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Resonant Interaction between Two Neutral Atoms*

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A standard formalism of quantum electrodynamics is used to investigate the interaction between two neutral atoms. The study is based on an approximate solution of the two-particle Green's function. The frequency distribution of the photon scattering is obtained. We also consider the self-stimulation effect when both atoms are in excited states.

I. INTRODUCTION

The two-atom problem has drawn considerable interest since the advent of lasers and masers. In fact, by using the Weisskopf-Wigner method, this problem has been solved exactly by Ernst and Stehle,[1] who deal with the atoms interacting with a common radiation field. The purpose of their work is partly directed toward constructing a rigorous description of the behavior of an ensemble of atoms and using some characteristics of the two-atom solution as a check for the approximate solution of the N-atom problem. In the present work, we shall use a different approach, based on the approximate solution of the Bethe-Salpeter equation,[2] to discuss the physical properties of the two-atom interaction. In particular, we shall pay some attention to the problem of the frequency shift of the emitted photons due to the resonant interaction; such a shift has been ignored in the Weisskopf-Wigner method of Ernst and Stehle. Several other features contained in our solution will be discussed in detail.

In Sec. II, we solve the Bethe-Salpeter equation in the ladder approximation. The approximate solution for the two-particle Green's function is then applied in Sec. III to discuss the interaction between two neutral atoms when no external field exists, and the results are compared with those obtained by others.[3-5] In Sec. IV, we consider low-intensity photon scattering, taking the two-atom interaction into account. We obtain an expression for the frequency distribution of the emitted photons. Our result is different from that derived by Fontana and Hearn[6] when the separation of the two atoms is not very small compared with the wavelength of the emitted light. Finally, in Sec. V, we study the decay of two excited atoms. Complete agreement with the work of Ernst and Stehle[1] on the two-atom problem is obtained. The extension of the present approach to the N-atom problem for a laser model is deferred to a future publication.

Appendixes A and B contain the necessary algebra to complete the derivations omitted in the text. Finally, a brief derivation of the natural line shape for a single isolated atom using the standard quantum-electrodynamical (QED) formalism is given in Appendix C; the result agrees with that of the Weisskopf-Wigner method.[7]

II. SOLUTION FOR THE TWO-PARTICLE GREEN'S FUNCTION IN THE LADDER APPROXIMATION

The two-atom problem has been considered extensively by Ernst and Stehle[1] using the method of Weisskopf and Wigner, and by Stephen[3] and many others using time-dependent perturbation methods. We now present a different method based on the Bethe-Salpeter equation and compare the results with the previous works. We solve the Bethe-Salpeter equation for the two-particle Green's function in the ladder approximation in this section, and apply the resulting two-particle Green's function to discuss the nature of interactions of the two neutral atoms in Sec. III. A further application of this Green's function to the photon scattering problem is given in Secs. IV and V.

Consider the integral form of the Bethe-Salpeter equation

$$G_{AB}(x_1, x_2; x_3, x_4) = G(x_1, x_3)G(x_2, x_4) - G(x_1, x_4) \times G(x_2, x_3) + \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 G(x_1, x_5)G(x_2, x_6) \times I_{AB}(x_5, x_6, x_7, x_8) G_{AB}(x_7, x_8, x_3, x_4) , \quad (1)$$

where $G(x_1, x_2)$ is the one-particle Green's function. We have

$$G(x_1, x_2) = S_F(x_1, x_2) - i\int S_F(x_1, x_3)M(x_3, x_4) \times G(x_4, x_2)\, d^4x_3 d^4x_4 , \quad (2)$$

in which $S_F(x_1, x_2)$ is the Green's function of the single electron in the prescribed static field and $M(x_3, x_4)$ is the mass operator. The mass operator includes the self-energy of the electron arising from its interaction with the electromagnetic field in the vacuum state, and when intense radiation fields are present, it includes contributions from the forward scattering of these fields. Here we are concerned only with low-intensity radiation fields. Writing

$$H_{nn}(\omega) = \int_{-\infty}^{\infty} \tilde{H}_{nn}(t)\, e^{i\omega t}\, dt ,$$
$$\tilde{H}_{nn}(t) = \int_{-\infty}^{\infty} \bar{\psi}_n(\vec{r}_1) M(\vec{r}_1, \vec{r}_2, t) \psi_n(\vec{r}_2)\, d^3r_1 d^3r_2 \quad (3)$$

we may use the known results,[8] that, to order e^2, $\mathrm{Re}H_{nn}(-E_n)$ is the Lamb shift of level n and $-\mathrm{Im}H_{nn}(-E_n) = \frac{1}{2}\gamma_0$ is the natural width of the level. For the ground state, $H_{nn}(-E_n)$ is, of course, real. Thus to order e^2 we have

$$G(x_1, x_2) = \frac{1}{2\pi i}\sum_n \bar{\psi}_n(\vec{r}_1)\psi_n(\vec{r}_2) \times \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t_1 - t_2)}}{\omega + E_n + H_{nn}(-\omega)} . \quad (4)$$

The I_{AB} in Eq. (1) is called the interaction operator, and in the first approximation, with respect to e^2, takes the form[9]

$$I_{AB}(x_1, x_2, x_3, x_4) = -ie^2 \gamma_\mu^A \gamma_\mu^B D_F(x_1, x_2)\,[\delta(x_1 - x_3)\delta(x_2 - x_4) - \delta(x_1 - x_4)\delta(x_2 - x_3)] , \quad (5)$$

where $D_F(x_1, x_2)$ is the photon propagator,

$$D_F(x_1, x_2) = \frac{-i}{(2\pi)^4}\int d^3k\, e^{-i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \times \int_{-\infty}^{\infty} dk_0 \frac{e^{ik_0(t_1 - t_2)}}{k_0^2 - \vec{k}^2 - i0} . \quad (6)$$

Throughout the paper, we use units such that

$$\hbar = c = 1 ,$$

and the metric is such that the scalar product of two four-vectors a_μ and b_μ is given by

$$a\cdot b = a_\mu b_\mu = a_0 b_0 - \vec{a}\cdot\vec{b} .$$

The indices A and B are used to distinguish the particles under consideration.

If we neglect the overlap of the wave functions between the particles, then the solution of the equation

$$G_{AB}(x_1, x_2; x_3, x_4) = G(x_1, x_3)G(x_2, x_4) - ie^2\int d^4x_5 d^4x_6\, G(x_1, x_5)G(x_2, x_6) \times \gamma_\mu^A \gamma_\mu^B D_F(x_5, x_6) G_{AB}(x_5, x_6; x_3, x_4) \quad (7)$$

corresponds to the solution including the Feynman diagrams of the ladder type shown in Fig. 1, in which the interaction due to the crossed photon graph of Fig. 2, corresponding to a term

$$-ie^4 \gamma_\mu^A \gamma_\mu^B G(x_1, x_3)G(x_4, x_2)\gamma_\nu^A \gamma_\nu^B \times D_F(x_1, x_4) D_F(x_2, x_3)$$

in I_{AB}, is not taken into account.

We note that the exact solution of Eq. (7) for G_{AB} depends on three different time arguments. We now make the assumption that the initial- and final-time coordinates of the two particles are the same, namely,

$$t = t_1 = t_2; \quad t' = t_3 = t_4 . \quad (8)$$

Since we are interested in the nonrelativistic limit in the subsequent discussions, the above assumption is consistent with applications.

