the integration over space-time coordinates, we get

$$C_{a\to b}^{(2n+1)}(t) = [(-1)^n/2\pi i]\,\xi^{2n+1}\int d\Omega_1\cdots d\Omega_{2n+1}\,e^{i\Omega_1 t} \times\delta(\Omega_2-\Omega_1+\omega)\,\delta(\Omega_3-\Omega_2+\omega)\cdots \times\delta(\Omega_{2n}-\Omega_{2n+1}+\omega)\,[(E_b+\Omega_1+H_{bb}) \times(E_a+\Omega_2+H_{aa})\cdots(E_a+\Omega_{2n}+H_{aa}) \times(E_b+\Omega_{2n+1}+H_{bb})(E_a+\Omega_{2n+1}+\omega)]^{-1}, \quad (71)$$

where H_{bb} in $E_b+\Omega_1+H_{bb}$ is a function of Ω_1, and a similar remark is true for other factors. For $n=0$, the above expression reduces to Eq. (69b). Because of the presence of the δ function, we may first perform the $\Omega_1, \Omega_2, \ldots, \Omega_{2n}$ integrations; without difficulty we get

$$C_{a\to b}^{(2n+1)}(t) = [(-1)^n/2\pi i]\,\xi^{2n+1}\int_{-\infty}^{\infty}dz\,[(z+a_1)\cdots \times(z+a_{2n+1})(z+b_2)\cdots(z+b_{2n})]^{-1}[z-g(z)]^{-1} \times\exp[-\tfrac{1}{2}i(E_a+E_b+\omega)t+\tfrac{1}{2}i a_{2n+1}t], \quad (72)$$

where a_j and b_j are defined in (58), and $g(z)$ in (61). Note that H_{aa}, H_{bb} occurring in (71) under the $\Omega_1,\ldots,\Omega_{2n}$ integrations have been ignored as they give rise to only inessential corrections.

We may sum (72) over n to get $C_{a\to b}(t)$ by restricting our attention to a term involving $z+a_1$

in the integral as in the evaluation of the integral in (60). The result is simply to modify (69a) by an extra factor $[J_0(\frac{1}{2}s)]^2$ in the numerator; the peak value of $P_{a\to b}(t)$ is still greater than unity due to the fact that we have omitted the iterated graph in the construction of the mass operator. However the approximation of Eq. (66), which is

$$g(z) \approx -\xi^2 J_0(s)/(z+a) , \tag{73}$$

can be used to justify Eq. (54) if (68) is satisfied. Thus one may use (55) to compute the amplitude for the $a \to a$ transition. It is

$$C_{a\to a}(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{izt}\,dz}{z - \xi^2 J_0(s)/(z+a)} , \tag{74}$$

where the irrelevant phase factor has been omitted. Computing the induced transition probability $P_{a\to b}(t)$ from

$$P_{a\to b}(t) = 1 - |C_{a\to a}(t)|^2 , \tag{75}$$

we get

$$P_{a\to b}(t) = \frac{4\xi^2 J_0(s)}{a^2 + 4\xi^2 J_0(s)} \sin^2\{\tfrac{1}{2}t[a^2 + 4\xi^2 J_0(s)]^{1/2}\} , \tag{76}$$

where

$$a \equiv a_1 = \omega_0 - \omega ,$$

which never exceeds unity and which gives the same width as Eq. (69a). Thus, the width d_1 for the single quantum transition is given by

$$d_1 = 2\xi\,[J_0(s)]^{1/2} . \tag{77}$$

In order to study the behavior of the width d_1 as a function of the power level, we define the function

$$f(s) \equiv d_1/\omega_0 = s\,[J_0(s)]^{1/2} . \tag{78}$$

The maximum of $f(s)$ is determined from that value of s satisfying

$$2J_0(s) = s\,J_1(s) ,$$

where $J_1(s)$ is the Bessel function of the first order.[39] The root can be approximated very accurately by the value

$$s \approx 1.60 .$$

For $s > 1.60$, $f(s)$ decreases to zero at $s \approx 2.40$. The width $f(s)$ is plotted in Fig. 6, for $0 \le s \le 2.40$.

The mathematical discussion of the function $f(s)$ related to the width d_1 clearly indicates that for $0 < s < 1.60$, we have a power broadening, while

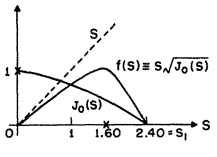

FIG. 6. Anomalous behavior of width for the single-quantum transition in a two-level system. The function is defined in Eq. (77). The dashed line is the semiclassical theory for the width.

for $1.60 < s < 2.4$, we get a power narrowing, in contrast with the semiclassical theory where one always has power broadening. Such anomalous behavior as demonstrated above is in qualitative agreement with Kusch's experiments.[11] A quantitative comparison requires the application of this method to an atom with five unequally spaced levels. This is now in preparation.

In closing this section, let us calculate the shift in the resonant frequency for the single-quantum transition. The calculation can be easily done using the consideration of Sec. IV for the evaluation of the Bloch-Siegert shift. In the beginning of this section, we were primarily interested in the width, and thus the part of the contribution due to the opposite rotating waves has been ignored. As we remarked earlier in Sec. IV, the shift is essentially a result of the antiresonant terms; therefore, we must add this type of terms to (56) with ω replaced by $-\omega$. To obtain the shift, we proceed in the same way as we have done in Sec. IV. Consider the amplitude for the $a \to a$ transition,

$$C_a(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{itz}}{z - f(z)} dz\, e^{-iE_a t} , \tag{79}$$

where $f(z)$ is similar to $g(z)$ defined in (61):

$$f(z) = \sum_{n=1}^{\infty} (-1)^n \xi^{2n} \left(\prod_{j=1}^{n} (z + a_j) \prod_{j=1}^{n-1} (z + b_j) \right)^{-1} , \tag{80}$$

and we have neglected the resonant term of the form

$$\xi^2/(z + a_1) ,$$

as it only contributes to the width and not to the shift.

