the impulse approximation.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Dr. O. K. Harling for making available his experimental results and for communications and discussions concerning this work.

*Work supported by NSF Grant No. GP 14058.

[1]F. London, *Superfluids* (Wiley, New York, 1954), Vol. II.

[2]O. Penrose and L. Onsager, Phys. Rev. 104, 576 (1956).

[3]R. A. Cowley and A. D. B. Woods, Phys. Rev. Letters 21, 767 (1968).

[4]O. K. Harling, Phys. Rev. Letters 24, 1046 (1970).

[5]Such experiments were suggested by P. C. Hohenberg and P. M. Platzman, Phys. Rev. 152, 198 (1966).

[6]R. D. Puff and J. S. Tenn, Phys. Rev. A 1, 125 (1970).

[7]L. Van Hove, Phys. Rev. 95, 249 (1954).

[8]J. Gavoret and P. Nozières, Ann. Phys. (N. Y.) 28, 349 (1964).

[9]See, e.g., A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, N. J., 1963), Chap. 1.

[10]D. G. Henshaw and A. D. B. Woods, Phys. Rev. 121, 1266 (1961).

[11]H. A. Gersch and J. M. Tanner, Phys. Rev. 139, 1769 (1965). Equations (81) and (110) of this paper provide the connection between n_p and $\mathcal{E}(p)$.

[12]W. L. McMillan, Phys. Rev. 138, A442 (1965).

[13]D. Schiff and L. Verlet, Phys. Rev. 160, 208 (1967).

[14]W. P. Francis, G. V. Chester, and L. Reatto, Phys. Rev. A 1, 86 (1970).

[15]R. B. Bernstein and F. A. Morse, J. Chem. Phys. 40, 917 (1964).

[16]O. K. Harling (private communication).

[17]J. F. Fernandez and H. A. Gersch, Phys. Letters 30A, 261 (1969).

[18]V. F. Sears, Phys. Rev. 185, 200 (1969).

[19]In support of this expectation, the wing contributions to the scattering peak, due to atoms with momentum near the roton bump and clearly present at 1.2 °K, seem to have disappeared at 4.3 °K. The situation is not as clear as one would like, owing to poor statistics and uncertain background subtractions.

PHYSICAL REVIEW A VOLUME 4, NUMBER 1 JULY 1971

Polarization Effects in Zeeman Lasers with x-y-Type Loss Anisotropies

Albert Le Floch and Roger Le Naour

Laboratoire d'Electronique Quantique, Faculté des Sciences, Avenue du Général Leclerc, Rennes, Ille et Vilaine, France

(Received 12 June 1970; revised manuscript received 29 October 1970)

It has already been reported that a laser subject to an axial magnetic field and having end mirrors which exhibit relatively high x-y-type loss anisotropy can be described by two theoretical methods. The first method uses self-consistent-field equations with distributed losses, whereas the second one is based on a resonance condition for a complete round-trip pass. The following study of a Zeeman laser with different x-y-type loss anisotropy at each mirror shows that the localization of the losses and the nonreciprocal character of the Farady rotation require a different formulation for each initial system. The rotation of the plane of polarization is different at each end of the laser, but it is shown that there always exists for central tuning a real or virtual average polarization vector which obeys Lamb's theory. The equivalence of the two methods is discussed.

I. INTRODUCTION

Lamb's self-consistent theory, which supposes distributed losses within the laser cavity, has been shown to be correct when extended to treat the electric field polarization properties of lasers with isotropic or weakly anisotropic cavities.[1-3]

In a recent paper, Greenstein[4] stated the difficulties of the self-consistent-field method, and compared the results with those given by a resonance condition for a complete round-trip pass[5] in a single centrally tuned laser, for a particular cavity where all x-y anisotropic losses were localized at one mirror. It was seen that, in general, the two sets of results did not agree except in the limit of small anisotropy. Consequently, Greenstein concluded that the two methods are not equivalent and only the resonance condition gives a correct description of the polarization for large cavity anisotropy. In particular, he points out that the maximum amount of rotation of the plane of polarization is always somewhere between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$, and approaches 90° in the limit of strong anisotropy ($\epsilon \to 1$); this result is inconsistent with that given by the self-consistent method, which however predicts much more in the limit of small anisotropy.