Equation (7) can be solved by the method of successive approximations if the second term in (7) satisfies the condition of the principle of contraction mapping.[10] In the first iteration, we have

$$G_{AB}(x_1, x_2; x_3, x_4) = G(x_1, x_3)G(x_2, x_4) - ie^2\int d^4x_5 d^4x_6\, G(x_1, x_5)G(x_2, x_6) \times \gamma_\mu^A \gamma_\mu^B D_F(x_5, x_6) G(x_5, x_3) G(x_6, x_4) . \quad (9)$$

On using Eqs. (4) and (8) to carry out the integration of Eq. (9), and neglecting the terms of the type

$$\sum_{n\neq n', m\neq m'} \psi_n(\vec{r}_1)\psi_m(\vec{r}_2)\bar{\psi}_{n'}(\vec{r}_3)\bar{\psi}_{m'}(\vec{r}_4) \int d\omega \cdots ,$$

which arise from part of the last term of Eq. (9), we obtain

$$G_{AB}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4; t - t') = \frac{1}{2\pi i}\sum_{n,m} \psi_n(\vec{r}_1)\psi_m(\vec{r}_2)\bar{\psi}_n(\vec{r}_3)$$

$$\times\,\bar{\psi}_m(\vec{r}_4)\int_{-\infty}^{\infty}\frac{e^{i\omega(t-t')}\,d\omega}{\omega+E_n'+E_m'+I_{n,m}(\omega_{nm})}\,, \quad (10)$$

where

$$E_n'=E_n+H_{nn}(-E_n)\,,\quad \omega_{nm}=|E_n-E_m|\,,$$

$$I_{n,m}(\omega)=\int_{-\infty}^{\infty}dt\,e^{i\omega t}\,\tilde{I}_{n,m}(t)\,, \quad (11)$$

$$\tilde{I}_{n,m}(t)=\int d^3r_1d^3r_2\,\bar{\psi}_n(\vec{r}_1)\bar{\psi}_m(\vec{r}_2)\,ie^2\gamma_\mu^A\gamma_\mu^B D_F(\vec{r}_1,\vec{r}_2;t)\times\psi_n(\vec{r}_2)\psi_m(\vec{r}_1)\,.$$

Here E_n is real, but $H_{nn}(-E_n)$ is complex.

In obtaining Eq. (10), we have used the following approximation:

$$1-x\approx 1/(1+x)\quad \text{if } |x|\ll 1\,,$$

which is consistent with the condition for the applicability of contraction mapping.

We now evaluate $I_{n,m}(\omega_0)$ defined in Eq. (11). Going to the nonrelativistic limit and the dipole approximation,[11] it is easy to see that

$$I_{n,n}(0)=0\,, \quad (12)$$

and by straightforward calculations we find[12]

$$I_{n,m}(\omega_0)=\omega_0^3e^{-i\omega_0R}\left[\frac{1}{\omega_0R}\left(\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}-\vec{q}_A\cdot\vec{q}_B\right)+\left(\frac{i}{\omega_0^2R^2}+\frac{1}{\omega_0^3R^3}\right)\left(\vec{q}_A\cdot\vec{q}_B-3\,\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}\right)\right], \quad (13)$$

where $\vec{R}=\vec{R}_A-\vec{R}_B$, $\vec{R}_A$ is the radius vector for the position of atom A. $\vec{q}_A$ is dipole moment for the atom A, $R=|\vec{R}|$ is the internuclear separation of the two atoms, and $\omega_0=|E_n-E_m|$.

III. INTERACTION BETWEEN TWO NEUTRAL ATOMS

In this section we shall apply expression (10) to discuss the interactions between two neutral atoms. This study is based on the following property of the Green's function. Suppose the system is initially described by the wave function $\psi_{AB}(\vec{r}_1,\vec{r}_2;0)$; then at a later time t, the wave function $\psi_{AB}(\vec{r}_1,\vec{r}_2;t)$ will be

$$\psi_{AB}(\vec{r}_1,\vec{r}_2;t)=\int G_{AB}(\vec{r}_1,\vec{r}_2,\vec{r}_1{}',\vec{r}_2{}';t)\times\psi_{AB}(\vec{r}_1{}',\vec{r}_2{}';0)\,d^3r_1'd^3r_2'\,. \quad (14)$$

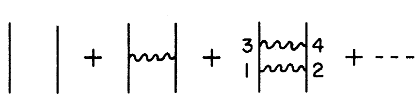

FIG. 1. Ladder diagram in the two-body interaction.

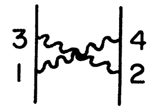

FIG. 2. Crossed photon graph.

By knowing $\psi_{AB}(\vec{r}_1,\vec{r}_2;0)$, one can compute $\psi_{AB}(\vec{r}_1,\vec{r}_2;t)$ with the aid of Eq. (10). Once $\psi_{AB}(\vec{r}_1,\vec{r}_2;t)$ is known, then by taking the inner product with $\psi_{AB}(\vec{r}_1,\vec{r}_2;0)$, one can calculate the probability amplitude of the system as a function of time. In order to do this, it is convenient to assume that each atom has only two levels between which a spectroscopic transition is allowed. The subscripts 0 and 1 will be used to denote the two levels of concern, where 0 is the ground state and 1 is the excited state. We also assume that the two atoms have the same structure.

As was pointed out above, $\mathrm{Re}H_{nn}(-E_n)$[9] is the Lamb shift, and $-\mathrm{Im}H_{11}(-E_1)=\frac{1}{2}\gamma_0>0$ is the natural width of the excited state; if we neglect the Lamb shift, we may write

$$E_0'=E_0\,,\quad E_1'=E_1-\tfrac{1}{2}i\gamma_0\,, \quad (15)$$

since the energy of the ground state, by definition, is sharp. Further, in carrying out the computation for the probability amplitude, the nonrelativistic limit and dipole approximation are used.

For the present two-level atoms, there are three different initial states of interest, namely, both atoms in the ground state, both atoms in the excited state, and one atom in the ground while the other in the excited state. In the first two cases, the system is nondegenerate, and according to Eq. (12), the interaction will be at least of order e^4. If we apply Eq. (10) to the case when both atoms are initially in the excited state,

$$\psi_{AB}(\vec{r}_1,\vec{r}_2;0)=\psi_{1A}(\vec{r}_1)\psi_{1B}(\vec{r}_2)\,, \quad (16)$$

then using Eqs. (10) and (14)–(16), carrying out the integration over ω by the theorem of residues, and taking the inner product of the resulting expression with (16), one finds the following expression for the probability amplitude:

$$a_{1,1}(t)=e^{-2iE_1t}\,e^{-\gamma_0t}\,. \quad (17)$$

In arriving at Eq. (17), we have made use of (12).

From (17), it follows that

$$|a_{1,1}(t)|^2=e^{-2\gamma_0t}\,. \quad (18)$$

This result coincides with that of Ernst and Stehle.[1] Equation (18) indicates that the system decays twice as fast as that for the single isolated atom. Since in our development the terms of order e^4 or higher have been neglected, it is clear that to the order e^2 the two atoms under consideration are independent of each other. Therefore, the probability for the system not to decay is the product of the probabilities of each atom not decaying.

On the other hand, if initially one atom is in the ground state while the other in the excited state, then the system is degenerate. Let

$$\psi_{AB}(\vec{r}_1, \vec{r}_2; 0) = \psi_{1A}(\vec{r}_1)\psi_{0B}(\vec{r}_2) .$$

On using Eqs. (14) and (10) to compute $\psi_{AB}(\vec{r}_1, \vec{r}_2; t)$, it is easily seen that the following Schrödinger equation holds:

$$i\frac{\partial}{\partial t}\psi_{AB}(\vec{r}_1, \vec{r}_2; t) = (H_{0A} + H_{0B} + I_{AB})\psi_{AB}(\vec{r}_1, \vec{r}_2; t) ,$$

where

$$H_{0A}\psi(\vec{r}_1, \vec{r}_2; 0) = E_A\psi_{AB}(\vec{r}_1, \vec{r}_2; 0) ,$$

and similarly for atom B, and I_{AB} is defined by

$$I_{0,1}(\omega_0) = \langle 1_A 0_B | I_{AB} | 1_B 0_A \rangle \equiv I_{0,1} ,$$

$$\omega_0 = E_1 - E_0 ,$$

as in Eq. (13), in the nonrelativistic limit and the dipole approximation.