Now assuming that the contribution to the integral is near the pole $z = 0$, we may approximate

$$f(z) \approx f(0) = \sum_{n=1}^{\infty} (-1)^n \xi^{2n} \left(\prod_{j=1}^{n} a_j \prod_{j=1}^{n-1} b_j \right)^{-1} \tag{81}$$

under the integral sign of (79). Further, assuming that $\omega \approx \omega_0$, as before for the evaluations of the single-quantum width, we get

$$f(0) \approx \sum_{n=1}^{\infty} (-1)^{n-1} \xi^{2n} [(n-1)!\,(n+1)!\,\omega_0^{2n-1}]^{-1}$$

$$= \omega_0 J_2(s) , \tag{82}$$

where

$$s = 2\xi/\omega_0 , \tag{83}$$

and $J_2(s)$ is the Bessel function of order 2. Thus the integral (79) becomes

$$C_a(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{itz}}{z + \omega_0 J_2(s)} dz\, e^{-iE_a t}$$

$$= e^{-it[E_a - \omega_0 J_2(s)]t} . \tag{84}$$

From (84), we see that the level a is shifted by the following amount:

$$-\omega_0 J_2(s) .$$

Similarly, we can calculate the shift to the level b by computing $C_b(t)$, assuming at $t=0$, the system is at the level b; we find

$$C_b(t) = e^{-i[E_b+\omega_0 J_2(s)]t} .$$

This expression implies that the level b is modified by the following shift:

$$+\omega_0 J_2(s) .$$

Hence by using the following notations for the two levels in the presence of the high-intensity field,

$$\overline{E}_a = E_a - \omega_0 J_2(s), \quad \overline{E}_b = E_b + \omega_0 J_2(s) , \tag{85}$$

and computing the Eqs. (21) and (23), we find

$$P_{a\to b}(t) = (\xi^2/\overline{\eta}^2)\sin^2 \overline{\eta} t , \tag{86}$$

where

$$\overline{\eta} = \tfrac{1}{2}[(\omega - \overline{\omega}_0)^2 + 4\xi^2]^{1/2}, \quad \overline{\omega}_0 = \overline{E}_b - \overline{E}_a .$$

Thus we find the following shift of the central maximum of (86):

$$\delta\omega = \omega_r - \omega_0 = 2\omega_0 J_2(s) . \tag{87}$$

For the case of a weak field, (87) reduces to

$$\delta\omega = 2\xi^2/2\omega_0 = \xi^2/\omega_0 ,$$

in agreement with (38) by taking $\omega' = -\omega_0$, $\xi' = \xi$. In order to study (87) as a function of the field amplitude, we define

$$F(s) \equiv \delta\omega/2\omega_0 = J_2(s) . \tag{88}$$

The function $F(s)$ is plotted in Fig. 7. It is seen that $F(s) < \frac{1}{2}$, and oscillates with decreasing amplitudes. This result is again in qualitative agreement with Kusch's experiments.[11]

VII. FURTHER APPLICATIONS TO EXPERIMENTS ON LASERS AND MASERS

In the foregoing sections, attention is restricted to experiments on molecular atomic beams. We should also consider the situations related to laser experiments. As far as the effect to order e^2 is concerned, the present approach offers a simple

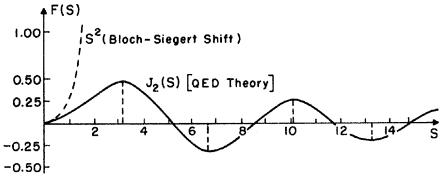

FIG. 7. Shift in the resonance frequency for a two-level system as function of field strength.

calculation for the physical quantities of interests. We wish to make a further theoretical application of Sec. IV to include the effect of the blackbody radiation field which always exists when the experiment is performed at temperature T. We shall derive the line shapes, frequency shift, and width due to the additional radiation fields.

We begin with the following expression for the $b \to b$ transition in a two-level system:

$$C_b(t) = (1/2\pi i)\, e^{-iE_b t}\int_{-\infty}^{\infty}[z - F(z)]^{-1} e^{itz}\, dz , \tag{89}$$

where, according to (30) and (32),

$$F(z) = \sum_{j=1}^{n}\frac{\xi_j^2}{z+\omega_0-\omega_j} + \sum_{j=1}^{n}\frac{\xi_j^2}{z+\omega_0+\omega_j} , \tag{90}$$

with the antiresonant terms added. If one includes the term arising from the contribution of the vacuum field, we must add one more term to (90):

$$\sum_{j=1}\frac{\eta_j^2}{z+\omega_0-\omega_j} , \tag{91}$$

where

$$\eta_j = (e^2\omega_j/2V)^{1/2}\,|\langle a|\hat{d}|b\rangle| .$$

First, let us make a simple calculation of the natural line shape by assuming $N_j = 0$ for all j. In this case $F(z)$ is simply

$$F(z) = \sum_{j=1}\frac{\eta_j^2}{z+\omega_0-\omega_j} . \tag{92}$$

Since the vacuum field possesses modes of all frequencies, we may convert

$$\frac{1}{V}\sum_j \to \frac{1}{(2\pi)^3}\int d^3k = \frac{1}{(2\pi)^3}\int \omega^2\, d\omega\, d\Omega$$

to get

$$F(z) = \frac{e^2}{2(2\pi)^3}\int d\Omega\,|\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2\int d\omega\,\omega^3(z+\omega_0-\omega)^{-1} . \tag{93}$$