Applying first-order degenerate perturbation theory, we find that the two states, symmetric and antisymmetric, in the atoms,

$$\psi_{AB}^{(\pm)} = 2^{-1/2}(\psi_{1A}\psi_{0B} \pm \psi_{1B}\psi_{0A}) ,$$

at $t > 0$, have the following energies[13]:

$$E_0' + E_1' \pm I_{0,1} = E_0 + E_1' \pm I_{0,1} ,$$

with the notation of Eq. (15). It is evident that the probability amplitudes for $\psi_{AB}^{(\pm)}$ are given by

$$a_{0,1}^{(\pm)}(t) = e^{-i(E_0 + E_1' \pm I_{0,1})t} .$$

Let

$$I_{0,1} = X + iY ;$$

we have

$$a_{0,1}^{(\pm)}(t) = e^{-i(E_0 + E_1 \pm X)t}\, e^{-(\gamma_0/2 \mp Y)t} . \tag{19}$$

According to Eq. (13), it is seen that X and Y are functions of $\vec{R}$, the internuclear separation between the two atoms.

We now investigate the following two limiting cases. First, if $\omega_0 R \ll 1$, then from Eq. (13) we have

$$X = V_{AB} = \frac{1}{R^3}\left(\vec{q}_A \cdot \vec{q}_B - 3\,\frac{(\vec{q}_A \cdot \vec{R})(\vec{q}_B \cdot \vec{R})}{R^2}\right) ,$$

$$Y = -\tfrac{2}{3}\,\omega_0^3(\vec{q}_A \cdot \vec{q}_B)(1 - \tfrac{1}{10}\,\omega_0^2 R^2)$$

$$= -\tfrac{1}{2}\gamma_0(1 - \tfrac{1}{10}\,\omega_0^2 R^2) ,$$

where

$$\gamma_0 = \tfrac{4}{3}\,\omega_0^3 q^2 , \quad \vec{q}_A \parallel \vec{q}_B \parallel \vec{R}, \quad |\vec{q}_A| = |\vec{q}_B| = q$$

have been used. Accordingly, we find

$$a^{(+)}(t) = e^{-i(E_0 + E_1 + V_{AB})t}\, e^{-\gamma_0 t} ,$$
$$a^{(-)}(t) = e^{-i(E_0 + E_1 - V_{AB})t}\, e^{-(\gamma_0/20)\omega_0^2 R^2 t} . \tag{20}$$

We see that the symmetric state $\psi_{AB}^{(+)}$ decays with a rate twice that of a single isolated atom, and the energy of the system for $t \ll \gamma_0^{-1}$ is $E_0 + E_1 + V_{AB}$. The antisymmetric state $\psi_{AB}^{(-)}$ is approximately stationary, and for $t \ll (\gamma_0\omega_0^2 R^2)^{-1}$, the energy of the system is given by $E_0 + E_1 - V_{AB}$. It is clear that only in the latter case is it appropriate to talk about the interaction energy between them.[3] This interaction is called the resonant interaction or the transfer of excitation energy.

For $\omega_0 R \gg 1$ it is easily seen that

$$\gamma_0 \gg |I_{1,0}| ,$$

and we have, according to Eq. (19),

$$a^{(\pm)}(t) = e^{-i(E_0 + E_1)t}\, e^{-(\gamma_0/2)t} ,$$

i.e., both states are damped with the width γ_0, which is the same as for a single isolated atom.

For a general separation R, it can be shown that

$$\tfrac{1}{2}\gamma_0 \pm Y > 0 .$$

Since γ_0 and Y are both of the order e^2, and γ_0 involves a single atom while Y requires the presence of the other atom, one would therefore expect the above inequality to hold. The decay law for a general separation R is given by

$$|a_{0,1}^{(\pm)}(t)|^2 = e^{-(\gamma_0 \mp 2Y)t} ,$$

where Y is a function of $\vec{R}$.

The width for the symmetric and antisymmetric state is given by

$$d^{(\pm)} = \gamma_0 \mp 2Y(\vec{R}) , \tag{21}$$

respectively. One can plot $d^{(\pm)}$ as functions of $\vec{R}$. This is in agreement with Hutchinson and Hameka[5] and with Power.[4]

IV. LOW-INTENSITY PHOTON SCATTERING FROM TWO INTERACTING ATOMS

As a further application of the two-particle Green's function given in Eq. (10), we shall derive an expression for the frequency distribution of scattered photons taking the effect of the interaction between the two atoms on the scattering into account.

We assume that both atoms are initially in the ground state. We shall consider their interaction to order e^2. In the final result, we shall specialize to two-level atoms. The frequency of the photon is taken to be near the Bohr frequency of the single atom, i.e., $\omega \approx E_1 - E_0$.

We follow the general QED formalism. In the presence of the external radiation field, the two-particle Green's function G_{AB}' satisfies the following integral equation:

$$G'_{AB}(12;34) = G_{AB}(12;34) - e^2 \int d5\,d6\, G_{AB}(12;56) \times [\hat{A}(5) + \hat{A}(6)]\, G'_{AB}(56;34),$$

$$\hat{A}(5) = \gamma_\mu A_\mu(5) \;, \tag{22}$$

where G_{AB} is the two-particle Green's function satisfying Eq. (1). A_μ is the potential of the external radiation field.

The second-order scattering matrix element $S^{(2)}_{i\to f}$ is given by

$$S^{(2)}_{i\to f} = e^2 \int d1\,d2\,d3\,d4 : \{\bar{\psi}_f(1,2)[\hat{A}_f(1) + \hat{A}_f(2)] \times G_{AB}(12;34)[\hat{A}_i(3) + \hat{A}_i(4)]\,\psi_i(3,4)\} : , \tag{23}$$

where: $\cdots$: denotes the normal-ordered product. In order to use the approximate solution of the two-particle Green's function given in Eq. (10), we assume

$$G_{AB}(12;34) = G_{AB}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4; t_1, t_3)\,\delta(t_1 - t_2)\,\delta(t_3 - t_4) \;. \tag{24}$$

We further write

$$A_i(x) = e_i (2\omega_i)^{-1/2} e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)},$$

$$A_f(x) = e_f (2\omega_f)^{-1/2} e^{-i(\vec{k}_f\cdot\vec{r} - \omega_f t)},$$

$$\psi_i(\vec{r}_1, \vec{r}_2, t) = \psi_i(\vec{r}_1, \vec{r}_2)\, e^{-iE_i t} \;,$$

$$\psi_f(\vec{r}_1, \vec{r}_2; t) = \psi_f(\vec{r}_1, \vec{r}_2)\, e^{-iE_f t} \;,$$

$$E_i = E_i^A + E_i^B \;, \quad E_f = E_f^A + E_f^B \;.$$

Integrating Eq. (23) with respect to t_1, t_2, t_3, and t_4 we have

$$S^{(2)}_{i\to f} = -2\pi i\, U_{i\to f}\, \delta(E_f + \omega_f - E_i - \omega_i) \;,$$

where

$$U_{i\to f} = \frac{2\pi\alpha}{(\omega_i\,\omega_f)^{1/2}} \sum_{n,m} \left(\frac{\langle f|\,\hat{e}_f (e^{-i\vec{k}_f\cdot\vec{r}_1} + e^{-i\vec{k}_f\cdot\vec{r}_2})\,|nm\rangle\langle nm|\,\hat{e}_i(e^{i\vec{k}_i\cdot\vec{r}_1} + e^{i\vec{k}_i\cdot\vec{r}_2})\,|i\rangle}{E'_n + E'_m + I_{n,m} - E_i - \omega_i} + i \leftrightarrow f \right) \;. \tag{25}$$