Now, the major contribution of the integral (89) arises from a pole around $z \approx 0$, we further ignore the principal value of (93), and approximate

$$F(z) \approx F(0) \approx \frac{e^2}{2(2\pi)^3}\int d\Omega\,\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2 \times\int d\omega\,\omega^3\,\pi i\delta(\omega-\omega_0) . \tag{94}$$

Here we write

$$\lim_{\epsilon\to 0}\frac{1}{x+i\epsilon} = P\frac{1}{x} - i\pi\delta(x) \tag{95}$$

and ignore the term P/x in (93). Thus we may approximate

$$F(z) \approx \tfrac{1}{2} i\gamma_0$$

under the integral sign of (89), where

$$\gamma_0=\frac{\pi e^2\omega_0^3}{(2\pi)^3}\int d\Omega\,|\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2\,. \tag{96}$$

Thus we find

$$|C_b(t)|^2=e^{-\gamma_0 t}\,. \tag{97}$$

For the $b\to a$ transition, we use a formula similar to (24). Using the same approximation of (94), we get

$$C_{b\to a}(t)=-e\left(\frac{\omega}{2V}\right)^{1/2}\langle a|\hat{d}|b\rangle\, e^{-i(\omega_0-\omega)t}\times\frac{e^{-(\gamma_0/2)t}-e^{+i(\omega_0-\omega)t}}{\omega_0-\omega-\frac{1}{2}i\gamma_0}\,. \tag{98}$$

Equations (96) and (97) are in agreement with the Weisskopf-Wigner theory.[4,7]

Next consider the system interacting with a blackbody radiation field at temperature T; then we can take

$$N(\omega)=1/(e^{\beta\omega}-1)\,, \tag{99}$$

where

$$\beta^{-1}=kT\,.$$

The $F(z)$ in this case can be written as

$$F(z)=\frac{e^2}{2V}\sum_{j=1}^{\infty}|\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2\,\omega_j\times\left(\frac{1+N(\omega_j)}{z+\omega_0-\omega_j}+\frac{N(\omega_j)}{z+\omega_0+\omega_j}\right)\,.$$

Since the continuous distribution of the modes is applicable in this case, we can apply (95) and ignore the functional dependence of $F(z)$ on z by taking $z=0$; we get

$$F(0)=\delta E_0'-\tfrac{1}{2}i\gamma_0'\,,$$

where

$$\delta E_0'=\frac{e^2}{2V}P\sum_{j=1}^{\infty}|\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2\,\omega_j\times\left(\frac{1+N(\omega_j)}{\omega_0-\omega_j}+\frac{N(\omega_j)}{\omega_0+\omega_j}\right), \tag{100}$$

$$\gamma_0'=\frac{\pi e^2\omega_0^3}{(2\pi)^3}\int d\Omega\,|\langle a|\hat{e}_\lambda\cdot\vec{r}\,|b\rangle|^2[1+N(\omega_0)]\,. \tag{101}$$

Here $\delta E_0'$ can be related to the level shift in the presence of the thermal field before mass renormalization. The effect of (100) on the hydrogen atom will be given elsewhere. γ_0' in (101) is obviously the width of the emission line [cf. Eq. (98)]. The temperature-dependent width is proportional to the factor

$$\omega_0^3(e^{\beta\omega_0}-1)^{-1}\,.$$

For low temperature, it is proportional to $\omega_0^3 e^{-\beta\omega_0}$, while for high temperature, to $\omega_0^2(kT)$. At room temperature, the magnitude of this contribution to the optical transition width is much less than the spontaneous width γ_0.

Finally, let us consider the situation when both the external monochromatic radiation field and the vacuum field act on the system simultaneously. We shall evaluate (89) with $F(z)$ written in the form

$$F(z)=V(z)+B(z)\,, \tag{102}$$

where $V(z)$ is the contribution due to the vacuum field (absorption and emission of virtual photons),

$$V(z)=\sum_{j=1}^{\infty}\frac{\eta_j^2}{z+\omega_0-\omega_j}+O(e^4)\,, \tag{103}$$

and $B(z)$ is the contribution arising from the forward scattering of the photon beam (absorption and emission of real photons at the same four-momentum k_μ, with $k_\mu^2=0=\omega^2-\vec{k}^2$),

$$B(z)=\frac{\xi^2}{z+\omega_0-\omega}+O(e^4)\,. \tag{104}$$

Thus the denominator of the integrand in (89) can be approximated by

$$\begin{aligned}[z-F(z)]^{-1}&=\{z-[V(z)+B(z)]\}^{-1}\\&=z^{-1}\left(1-\frac{V(z)}{z}-\frac{B(z)}{z}\right)^{-1}\\&\approx z^{-1}\left[\left(1-\frac{V(z)}{z}\right)\left(1-\frac{B(z)}{z}\right)\right]^{-1}\,.\end{aligned} \tag{105}$$

The term omitted in the approximation of (105) involved the product of $V(z)B(z)$ which is of order e^4. We write the approximate expression (105) in the following alternative form:

$$[z-F(z)]^{-1}\approx z\{[z-V(z)][z-B(z)]\}^{-1}\,. \tag{106}$$

Thus the integral to be computed in (89) becomes

$$C_b(t)=(1/2\pi i)\,e^{-iE_b t}\times\int_{-\infty}^{\infty}z\{[z-V(z)][z-B(z)]\}^{-1}e^{itz}\,dz\,. \tag{107}$$

In order to evaluate (107), we use the approximation developed earlier in this section for $V(z)$; namely, we set

$$V(z)\approx\tfrac{1}{2}i\gamma_0\,, \tag{108}$$

where γ_0 is the spontaneous decay width given in (96). The integral in (107) can be calculated easily to get

$$C_b(t) = e^{-iE_b t}\left[A_0 e^{-\gamma_0 t/2} + A_+ e^{i\alpha_+ t} + A_- e^{i\alpha_- t}\right] , \quad (109)$$

where

$$A_0 = \frac{\frac{1}{2} i\gamma_0 (\frac{1}{2} i\gamma_0 + a)}{(\frac{1}{2} i\gamma_0 - \alpha_+)(\frac{1}{2} i\gamma_0 - \alpha_-)} ,$$

$$A_\pm = \frac{\alpha_\pm (\alpha_\pm + a)}{(\alpha_\pm - \frac{1}{2} i\gamma_0)(\alpha_\pm - \alpha_\mp)} , \quad (110)$$

$$a = \omega_0 - \omega ,$$

are $\alpha_\pm$ are the roots of the equation

$$z(z+a) - \xi^2 = 0 ,$$

and are given by

$$\alpha_\pm = \tfrac{1}{2}\left[-a \pm (a^2 + 4\xi^2)^{1/2}\right] . \quad (111)$$

Expression (109) indicates that $|C_b(\infty)|^2 \neq 0$ because of the existence of the external radiation field which acts on the atom at all times and which can induce transitions from the ground state a, thus preventing the decay of the excited state to some extent.

We now evaluate the induced transition probability for the system in level a at time t. Using (24) and the above approximation, we get

$$C_a(t) = \left(\frac{e^2\omega N}{2V}\right)^{1/2} \langle b|\hat{d}|a\rangle \frac{1}{2\pi i} e^{-iE_a t} \times \int_{-\infty}^{\infty} dz \frac{z\, e^{itz}}{(z+a)\left[z - \xi^2/(z+a)\right](z - \frac{1}{2} i\gamma_0)} . \quad (112)$$

Computing this integral by theorem of residues, we obtain

$$C_a(t) = (e^2\omega N/2V)^{1/2} \langle b|\hat{d}|a\rangle e^{-iE_a t} \times (B_0 e^{-\gamma_0 t/2} + B_+ e^{i\alpha_+ t} + B_- e^{i\alpha_- t}) , \quad (113)$$

where

$$B_0 = \frac{\frac{1}{2} i\gamma_0}{\frac{1}{2} i\gamma_0 (\frac{1}{2} i\gamma_0 + a) - \xi^2} ,$$

$$B_\pm = \frac{\alpha_\pm}{(\alpha_\pm - \frac{1}{2} i\gamma_0)(\alpha_\pm - \alpha_\mp)} . \quad (114)$$

As $t \to \infty$, the terms in $|C_a(t)|^2$ involving B_0 vanish as can be easily seen from (113). Furthermore, the oscillating terms in $|C_a(t)|^2$ can be easily calculated to be

$$\frac{2\xi^2}{(\frac{1}{4}\gamma_0^2 - \xi^2)^2 + \frac{1}{4}\gamma_0^2 (a^2 + 4\xi^2)} \{(\tfrac{1}{4}\gamma_0^2 - \xi^2) \times \cos[(a^2 + 4\xi^2)^{1/2} t] - \tfrac{1}{2}\gamma_0 (a^2 + 4\xi^2)^{1/2} \times \sin[(a^2 + 4\xi^2)^{1/2}]\} . \quad (115)$$

For very high intensity (that is, $\xi \to \infty$), the above term is rapidly oscillating, and thus can be averaged out. Hence, for large times and high-intensity field, we are left with the following expression for $|C_a(\infty)|^2$:

$$|C_a(\infty)|^2 = (e^2\omega N/2V)\,|\langle b|\hat{d}|a\rangle|^2 f(x) , \quad (116)$$

where

$$f(x) = \frac{x^2 + \alpha^2}{(x^2 + 4\xi^2)(x^2 + \beta^2)} , \quad (117)$$

and

$$x \equiv a = \omega_0 - \omega ,$$

$$\alpha^2 = (8\xi^2/\gamma_0^2)(\xi^2 + \tfrac{1}{4}\gamma_0^2) ,$$

$$\beta^2 = (4/\gamma_0^2)(\xi^2 + \tfrac{1}{4}\gamma_0^2)^2 .$$

Equation (117) determines the probability distribution as a function of frequency detuning for the laser field. The detailed structure of the function $f(x)$ depends on ξ and γ_0, the field amplitude and the spontaneous decay width of the excited state, respectively. The extrema of $f(x)$ are determined from the following algebraic equation:

$$\frac{x}{x^2 + \alpha^2} = \frac{x}{x^2 + \beta^2} + \frac{x}{x^2 + 4\xi^2} . \quad (118)$$