In order to carry out the summation over the intermediate states, we use the interpretation of Sec. III. The condition for resonant scattering gives two terms in the sum of Eq. (25). These two terms are due to the two intermediate states

$$|\pm\rangle = \frac{1}{2^{-1/2}} (|1_A\rangle|0_B\rangle \pm |1_B\rangle|0_A\rangle) \;,$$

associated with the energy denominators

$$E'_1 + E'_0 \pm I_{1,0} \;.$$

Going to the nonrelativistic limit and the dipole approximation, and using the notation of Sec. III, we have

$$U_{i\to f} = 4\pi^2 (\omega_i\,\omega_f)^{1/2} (\vec{q}\cdot\vec{e}_i)(\vec{q}\cdot\vec{e}_f)(1 + e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{R}}) \times \frac{\omega_0 - \omega_i - \frac{1}{2} i\gamma_0 - (X+iY)\eta}{(\omega_0 - \omega_i - \frac{1}{2} i\gamma_0)^2 - (X+iY)^2} \;, \tag{26}$$

where $\vec{q}$ is the transition dipole moment for either atom, and

$$\eta = \frac{\cos\vec{k}_i\cdot\vec{R} + \cos\vec{k}_f\cdot\vec{R}}{1 + \cos(\vec{k}_i - \vec{k}_f)\cdot\vec{R}} \;.$$

Computing $|U_{i\to f}|^2$, and averaging over the orientations and polarizations of the incoming and outgoing photons, we get the frequency distribution

$$|U_{i\to f}|^2 \propto \frac{Cx^2 + Dx + E}{(x^2 + A)^2 + (x + B)^2} \;, \tag{27}$$

where

$$x = \frac{\omega - \omega_0}{\gamma_0} \;, \quad A = \frac{Y^2 - X^2}{\gamma_0^2} - \frac{1}{4},$$

$$B = -2\frac{XY}{\gamma_0^2} \;, \quad C = 1 + \zeta^2,$$

$$D = 4\zeta\frac{X}{\gamma_0}, \quad E = (1+\zeta^2)\left(\frac{X^2 + Y^2}{\gamma_0^2} + \frac{1}{4}\right) + 2\zeta\frac{Y}{\gamma_0} \;, \tag{28}$$

$$\zeta = \frac{3(\sin\omega_0 R - \omega_0 R\cos\omega_0 R)}{(\omega_0 R)^3} \;.$$

In the limiting case when $\omega_0 R \ll 1$, we have

$$X = V_{AB} \;, \quad Y = -\tfrac{1}{2}\zeta\gamma_0 \;, \tag{29}$$

since in this limit we have

$$\zeta = 1 - \tfrac{1}{10}\omega_0^2 R^2 \;, \quad Y = -\tfrac{1}{2}\gamma_0(1 - \tfrac{1}{10}\omega_0^2 R^2) \;.$$

From Eqs. (27) and (29), we see that our result is the same as that obtained by Fontana and Hearn.[6] For $\omega_0 R \gg 1$, Eq. (27) reduces to the Kramers-Heisenberg dispersion formula as it should.

However, when $\omega_0 R \approx 1$, Eq. (27) differs from the expression derived by Fontana and Hearn.[6] This discrepancy arises from the fact that they assume the interaction between the excited- and the ground-state atoms as the static dipole-dipole interaction *a priori*. In our treatment, such a static interaction appears as a consequence of the

TABLE I. Positions of maxima of Eq. (27) in units of γ_0, the natural linewidth.

$\omega_0 R$	x
1.5	−0.696 583
1.0	−2.072 525
0.7	−5.316 544
0.5	−13.407 516
0.3	−57.999 802
0.1	−1507.481 003
0.08	−2939.047 607
0.06	−6956.933 167

transfer of the excitation due to the external radiation field when both atoms are sufficiently close so that they can exchange virtual photons many times.

In Table I, we show the positions of the maxima of Eq. (27) for various values of $\omega_0 R$. It is seen that when $\omega_0 R \leq 0.1$, the peak value is very close to those estimated by Fontana and Hearn.[6] For $\omega_0 R > 0.1$, we find that the positions of the maxima are different.

V. FURTHER INVESTIGATION OF THE PROBABILITY DISTRIBUTIONS OF THE EMITTED PHOTONS

In the previous sections, we have investigated the problem of resonance scattering when the atomic system is in the ground state. We now turn our attention to the situation when both atoms are in the excited state and no field is present. This problem has been done partly in Sec. III. We carry out the detail calculations here. Our purpose is to compare our results with the results of Ernst and Stehle[1] obtained by using the Weisskopf-Wigner method.

We shall use the same notations for the amplitudes $\alpha_{k_1}(t)$, $\alpha_{k_1 k_2}(t)$, etc., as were given in the paper of Ernst and Stehle. The level scheme of the two-atom system is shown in Fig. 3.

Assume that at $t=0$, the state of the system is

$$\psi_{AB}(\vec{r}_1, \vec{r}_2; 0) = |1_A, 1_B; 0\rangle = |i; 0\rangle \equiv \psi_i(\vec{r}_1, \vec{r}_2) .$$

The amplitude

$$\alpha_0(t) = \langle \psi(t) | 1_A, 1_B; 0\rangle = e^{-\gamma_0 t} e^{-i(E_1 + E_1)t}$$

has been evaluated in Sec. III. We now evaluate the amplitude for the system to be in the state with one photon $\vec{k}$ present,

$$|f^{\pm}; \vec{k}\rangle = 2^{-1/2}(|1_A 0_B\rangle \pm |1_B 0_A\rangle)|\vec{k}\rangle \equiv \psi^{(\pm)}(\vec{r}_1, \vec{r}_2) .$$

This amplitude is given by

$$\alpha_{\vec{k}}^{(\pm)}(t) = -i\langle f^{\pm}; \vec{k} | S | i; 0\rangle$$
$$= i e (2kV)^{-1/2} \int d^3 r_1 \cdots d^3 r_4 \, dt' \, \theta(t') \, \bar{\psi}_f^{(\pm)}(\vec{r}_1, \vec{r}_2)$$
$$\times G_{AB}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4; t-t')(\hat{e}\, e^{-ikt' + i\vec{k}\cdot\vec{r}_3}$$
$$+ \hat{e}\, e^{ikt' + i\vec{k}\cdot\vec{r}_4}) \, \psi_i(\vec{r}_3, \vec{r}_4) \, e^{-iE_i t'} , \qquad (30)$$

where

$$\psi_i(\vec{r}_1, \vec{r}_2) = \psi_1(\vec{r}_1)\, \psi_1(\vec{r}_2) ,$$
$$\psi_f^{(\pm)}(\vec{r}_1, \vec{r}_2) = 2^{-1/2}[\psi_1(\vec{r}_1)\, \psi_0(\vec{r}_2) \pm \psi_0(\vec{r}_1)\, \psi_1(\vec{r}_2)] ,$$
$$E_i = E_i'^{(1)} + E_1'^{(2)} = 2E_1 - i\gamma_0 .$$

Carrying out the integration over $\vec{r}_1$ and $\vec{r}_2$, we get

$$\alpha_{\vec{k}}^{(\pm)}(t) = \frac{ie}{(2kV)^{1/2}} \int dt' \, \theta(t')$$
$$\times \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\Omega(t-t')}\, d\Omega}{E_0' + E_1' \pm I_{0,1}(-\Omega) + \Omega}$$
$$\times e^{ikt' - 2iE_1 t' - \gamma_0 t'} \, 2^{-1/2}(\langle 0_B | \hat{e}\, e^{-i\vec{k}\cdot\vec{r}_4} | 1_B\rangle$$
$$\pm \langle 1_A | \hat{e}\, e^{-i\vec{k}\cdot\vec{r}_3} | 0_A\rangle) , \qquad (31)$$

where we have used the explicit form of $G_{AB}(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4; t-t')$ as given in Eq. (10), with the observation that $+I_{0,1}(-\Omega)$ in the energy denominator of Eq. (31) is to be associated with the symmetric state and $-I_{0,1}(-\Omega)$ with the antisymmetric state, as was explained in Sec. III. Let

$$C_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{R}_j} \equiv \langle 0_j | \hat{e}\, e^{i\vec{k}\cdot\vec{r}} | 1_j\rangle$$
$$= \int d^3 r \, \bar{\psi}_1(\vec{r} - \vec{R}_j)\, \hat{e}\, e^{-i\vec{k}\cdot\vec{r}} \psi_0(\vec{r} - \vec{R}_j) , \qquad (32)$$

where the subscript j can be A or B; then Eq. (32) can be put into the form

$$\alpha_{\vec{k}}^{(\pm)}(t) = \frac{e}{(2kV)^{1/2}} \frac{1}{\sqrt{2}}$$
$$\times (C_k^* e^{-i\vec{k}\cdot\vec{R}_B} \pm C_k e^{-i\vec{k}\cdot\vec{R}_A}) f_k^{(\pm)}(t) , \qquad (33)$$

where

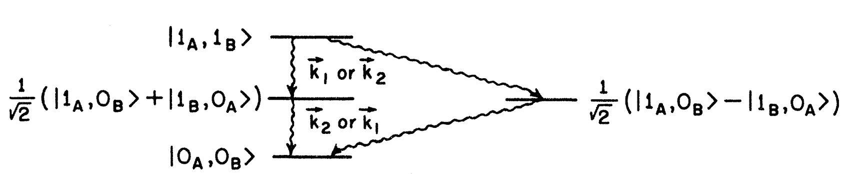

FIG. 3. Transition scheme for two-atom system.

$$f_k^{(\pm)}(t) = i\int_{-\infty}^{\infty} dt'\,\theta(t')\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\Omega \times \frac{e^{i\Omega(t-t')}\,e^{i(k-2E_1)t'-\gamma_0 t'}}{E_1+E_0-\frac{1}{2}i\gamma_0 \pm I_{0,1}(-\Omega)+\Omega} \quad (34)$$

Now write

$$I_{0,1}(-\Omega) = V_{AB} - \tfrac{1}{2}i\gamma_{12}\,,$$

$$\gamma_0 \geq \gamma_{12} \geq 0\,, \quad \gamma_{12} \equiv 2Y(\vec{R})\,,$$

and assume that V_{AB} and γ_{12} are real constants independent of Ω, the integration variable in Eq. (34). We can then carry out the integration in (34) to get

$$f_k^{(\pm)}(t) = \frac{e^{-i(2E_1-k)t}}{k-\omega_0 \pm V_{AB} + \frac{1}{2}i(\gamma_0 \mp \gamma_{12})} \times [e^{-\gamma_0 t} - e^{i(\omega_0 - k \pm V_{AB})t - \frac{1}{2}(\gamma_0 \pm \gamma_{12})t}]\,, \quad (35)$$

since γ_{12} varies between 0 and γ_0 and $\gamma_{12} \approx \gamma_0$ when $\omega_0 R \ll 1$. Thus, in this case we get

$$f_k^{(+)}(\infty) = 0\,,$$

$$|f_k^{(-)}(\infty)|^2 = [(k-\omega_0 - V_{AB})^2 + \gamma_0^2]^{-1}\,. \quad (36)$$

Note that (36) holds only in the limiting case $\omega_0 R \ll 1$. In reality, $f_k^{(\pm)}(t)$ both go to 0 with increasing time as indicated by the exponential factor $e^{-(\gamma_0 \pm \gamma_{12})t/2}$ in (35). Due to the appearance of $f_k^{(-)}(\infty)$ in (36), the antisymmetric state may be regarded as a metastable state, in agreement with Sec. III.

Finally, we evaluate the amplitude for both atoms to be in the ground state and two photons of momenta $\vec{k}_1$ and $\vec{k}_2$ to be present. This amplitude can be written in the form

$$\alpha^{12}_{\vec{k}_1\vec{k}_2}(t) = \langle f; \vec{k}_1\vec{k}_2 | S(t) | i; 0\rangle\,, \quad (37)$$

where

$$|f\rangle = |0_A, 0_B\rangle\,, \quad |i\rangle = |1_A, 1_B\rangle\,.$$

The amplitude defined in (37) can be rewritten in the following form with the aid of the two-particle Green's function:

$$\alpha^{12}_{\vec{k}_1\vec{k}_2}(t) = \frac{i^2 e^2}{(2k_1V)^{1/2}(2k_2V)^{1/2}}\int d^3r_1\cdots d^3r_6\,dt'\,dt''\,\theta(t'')\,\bar{\psi}_f(\vec{r}_1,\vec{r}_2)\,G_{AB}(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4;t-t')$$
$$\times \hat{e}_2(e^{ik_2t'-i\vec{k}_2\cdot\vec{r}_3} + e^{ik_2t'-i\vec{k}_2\cdot\vec{r}_4})\,G_{AB}(\vec{r}_3,\vec{r}_4,\vec{r}_5,\vec{r}_6;t'-t'')$$
$$\times \hat{e}_1(e^{ik_1t''-i\vec{k}_1\cdot\vec{r}_5} + e^{ik_1t''-i\vec{k}_1\cdot\vec{r}_6})\,\psi_i(\vec{r}_5,\vec{r}_6) + (\text{term with } \vec{k}_1 \leftrightarrow \vec{k}_2,\ \ k_1 \leftrightarrow k_2)\,, \quad (38)$$

where $\psi_i(\vec{r}_1,\vec{r}_2)$ has been given already, and $\psi_f(\vec{r}_1,\vec{r}_2) = \psi_0(\vec{r}_1)\psi_0(\vec{r}_2)$. The symbol $(\vec{k}_1 \leftrightarrow \vec{k}_2,\ k_1 \leftrightarrow k_2)$ in (38) means a term must be added such that $\alpha^{12}_{\vec{k}_1\vec{k}_2}(t)$ is symmetric in $\vec{k}_1$ and $\vec{k}_2$, and in k_1 and k_2.

To carry out the integrations in (38), two situations occur due to the presence of two different intermediate states which are implicit in G_{AB} as was noted before. With this interpretation, (38) can be written in the form

$$\alpha^{12}_{\vec{k}_1\vec{k}_2}(t) = \alpha^{(+)}_{\vec{k}_1\vec{k}_2}(t) + \alpha^{(-)}_{\vec{k}_1\vec{k}_2}(t)\,, \quad (39)$$

where

$$\alpha^{(\pm)}_{\vec{k}_1\vec{k}_2}(t) = \frac{i^2e^2}{(2k_1V)^{1/2}(2k_2V)^{1/2}}\int d^3r_1\cdots d^3r_4\,dt'\,dt''\,\theta(t'')\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\Omega\,\frac{e^{i\Omega(t-t')}}{\Omega+2E_0-i0}\,\bar{\psi}_0(\vec{r}_1)\bar{\psi}_0(\vec{r}_2)\,\hat{e}_1$$
$$\times(e^{ik_2t'-i\vec{k}_2\cdot\vec{r}_1} + e^{ik_2t'-i\vec{k}_2\cdot\vec{r}_2})\,\frac{1}{\sqrt{2}}[\psi_0(\vec{r}_1)\psi_1(\vec{r}_2) \pm \psi_1(\vec{r}_1)\psi_0(\vec{r}_2)]$$
$$\times\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\Omega'\,\frac{e^{i\Omega'(t'-t'')}}{E_0+E_1' \pm I_{1,0}+\Omega'}\,\frac{1}{\sqrt{2}}\,[\bar{\psi}_0(\vec{r}_3)\bar{\psi}_1(\vec{r}_4) \pm \bar{\psi}_1(\vec{r}_3)\bar{\psi}_0(\vec{r}_4)]$$
$$\times\hat{e}_2(e^{ik_1t''-i\vec{k}_1\cdot\vec{r}_3} + e^{ik_1t''-i\vec{k}_1\cdot\vec{r}_4})\,\psi_1(\vec{r}_3)\psi_1(\vec{r}_4)\,e^{-2iE_1't''} + (\text{term with } \vec{k}_1 \leftrightarrow \vec{k}_2\,,\ \ k_1 \leftrightarrow k_2)\,. \quad (40)$$