The roots are

$$x = 0 ,$$

$$x^2 = -\frac{8\xi^2}{\gamma_0^2}(\xi^2 + \tfrac{1}{4}\gamma_0^2) \pm \frac{4\sqrt{2}}{\gamma_0^2}\,\xi\,(\xi^2 + \tfrac{1}{4}\gamma_0^2)^{1/2}(\tfrac{1}{4}\gamma_0^2 - \xi^2) .$$

If

$$\gamma_0 > 2(2 + \sqrt{5})^{1/2}\,\xi \approx 4.116\,\xi , \quad (119)$$

then Eq. (118) has three real roots,

$$x = 0 ,$$

$$x = \pm (2/\gamma_0)\left[2\xi^2(\xi^2 + \tfrac{1}{4}\gamma_0^2)\right]^{1/4} \times \left[\tfrac{1}{4}\gamma_0^2 - \xi^2 - \xi(2\xi^2 + \tfrac{1}{2}\gamma_0^2)^{1/2}\right]^{1/2} . \quad (120)$$

The results of (119) and (120) for the behavior of $f(x)$ are plotted in Fig. 8. From the figure we see that there is a dip when the laser frequency is tuned at the atomic frequency. This result can be understood by regarding the combined effect of spontaneous and stimulated emission. The proper description of such a combined effect given here indicates that the spontaneous and stimulated emissions are not taking place independently. The interference between these two processes gives rise to an apparent modification of the induced transition probability. If one takes two modes (e.g., a moving atom in a standing wave field), one would expect the occurrence of two dips. The calculation is straightforward. We hope that the calculation explicitly performed here will lead to further understanding of the detailed dynamics of an atom interacting with radiation field.

VIII. CONCLUSION

Most of the calculations given in this paper refer to some particular process. In spite of this, it appears that some general conclusions can be inferred from the results.

One would expect, from the most elementary form of the correspondence principle, that the method of QED and of the semiclassical treatment should have some region of agreement, in particular for situations where weak low-frequency fields are involved. This is indeed the case. In those systems where transitions involve the emission or absorption of a single photon, the results of the two methods coincide when the QED calculation is made to order e^2. When many-level atoms are present and transitions between nonadjacent levels are induced by many-photon absorption or emission, then there appear to be differences between the QED and the semiclassical methods of calculations. The one explicitly shown here is the dependence of line width on incident power. In the semiclassical theory of Salwen[38] this is a linear dependence, while in the QED description given here the line width reaches a maximum and then decreases with increasing incident power. This is in qualitative agreement with experiment.[11] These differences do not arise from radiative corrections. These have been accounted for by using the observed (renormalized) charge and mass of the electron, and then invoking the low-energy theorem of Low[16] and of Gell-Mann and Goldberger.[17] This theorem asserts that in the low-frequency limit the scattering amplitude and its first derivative are completely renormalized by using the observed charge and mass of the electron. Thus radiative corrections to microwave or rf processes are negligible. The differences are caused simply by the fact that QED and the semiclassical treatments are two different theories in general.

While the formalism of QED in this context is no harder to use than the semiclassical treatment, and indeed may be easier when attention is restricted to order e^2, the calculations become rapidly more tedious as the order increases. They are all finite calculations, however, and are therefore accessible to numerical computation, unlike those coming from radiative corrections.

It is sometimes asserted that the results of the semiclassical treatment are "exact." By this it is meant that the solution of the equations of motion for some simple systems resulting from using a classical electromagnetic field can be solved without mathematical approximations. In this paper perturbation theory has been used consistently, and no "exact" solutions have been obtained. In one case an approximate theory is solved exactly and in the other an accurate theory has been solved approximately. Only the latter process leads in practice to a sequence of more and more accurate calculations of physical effects.

APPENDIX A: TIME-DEPENDENT PERTURBATION METHOD OF HEITLER AND MA

The purpose of this Appendix is to solve the problems similar to Secs. V and III using another approach. Since the resulting expressions from the present Appendix are in many ways similar to the QED results, the comparison can be easily done. It will be seen below, that the method of the present Appendix is equivalent to the semiclassical treatment.

Consider the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=H\psi\ , \tag{A1}$$

where

$$H=H_0+H_I=H_E+H_R+H_I\ ;$$

H_E is the Hamiltonian for the electron, H_R for the radiation field, and H_I the interaction between them.

In the interaction representation, we have

$$i\frac{\partial\psi'}{\partial t}=H_I'\psi'\ , \tag{A2}$$

where

$$H_I'=e^{iH_0t}H_I\,e^{-iH_0t}\ .$$

The exact solution of Eq. (A2) can be expanded in the form

$$\psi'(t)=\sum_n b_n(t)\psi_n\ . \tag{A3}$$

We then obtain

$$i\dot{b}_n(t)=\sum_m H_{n|m}\,e^{i(E_n-E_m)t}\,b_m(t)\ , \tag{A4}$$

where $H_{n|m}$ are the matrix elements in the interaction representation, and are independent of time; E_n are the eigenvalues for H_0. The functions $b_n(t)$ represent the probability amplitudes at time t. Note that the label n includes the electron and the radiation field.

The solution of Eq. (A4) is to be subject to the following initial conditions:

$$b_0(+0)=1,\quad b_n(+0)=0\ .$$

We further impose the following conditions for analytical reasons:

$$b_n(t)=b_0(t)=0\quad\text{for } t<0\ .$$

Then Eq. (A4) reduces to

$$i\dot{b}_n=\sum_m H_{n|m}\,b_m\,e^{i(E_n-E_m)t}+i\,\delta_{n0}\delta(t)\ . \tag{A5}$$

Let

$$b_n(t)=-(1/2\pi i)\int_{-\infty}^{\infty}dE\,G_{n|0}(E)\,e^{i(E_n-E)t}\ , \tag{A6}$$

we have

$$(E - E_n) G_{n|0}(E) = \sum_m H_{n|m} G_{m|0}(E) + \delta_{n0} . \tag{A7}$$

The system of algebraic equations (A7) determines $G_{n|0}(E)$.

Since the solution of (A7) involves a denominator of the type

$$E - E_n - H_{n|n} ,$$

we may take

$$E = E + i0 ,$$

so that we can avoid a zero denominator when $E - E_n - H_{n|n}$ becomes small. The limiting case is defined in the following way:

$$\zeta(x) = \lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} = P \frac{1}{x} - i\pi \delta(x) .$$

In order to apply the above formalism to the system we are going to consider, we use the following notation. Since E_n occurring in Eq. (A4)–(A7) is the energy for the whole system, namely, the sum of energy for the electron and radiation field, we shall use the notation E_n' to denote the eigenvalues of $H_0 = H_E + H_R$, while E_n is the eigenvalue of H_E, where $n = 1, 2$ for the two-level system and $n = 1, 2, 3$ for the three-level system and so on, and can be regarded as the label for the atomic levels under consideration.

For the two-level system, suppose at $t = 0$, the system is in the lower level 1, no photon present; at time t, the system is in the level 2, a photon of frequency ω is emitted. Then

$$E_1' = E_1, \quad E_2' = E_2 - \omega . \tag{A8}$$

The system of algebraic equations (A7) can be written out explicitly,

$$\begin{aligned} (E - E_1') G_{1|1}(E) &= 1 + H_{1|1} G_{1|1}(E) + H_{1|2} G_{2|1}(E) , \\ (E - E_2') G_{2|1}(E) &= H_{2|1} G_{1|1}(E) + H_{2|2} G_{2|1}(E) \end{aligned} \tag{A9}$$

with the following solutions:

$$\begin{aligned} G_{1|1}(E) &= \left(E - E_1 - H_{1|1} - \frac{H_{1|2} H_{2|1}}{E - E_2 - H_{2|2}} \right)^{-1} , \\ G_{2|1}(E) &= - \frac{H_{2|1}}{E - E_2 - H_{2|2}} \\ &\quad \times \left(E - E_1 - H_{1|1} - \frac{H_{1|2} H_{2|1}}{E - E_2 - H_{2|2}} \right)^{-1} . \end{aligned} \tag{A10}$$

As was discussed in detail by Heitler, the diagonal matrix elements of H represent the self-energy correction to the atomic level; we may ignore them,

$$H_{1|1} = H_{2|2} = 0 ,$$

and put

$$H_{1|2} = H_{2|1} = \xi \tag{A11}$$

for the allowed transitions. It turns out that ξ in Eq. (A11) is the same as the ξ defined in Eq. (18).

From Eq. (A5) we find the following expressions for the probability amplitudes:

$$\begin{aligned} C_{1\to1}(t) = b_1(t) &= - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(z - \frac{\xi^2}{z + \omega - \omega_0} \right)^{-1} e^{-izt} \, dz , \\ C_{1\to2}(t) = b_2(t) &= - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-izt} \, dz}{z(z + \omega - \omega_0) - \xi^2} , \end{aligned} \tag{A12}$$

where the contour of the integrations is to be carried out in the lower half-plane. It is easily seen that Eqs. (A12) are in agreement with Sec. III.

Next consider the three-level system in the normal Zeeman effect. Suppose that the system at $t = 0$ is at level 2 and no photon is present. We may put

$$E_1' = E_1 + \omega , \quad E_2' = E_2 , \quad E_3' = E_3 - \omega .$$

Ignoring the self-energy corrections as before,

$$H_{1|1} = H_{2|2} = H_{3|3} = 0 ,$$

and using the dipole selection rules $\Delta m = \pm 1$,

$$\begin{aligned} H_{1|3} &= H_{3|1} = 0 , \\ H_{1|2} &= H_{1|2} = H_{2|3} = H_{3|2} = \sqrt{2} \, \xi , \end{aligned}$$

as in Sec. V, then the solutions of Eq. (A7) for $G_{2|2}(E)$, $G_{3|2}(E)$, and $G_{1|2}(E)$ are the following:

$$\begin{aligned} G_{2|2}(y) &= (1/\Delta) [y^2 - (\omega_0 - \omega)^2] , \\ G_{3|2}(y) &= (1/\Delta) (y + \omega_0 - \omega) , \\ G_{1|2}(y) &= (1/\Delta) (y - \omega_0 + \omega) , \end{aligned} \tag{A13}$$

where

$$\begin{aligned} \Delta &= y [y^2 - (\omega_0 - \omega)^2 - 4\xi^2] , \\ y &= E - E_2 , \\ \omega_0 &= E_3 - E_2 = E_2 - E_1 . \end{aligned}$$