The amplitude $\alpha^{(+)}_{\vec{k}_1\vec{k}_2}(t)$ can be understood by considering the following transition scheme:

$$|1_A, 1_B; 0\rangle \xrightarrow[(\vec{k}_1, k_1)]{} 2^{-1/2}(|1_A, 0_B; \vec{k}_1\rangle + |1_B, 0_A; \vec{k}_1\rangle) \xrightarrow[(\vec{k}_2, k_2)]{} |0_A, 0_B; \vec{k}_1, \vec{k}_2\rangle$$

and

$$|1_A, 1_B; 0\rangle \xrightarrow[(\vec{k}_2, k_2)]{} 2^{-1/2}(|1_A, 0_B; \vec{k}_2\rangle + |1_B, 0_A; \vec{k}_2\rangle) \xrightarrow[(\vec{k}_1, k_1)]{} |0_A, 0_B; \vec{k}_1, \vec{k}_2\rangle \ .$$

Similarly, one can explain the form for $\alpha^{(-)}_{\vec{k}_1\vec{k}_2}(t)$ given in (40).

We may carry out the integrations of (40) as follows:

$$\alpha^{(\pm)}_{\vec{k}_1\vec{k}_2}(t) = \frac{i^2 e^2}{(2k_1V)^{1/2}(2k_2V)^{1/2}} \frac{1}{2}(\langle 0|\hat{e}_1 e^{-i\vec{k}_2\cdot\vec{r}_2}|1\rangle_B \pm \langle 0|\hat{e}_1 e^{i\vec{k}_2\cdot\vec{r}_1}|1\rangle_A)$$
$$\times(\langle 0|\hat{e}_2 e^{-i\vec{k}_1\cdot\vec{r}_3}|1\rangle_A \pm \langle 0|\hat{e}_2 e^{-i\vec{k}_1\cdot\vec{r}_4}|1\rangle_B)\, F^{(\pm)}_{k_1k_2}(t) \ , \qquad (41)$$

where

$$F^{(\pm)}_{k_1k_2}(t) = e^{-2iE_0t}\{[\omega_0 - k_2 \pm V_{AB} - \tfrac{1}{2}i(\gamma_0 \pm \gamma_{12})]^{-1}(\omega_0 - k_1 + \omega_0 - k_2 - i\gamma_0)^{-1} - e^{-i(E_1 + E_0 - k_2)t - 1/2(\gamma_0 \mp \gamma_{12})t}$$
$$\times[\omega_0 - k_2 \pm V_{AB} - \tfrac{1}{2}i(\gamma_0 \pm \gamma_{12})]^{-1}\,[\omega_0 - k_1 \pm V_{AB} - \tfrac{1}{2}i(\gamma_0 \mp \gamma_{12})]^{-1} + e^{i(\omega_0 - k_1 + \omega_0 - k_2)t - \gamma_0 t}$$
$$\times(\omega_0 - k_1 + \omega_0 - k_2 - i\gamma_0)^{-1}[\omega_0 - k_1 \mp V_{AB} - \tfrac{1}{2}i(\gamma_0 \mp \gamma_{12})]^{-1}\} + (\vec{k}_1 \leftrightarrow \vec{k}_2 \ , \ \ k_1 \leftrightarrow k_2) \ . \qquad (42)$$

In obtaining (42), $E_1' = E_1 - \frac{1}{2}i\gamma_0$ has been used. We now examine the behavior of the function $F^{(+)}_{k_1k_2}(t)$ when t is large. We see from (42) that there is one term with a long lifetime $(\gamma_0 - \gamma_{12})^{-1}$ which is related to the trapping effect discussed before. If we ignore the trapping effect, then we get

$$F^{(+)}_{k_1k_2}(\infty) = \{[\omega_0 - k_2 \pm V_{AB} - \tfrac{1}{2}i(\gamma_0 + \gamma_{12})]$$
$$\times(\omega_0 - k_1 + \omega_0 - k_2 - i\gamma_0)\}^{-1} + (k_1 \leftrightarrow k_2) \ , \quad (43)$$

where we have set $E_0 = 0$. (43) can be rewritten in the form

$$F^{(+)}_{k_1k_2}(\infty) = \left(1 + \frac{V_{AB} - i\gamma_{12}}{\omega_0 - k_1 + \omega_0 - k_2 - i\gamma_0}\right)$$
$$\times\{[\omega_0 - k_1 + V_{AB} - \tfrac{1}{2}i(\gamma_0 + \gamma_{12})]$$
$$\times[\omega_0 - k_2 + V_{AB} - \tfrac{1}{2}i(\gamma_0 + \gamma_{12})]\}^{-1} \ . \quad (44)$$

By setting $V_{AB} = 0$, (44) is identical with the function $\tilde{\beta}_{k_1k_2}$ obtained by Ernst and Stehle.[14] The function defined in (44) determines the line shape of the two emitted photons through the decay of the symmetric states of the two-atom system.

Finally, we discuss the spatial correlation of the emitted photons. We are interested in the case when the atoms are far apart, and thus γ_{12} will be small compared with γ_0, and we can ignore V_{AB} completely. In order to study the physical content of such a correlation, we shall evaluate the quantity

$$R_{\vec{k}_1\vec{k}_2\cdots\vec{k}_N} = \frac{\epsilon^2(\vec{k}_1\cdots\vec{k}_N)}{N!}\,\frac{|\alpha^{12\cdots N}_{\vec{k}_1\cdots\vec{k}_N}(\infty)|^2}{\prod_{\nu=1}^{N}|\alpha^1_{\vec{k}_\nu}(\infty)|^2} \ , \qquad (45)$$

where

$$\epsilon(\vec{k}_1\cdots\vec{k}_N) = \left(\prod_{i=1}^{N}(n_i!)\right)^{1/2} \ ,$$

and $|\alpha^1_{\vec{k}_\nu}(\infty)|^2$ is the probability distribution for a photon emitted by a single isolated excited atom. Its derivation is given in Appendix C. The physical meaning of the form (45) given by Ernst and Stehle[15] is that $R_{\vec{k}_1\cdots\vec{k}_N}$ is a measure for a "self-stimulation" effect occurring in the real physical system. Using (32), (39), (41), (42) and $|\alpha^1_k(\infty)|^2$ given in Appendix C, we find that

$$R_{k_1k_2} = [1 + \cos(\vec{k}_1 - \vec{k}_2)\cdot\vec{R}] - \eta(\cos\vec{k}_1\cdot\vec{R} + \cos\vec{k}_2\cdot\vec{R})$$
$$\times \mathrm{Re}\left[\frac{-i\gamma_0}{\omega_0 - k_1 + \omega_0 - k_2 - i\gamma_0}\left(\frac{\omega_0 - k_2 - \frac{1}{2}i\gamma_0}{\omega_0 - k_1 - \frac{1}{2}i\gamma_0} + \frac{\omega_0 - k_1 - \frac{1}{2}i\gamma_0}{\omega_0 - k_2 - \frac{1}{2}i\gamma_0}\right)\right] + O(\eta^2) \ , \qquad (46)$$

where

$$\eta \equiv \gamma_{12}/\gamma_0 \ll 1 \ , \quad \vec{R} = \vec{R}_A - \vec{R}_B \ .$$

It is seen that (46) is exactly the same as that obtained by Ernst and Stehle.[16] Note that the first bracket in (46) is not just the product of the probabilities of emitting photons of momenta $\vec{k}_1$ and $\vec{k}_2$ separately. For the physical interpretation of (46), we refer to the paper by Ernst and Stehle.[1] It appears worthwhile to extend the present approach to the N-atom problem. This will be done in the near future.