We end up with the following probability amplitudes:

$$\begin{aligned} C_{2\to2}(t) &= - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[y^2 - (\omega_0 - \omega)^2] e^{-iyt}}{y [y^2 - (\omega_0 - \omega)^2 - 4\xi^2]} dy , \\ C_{2\to3}(t) &= - \frac{\sqrt{2} \, \xi}{2\pi i} \int_{-\infty}^{\infty} \frac{(y + \omega_0 - \omega) e^{-iyt}}{y [y^2 - (\omega_0 - \omega)^2 - 4\xi^2]} dy \, e^{i\omega_0 t} , \\ C_{2\to1}(t) &= - \frac{\sqrt{2} \, \xi}{2\pi i} \int_{-\infty}^{\infty} \frac{(y - \omega_0 + \omega) e^{-iyt}}{y [y^2 - (\omega_0 - \omega)^2 - 4\xi^2]} dy \, e^{-i\omega_0 t} , \end{aligned} \tag{A14}$$

where the contour integration is carried out in the lower half-plane. It is not difficult to see that Eqs. (A14) are in agreement with Eqs. (50) and (51).

Similarly, if we start at level 3 at $t=0$, then we may take

$$E_3' = E_3 , \quad E_2' = E_2 + \omega , \quad E_1' = E_3 + 2\omega ,$$

and using the same arguments as before, we have

$$G_{3|3}(z) = \frac{z(z+a) - 2\xi^2}{z[z^2 - (a^2+4\xi^2)]} ,$$

$$G_{2|3}(z) = \frac{\sqrt{2}\,\xi\,(z+a)}{z[z^2 - (a^2+4\xi^2)]} , \qquad \text{(A15)}$$

$$G_{1|3}(z) = \frac{2\xi^2}{z[z^2 - (a^2+4\xi^2)]} ,$$

where ξ is the same as before and

$$z = E - E_2 , \quad a = \omega_0 - \omega .$$

From Eq. (A15), we deduce the following probability amplitudes:

$$C_{3\to 3}(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[z(z+a) - 2\xi^2]\,e^{-itz}}{z[z^2 - (a^2+4\xi^2)]}\,dz\,e^{-i\omega_0 t} ,$$

$$C_{3\to 2}(t) = -\frac{\sqrt{2}\,\xi}{2\pi i}\int_{-\infty}^{\infty} \frac{(z+a)\,e^{-itz}}{z[z^2 - (a^2+4\xi^2)]}\,dz , \qquad \text{(A16)}$$

$$C_{3\to 1}(t) = -\frac{2\xi^2}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-itz}}{z[z^2 - (a^2+4\xi^2)]}\,dz\,e^{i\omega t} .$$

By closing the contour of integration in the lower half-plane, we find

$$C_{3\to 3}(t) = \left(\frac{2\xi^2}{a^2+4\xi^2} + \frac{a^2+2\xi^2}{a^2+4\xi^2}\cos[t(a^2+4\xi^2)^{1/2}] + \frac{ia}{(a^2+4\xi^2)^{1/2}}\sin[t(a^2+4\xi^2)^{1/2}]\right)e^{-i\omega_0 t} ,$$

$$C_{3\to 2}(t) = \sqrt{2}\,\xi\left(-\frac{a}{a^2+4\xi^2} + \frac{a}{a^2+4\xi^2}\cos[t(a^2+4\xi^2)^{1/2}] + \frac{i}{(a^2+4\xi^2)^{1/2}}\sin[t(a^2+4\xi^2)^{1/2}]\right) , \qquad \text{(A17)}$$

$$C_{3\to 1}(t) = -\frac{4\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\,e^{i\omega t} ,$$

and the probabilities for each transition can be found easily:

$$P_{3\to 3}(t) = |C_{3\to 3}(t)|^2 = \left(1 - \frac{4\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\right)^2 ,$$

$$P_{3\to 2}(t) = |C_{3\to 2}(t)|^2 = \frac{8\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\left(1 - \frac{4\xi^2}{a^2+4\xi^2}\sin^2[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}]\right) , \qquad \text{(A18)}$$

$$P_{3\to 1}(t) = |C_{3\to 1}(t)|^2 = \frac{16\xi^4}{(a^2+4\xi^2)^2}\sin^4[\tfrac{1}{2}t(a^2+4\xi^2)^{1/2}] .$$

The results of Eqs. (A18) reproduce the Majorana formula for spin-1 system.[36]

Similarly, one can derive the expressions for various transitions starting with the level 1; we omit it. It may be stated that the present approach can be applied to a system with a general spin, for the normal and anomalous Zeeman effects.

APPENDIX B: PROOFS OF SOME MATHEMATICAL ASSERTIONS

(i) In the following paragraphs, we will investigate the asymptotic expansion of Eq. (26). Define

$$F(t,\zeta) = \int_{-\infty}^{\infty} \frac{\cos[t(a^2+x^2)^{1/2}]\,dx}{(a^2+x^2)[\delta^2 + (x-\zeta)^2]} , \qquad \text{(B1)}$$

where a, δ, ζ are real. Let

$$u = (a^2+x^2)^{1/2} ;$$

then

$$F(t,\zeta) = 2\int_{-\infty}^{\infty} \frac{(u^2 - a^2 + \delta^2 + \zeta^2)\cos ut\,du}{u(u^2-a^2)^{1/2}[(u^2+\delta^2-\zeta^2+a^2)^2 + 4\xi^2\zeta^2]} . \qquad \text{(B2)}$$

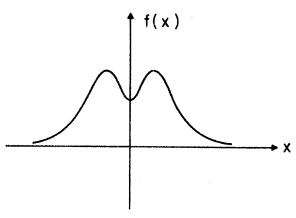

FIG. 8. Induced transition probability for a two-level atom in a laser field of single mode.

The integral

$$\int_{-\infty}^{\infty} \frac{(u^2-a^2+\delta^2+\zeta^2)\,du}{u(u^2-a^2)^{1/2}[(u^2+\delta^2-\zeta^2+a^2)^2+4\delta^2\zeta^2]}$$

$$= \int_{-\infty}^{\infty} \frac{du}{(a^2+u^2)\,[\delta^2+(\zeta-u)^2]}$$

$$= \frac{\pi(a+\delta)}{a\delta\,[\zeta^2+(\delta+a)^2]}$$

is absolutely convergent; by the Riemann-Lebesgue lemma,[41] we have

$$\lim_{t\to\infty} F(t,\zeta) = 0 \,. \tag{B3}$$

This proves the existence of the asymptotic expansion for Eq. (B1).

In order to estimate the leading term of $F(t,\zeta)$ when t is large, we make the following changes of integration variables in Eq. (B2):

$$u = a\sec\theta \,, \quad z = \tan\tfrac{1}{2}\theta \,.$$

Then

$$F(t,\zeta) = 2\int_0^1 K(z)\,e^{tf(z)}\,dz + 2\int_0^1 K(z)\,e^{-tf(z)}\,dz\,; \tag{B4}$$

$$K(z) = \frac{az^2(1-z^2)^2+a_4(1-z^2)^4}{(1+z^2)\,[a_1(1-z^2)^4+a_2z^2(1-z^2)^2+a_3z^4]} \,,$$

$$f(z) = ia\,(1+z^2)\,(1-z^2)^{-1} \,,$$

$$a_1 = (\delta^2+\zeta^2)^2, \quad a_2 = 8a^2(\delta^2-\zeta^2),$$

$$a_3 = 16a^4, \quad a_4 = a^{-1}(\delta^2+\zeta^2) \,.$$

Let

$$f(z) = \varphi(x,y) + i\,\psi(x,y) \,, \quad z = x+iy \,.$$

Since the integral of Eq. (B4) is along the real axis in which $\varphi = 0$, $y = 0$, we may write

$$F(t,\zeta) = 2\int_0^1 K(x)\,e^{it\psi(x)}\,dx + 2\int_0^1 K(x)\,e^{-it\psi(x)}\,dx \,.$$

Let x_0 be the saddle point, i.e.,

$$\psi'(x_0) = 0 \,;$$

we find

$$x_0 = 0 \,.$$

The method of the stationary phase[27] gives, for large t,

$$F(t,\zeta) = \left(\frac{2\pi}{t\,|\psi''(x_0)|}\right)^{1/2} K(x_0)$$

$$\times \exp[it\psi(x_0) + \tfrac{1}{4}\pi i\,\psi''(x_0)] + O(1/t) + \text{c.c.}$$

We find

$$F(t\zeta) = \frac{(2\pi)^{1/2}}{(a^3t)^{1/2}(\delta^2+\zeta^2)}\cos(at+\pi a) + O(1/t) \,. \tag{B5}$$

(ii) In the following paragraphs, we prove the algebraic assertions made in Sec. IV. To prove that

$$f(z) = z - \sum_{j=1}^{n} \frac{b_j^2}{z+a_j} = 0 \tag{B6}$$

can have only real roots for real b_j, a_j, let

$$z = \alpha + i\beta$$

be one of the roots where α, β are real; then $z = \alpha - i\beta$ is also one of the root; thus

$$\alpha + i\beta - \sum_{j=1}^{n} \frac{b_j^2}{\alpha+a_j+i\beta} = 0 \,,$$

$$\alpha - i\beta - \sum_{j=1}^{n} \frac{b_j^2}{\alpha+a_j-i\beta} = 0 \,.$$

Subtracting the last two equations, we have

$$2i\beta\left(1+\sum_{j=1}^{n}\frac{b_j^2}{(\alpha+a_j)^2+\beta^2}\right) = 0$$

but

$$1+\sum_{j=1}^{n}\frac{b_j^2}{(\alpha=a_j)^2+\beta^2} > 0 \,,$$

so we get

$$\beta = 0. \tag{B7}$$

To show that

$$\sum_{j=0}^{n} [f'(\alpha_j)]^{-1} = 1 \,, \tag{B8}$$

where $f(z)$ is defined in Eq. (B6), and

$$f'(z) = \frac{df(z)}{dz} \,.$$

In order to prove Eq. (B8), we assume $n = 2$ in Eq. (B6). Let the roots of Eq. (B6) be given by α_1, α_2, α_3; all are real according to Eq. (B7); then

$$f'(\alpha_1) = \lim_{z\to\alpha_1}\frac{f(z)}{z-\alpha_1} = \frac{(\alpha_1-\alpha_2)\,(\alpha_1-\alpha_3)}{(\alpha_1+a_1)\,(\alpha_1+a_2)} \,.$$

Similarly one can compute $f'(\alpha_2)$, $f'(\alpha_3)$.

Utilizing the explicit expressions for $f'(\alpha_j)$, a straightforward algebra gives

$$\sum_{j=1}^{3} [f'(\alpha_j)]^{-1} = 1 \,. \tag{B9}$$

One can generalize Eq. (B9) to an arbitrary natural number n.

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†Present address: Department of Physics, MIT, Cambridge, Mass. 02139.

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