APPENDIX A: CALCULATIONS OF THE FIRST-ORDER DISPERSION FORCES

Consider the two-level atoms, $E_1 - E_0 = \omega_0$. We evaluate

$$I_{1,0}(\omega_0)=\frac{e^2}{4\pi}\int\frac{d^3k}{(2\pi)^3}\,d^3r\,d^3r'\,\bar{\psi}_1(\vec{r}\,)\gamma^A_\mu e^{-i\vec{k}\cdot\vec{r}}$$

$$\times\psi_0(\vec{r})\bar{\psi}_0(\vec{r}+\vec{r}\,')\gamma^B_\mu\psi_1(\vec{r}+\vec{r}\,')(\vec{k}^2-\omega_0^2+i0)^{-1}\;. \quad \text{(A1)}$$

Going to the nonrelativistic limit and making the dipole approximation we have

$$I_{1,0}(\omega_0)=(\vec{q}_A\cdot\vec{q}_B)\,\omega_0^2\,e^{-i\omega_0R}R^{-1}-\frac{1}{2\pi^2}\int d^3k(\vec{k}\cdot\vec{q}_A)$$

$$\times(\vec{k}\cdot\vec{q}_B)\,e^{-i\vec{k}\cdot\vec{R}}\,(k^2-\omega_0^2+i0)^{-1}\;, \quad \text{(A2)}$$

where $\vec{R}=\vec{R}_A-\vec{R}_B$, $R=|\vec{R}|$, $k=|\vec{k}|$. The magnetic dipole term has been ignored. By choosing $\vec{R}$ along the polar axis, as shown in Fig. 4, we have

$$I_{1,0}(\omega_0)=\omega_0^2(\vec{q}_A\cdot\vec{q}_B)\,e^{-i\omega_0R}R^{-1}-(q_Aq_B/2\pi^2)$$

$$\times\int k^4\,dk\,\sin\theta\,d\theta\,d\varphi\cos\theta_1\cos\theta_2$$

$$\times e^{-ikR\cos\theta}(k^2-\omega_0^2+i0)^{-1}\;. \quad \text{(A3)}$$

Let the angular coordinates of $\vec{q}_A,\vec{q}_B$ be $\theta_1'\varphi_1'$, $\theta_2'\theta_2'$, respectively. Then we have

$$\cos\theta_i=\cos\theta\cos\theta_i'+\sin\theta\sin\theta_i'\cos(\varphi-\varphi_i')\;,$$

where $i=1,2$. Integrating (A3) over φ, we have

$$I_{1,0}(\omega_0)=(\vec{q}_A\cdot\vec{q}_B)\,\omega_0^2\,k^{-1}\,e^{-i\omega_0R}-(q_Aq_B/2\pi)$$

$$\times\int_{-\infty}^{\infty}k^4(k^2-\omega_0^2-i0)^{-1}dk\int_0^{\pi}d\theta\sin\theta$$

$$\times e^{-ikR\cos\theta}[2\cos^2\theta\cos\theta_1'\cos\theta_2'$$

$$+\sin^2\theta\sin\theta_1'\sin\theta_2'\cos(\varphi_1'-\varphi_2')]\;. \quad \text{(A4)}$$

Integrating (A4) over θ, we obtain

$$I_{1,0}(\omega_0)=(\vec{q}_A\cdot\vec{q}_B)\,\omega_0^2R^{-1}e^{-i\omega_0R}+\pi^{-1}\int_0^{\infty}R^{-1}(k^2-\omega_0^2+i0)^{-1}k^3\sin(kR)\left(\vec{q}_A\cdot\vec{q}_B-\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}\right)dk$$

$$+\pi^{-1}\int_0^{\infty}k^4(k^2-\omega_0^2+i0)^{-1}\left(\frac{\sin kR}{kR}-2\frac{\cos kR}{k^2R^2}+2\frac{\sin kR}{k^3R^3}\right)\left(\vec{q}_A\cdot\vec{q}_B-3\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}\right)dk\;. \quad \text{(A5)}$$

Using the formula

$$\int_0^{\infty}k\sin(kR)\,(k^2-\omega_0^2+i0)^{-1}dk=\tfrac{1}{2}\pi e^{-i\omega_0R}\;,$$

and differentiating the above expression on both sides with respect to R once and twice, respectively,

$$\int_0^{\infty}k^2\cos(kR)\,(k^2-\omega_0^2+i0)^{-1}dk=-\tfrac{1}{2}i\pi\,\omega_0\,e^{-i\omega_0R}\;,$$

$$\int_0^{\infty}k^3\sin(kR)\,(k^2-\omega_0^2+i0)^{-1}dk=\tfrac{1}{2}\pi\,\omega_0^2\,e^{-i\omega_0R}\;,$$

we obtain from (A5),

$$I_{1,0}(\omega_0)=\omega_0^3\,e^{-i\omega_0R}\left[\frac{1}{\omega_0R}\left(\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}-(\vec{q}_A\cdot\vec{q}_B)\right)\right.$$

$$\left.+\left(\frac{i}{\omega_0^2R^2}+\frac{1}{\omega_0^3R^3}\right)\left((\vec{q}_A\cdot\vec{q}_B)-3\frac{(\vec{q}_A\cdot\vec{R})(\vec{q}_B\cdot\vec{R})}{R^2}\right)\right]\;. \quad \text{(A6)}$$

APPENDIX B: RIGOROUS DERIVATION OF THE TWO-PARTICLE GREEN'S FUNCTION

(i) We apply the principle of contraction mapping[10] to the integral equation (7). This principle asserts that if we have

$$G(x_1,x_3)G(x_2,x_4)\in L_2(-\infty,\infty)$$

$$K^2=\int d1\,d2\,d3\,d4[G(1,3)G(2,4)\gamma^A_\mu\gamma^B_\mu D_F(3,4)]^{\dagger}$$

$$\times[G(1,3)G(2,4)\gamma^A_\nu\gamma^B_\nu D_F(3,4)]<\infty\;, \quad \text{(B1)}$$

and the condition

$$e^2|K|<1 \quad \text{(B2)}$$

is fulfilled, then the iteration method can be applied to Eq. (7) to yield a unique solution for $G_{AB}(x_1,x_2;x_3,x_4)\in L_2(-\infty,\infty)$.

To see whether (B2) is satisfied, consider the two-level system for simplicity, and evaluate (B1) in the nonrelativistic limit and in the dipole approximation; we find

$$K^2=|I_{1,0}(\omega_0)|^2/e^4\,\omega_0^2\;,$$

where $\omega_0=E_1-E_0$, and $I_{1,0}(\omega_0)$ is given in Eq. (13). Thus, we have

$$e^2|K|=|I_{1,0}(\omega_0)|/\omega_0\;. \quad \text{(B3)}$$

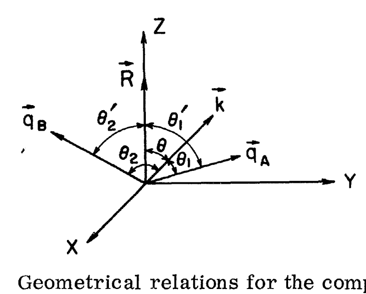

FIG. 4. Geometrical relations for the computations of the first-order dispersion forces.

As discussed in Sec. III, $|I_{1,0}(\omega_0)|$ is always less than ω_0 for reasonable internuclear separation R between the two atoms. In particular, for $\omega_0 R \approx 1$,

$$|I_{1,0}(\omega_0)| \approx \gamma_0 ,$$

where γ_0 is the natural linewidth for the atomic excited state. In the hydrogen atom, $\gamma_0 \sim 10^7$ cps, and $\omega_0 \sim 10^{15}$ cps for the transition between the first excited state and the ground state; we thus see

$$e^2|K| \ll 1 ,$$

and the first iteration of Eq. (7) gives a good result.

(ii) In order to justify the result of Eq. (10), we expand the two-particle Green's function as follows:

$$G_{AB}(x_1, x_2, x_3, x_4) = \sum_{m,n} \psi_n(\vec{r}_1)\psi_m(\vec{r}_2)\overline{\psi}_n(\vec{r}_3)\overline{\psi}_m(\vec{r}_4)$$

$$\times \int \frac{d\omega_1}{2\pi i}\frac{d\omega_2}{2\pi i} F_{n,m}(\omega_1, \omega_2)\, e^{i\omega_1(t_1-t_3)+i\omega_2(t_2-t_4)}$$

$$+ \sum_{n\neq n', m\neq m'} \psi_n(\vec{r}_1)\psi_m(\vec{r}_2)\overline{\psi}_{n'}(\vec{r}_3)\overline{\psi}_{m'}(\vec{r}_4)$$

$$\times \int \frac{d\omega_1}{2\pi i}\frac{d\omega_2}{2\pi i} F_{nn',mm'}(\omega_1, \omega_2)\, e^{i\omega_1(t_1-t_3)+i\omega_2(t_2-t_4)} , \tag{B4}$$

with the same assumption as Sec. II on the time coordinates. Substituting (B4) into Eq. (7) and taking the inner product with

$$\overline{\psi}_n(\vec{r}_1)\overline{\psi}_m(\vec{r}_2)\beta^A\beta^B \cdots \beta^A\beta^B \psi_n(\vec{r}_3)\psi_m(\vec{r}_4) ,$$

we have

$$F_{n,m}(\omega_1, \omega_2) = \frac{1}{(E_n' + \omega_1)(E_m' + \omega_2)} + h_{n,m} F_{n,m}(\omega_1, \omega_2)$$

$$+ \sum_{p\neq n, q\neq m} h_{np,mq} F_{pn,qm}(\omega_1, \omega_2) . \tag{B5}$$

Similarly, taking the inner product with

$$\overline{\psi}_n(\vec{r}_1)\overline{\psi}_m(\vec{r}_2)\beta^A\beta^B \cdots \beta^A\beta^B \psi_{n'}(\vec{r}_3)\psi_{m'}(\vec{r}_4) ,$$

we have

$$F_{nn',mm'}(\omega_1, \omega_2) = h_{nn',mm'} F_{n,m}(\omega_1, \omega_2)$$

$$+ \sum_{p\neq n', q\neq m'} h_{np,mq} F_{pn',qm'}(\omega_1, \omega_2) , \tag{B6}$$

where

$$h_{n,m} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{I_{n,m}(\omega_2 - \omega)}{(E_n' + \omega)(E_m' + \omega_1 + \omega_2 + \omega)} ,$$

$$\tilde{I}_{n,m}(t) = \int d^3r_1\, d^3r_2\, \overline{\psi}_n(\vec{r}_1)\overline{\psi}_m(\vec{r}_2)$$

$$\times I_{AB}(\vec{r}_1, \vec{r}_2, t)\psi_n(\vec{r}_2)\psi_m(\vec{r}_1) ,$$

$$I_{n,m}(\omega) = \int dt\, e^{i\omega t}\tilde{I}_{n,m}(t) ,$$

and similar expressions for $h_{n'm',nm}$, $\tilde{I}_{n'm',nm}(t)$, and $I_{n'm',nm}(\omega)$.

In order to solve (B5) and (B6), we introduce the following notation:

$$(F_{n,m})^{-1} \equiv Q_{n,m} ,$$

$$(F_{nn',mm'})(F_{n,m})^{-1} \equiv R_{nn',mm'} ,\quad n\neq n',\quad m\neq m' ;$$

then

$$Q_{n,m} = (E_n' + \omega_1)(E_m' + \omega_2)$$

$$\times\left(1 - h_{n,m} - \sum_{p\neq n, q\neq m} h_{np,mq}R_{pn,qm}\right) ,$$

$$R_{nn',mm'} = -h_{nn',mm'} - \sum_{p\neq n', q\neq m'} h_{np,mq}R_{pn',qm'} . \tag{B7}$$

Assuming that $h_{np,mq}$ are small, and using the method of the successive approximations, we have,

$$Q_{n,m} = (E_n' + \omega_1)(E_m' + \omega_2)$$

$$\times\left(1 - h_{n,m} + \sum_{n\neq p, m\neq q} h_{np,mq}h_{pn,qm} + \cdots\right) , \tag{B8}$$

$$R_{nn',mm'} = -h_{nn',mm'} - \sum_{p\neq n', q\neq m'} h_{np,mq}h_{pn',qm'} - \cdots .$$

Consequently, we get

$$F_{n,m}(\omega_1, \omega_2) = [(E_n' + \omega_1)(E_m' + \omega_2)(1 - h_{n,m} + \cdots)]^{-1} ,$$

$$F_{nn',mm'}(\omega_1, \omega_2) = (-h_{nn',mm'})[(E_n' + \omega_1)(E_m' + \omega_2)$$

$$\times(1 - h_{n,m} + \cdots)]^{-1} . \tag{B9}$$

From (B9), we see that

$$|F_{n,m}| \gg |F_{nn',mm'}| . \tag{B10}$$

Therefore, the nondiagonal term in the expansion (B4) can be neglected. Note that (B10) is also consistent with condition (B2) for the atomic system.

APPENDIX C: THE EMISSION LINE SHAPE OF A SINGLE ISOLATED ATOM

The Green's function in the presence of radiative correction can be approximated by

$$G(x_1, x_2) = \frac{1}{2\pi i}\sum_n \psi_n(\vec{r}_1)\overline{\psi}_n(\vec{r}_2)$$

$$\times\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t_1-t_2)}}{E_n(1-i0) + \omega + H_{nn}(-\omega)} , \tag{C1}$$

as was derived by Low.[8] We can approximate $H_{nn}(-\omega)$ by

$$H_{nn}(-\omega) \approx \delta E_n - \tfrac{1}{2} i\gamma_n ,$$

where δE_n and γ_n are real constants, and $\gamma_n \geq 0$.

We wish to calculate the amplitude for the emission of one photon $\vec{k}$. It is given by

$$\alpha_k^1(t) = -i \langle 0;\ k| S |i;\ 0\rangle$$

$$= \frac{ie}{(2kV)^{1/2}} \int d^3r\, d^3r'\, dt'\, \theta(t')\bar{\psi}_0(\vec{r}) G(\vec{r},\vec{r}';t-t') \times \hat{e}\, e^{ikt'-i\vec{k}\cdot\vec{r}'} \psi_1(\vec{r}')\, e^{iE_1' t'} , \qquad (C2)$$

where

$$E_1' = E_1 - \tfrac{1}{2} i\gamma_0 .$$

Carrying out the integrations in (C2), we obtain

$$\alpha_{\vec{k}}^1(t) = \frac{e}{(2kV)^{1/2}} \langle 0|\hat{e}\, e^{-i\vec{k}\cdot\vec{r}}|1\rangle e^{-i(\omega_0-k)t} \beta_k(t) ,$$

where

$$\beta_k(t) = \frac{e^{-\gamma_0 t/2} - e^{-i(\omega_0-k)t}}{\omega_0 - k - \frac{1}{2} i\gamma_0} .$$

For $t \to \infty$, we have

$$|\alpha_{\vec{k}}^1(\infty)|^2 = \frac{e^2}{2kV} |\langle 0|\hat{e}\, e^{i\vec{k}\cdot\vec{r}}|1\rangle|^2 [(\omega_0-k)^2 + \tfrac{1}{4}\gamma_0^2]^{-1} . \qquad (C3)$$

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[11]See, e.g., Ref. 8(b), pp. 144–145.

[12]The detailed algebra is given in Appendix A.

[13]Perturbation by a non-Hermitian operator under certain conditions has the same results as with a Hermitian operator. See J. Wong, J. Math. Phys. 8, 2039 (1967).

[14]See Ref. 1, Eq. (4.12).

[15]See Ref. 1, Eq. (3.21).

[16]See Ref. 1, Eq. (4.14).