The following paper discusses the general case of a cavity with an anisotropic x-y mirror at each

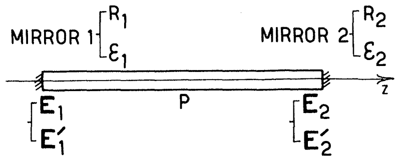

FIG. 1. Schematic diagram of a resonator with two anisotropic mirrors, M_1 and M_2.

end of the laser. It shows that the two methods do not really give different results but instead complementary ones, and that there always exists an average polarization vector which strictly obeys the self-consistent method. The localization of the losses and the nonreciprocal character of the Faraday rotation seem important, however, for the resolution of this type of problem.

We show that the theoretical maximum-rotation value of the polarization can be anywhere between 0 and $\frac{1}{2}\pi$, that rotations are different at each end, and that it is always possible to define an average polarization vector, real or virtual, according to the laser cavity, which we shall call "Lamb's vector."

The application of our results to the particular case of cavity considered by Greenstein shows that, in this case, the average polarization vector is located on the isotropic mirror.

II. RESONANCE CONDITION

Let us consider a centrally tuned single-mode laser in a homogeneous axial magnetic field with anisotropic losses at each end, either on mirrors or on Brewster windows. It has been shown theoretically[1,6-11] for very weak anisotropies that the initial linear polarization rotates through an angle of $\frac{1}{4}\pi$, this limiting angle arising for a critical H_c value of the magnetic field. Furthermore, this result has been verified experimentally,[3,5] for example in the case of the 1.15-μ line of neon (a $J=1 \rightarrow J=2$ transition). Since the medium rotates the initial polarization vector, and the unequal x-y losses of the end mirrors counter-rotate it, we suppose that there exists a polarized standing wave in the anisotropic cavity. To facilitate the comparison of our results with those of Greenstein, we shall use the same notation, extending it to the more general case studied here.

In a circular basis, we write the two σ^+ and σ^- polarized components $\vec{E}_+$ and $\vec{E}_-$ of the electric field $\vec{E}$ in terms of a rectangular coordinate system

$$\vec{E}_+ = -2^{-1/2}(\vec{E}_x + i\vec{E}_y) \quad , \quad \vec{E}_- = 2^{-1/2}(\vec{E}_x - i\vec{E}_y) \quad .$$

For any laser cavity the resonance condition can be written at one end as either

$$PR_2PR_1\vec{E}_1 = \vec{E}_1 \quad , \tag{1}$$

or

$$R_1PR_2P\vec{E}_1' = \vec{E}_1' \quad , \tag{2}$$

where R_1 and R_2 are the reflection matrices for each mirror and where P represents the propagation matrix relative to a cavity of length L (Fig. 1). The eigenvector $\vec{E}_1$ of Eq. (1) represents the electric vector before reflection at mirror 1: By multiplying by the transmission matrix of mirror 1, one obtains the outward observable vector. The eigenvector $\vec{E}_1'$ of Eq. (2) represents the electric vector reflected from the mirror 1. Similarly at the other end we have

$$PR_1PR_2\vec{E}_2 = \vec{E}_2 \quad , \tag{3}$$

$$R_2PR_1P\vec{E}_2' = \vec{E}_2' \quad . \tag{4}$$

In the presence of the magnetic field, the propagation along the z axis of the right and left circular components σ^+ and σ^- is described by $e^{-j\beta_\pm z}$. For central tuning, let

$$P = \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix} \quad ,$$

with

$$p_\pm = e^{\beta''_\pm L}\, e^{-j\beta'_\pm L} \quad ,$$

$$\beta'_\pm = (\omega/c)(1 + \tfrac{1}{2}\chi'_\pm) \quad , \quad \beta''_\pm = \tfrac{1}{2}(\omega/c)\chi''_\pm - l_0 \quad ,$$

where ω is the frequency of the electric field, $\chi'_\pm$ and $\chi''_\pm$ are the real and imaginary parts of the susceptibility of the circular waves, and l_0 is a phenomenological isotropic loss term for the cavity.

In the general case, we notice that there would be off-diagonal contributions to P, but in the special case of central tuning ($\omega = \omega_0$, where ω_0 is the atomic line center frequency) one can avoid complications by the assumption that the two intensities of opposite circular polarizations are equal, i.e., we assume that $\chi'_\pm(\omega_0)$ and $\chi''_\pm(\omega_0)$ obey the symmetry relations

$$\chi''_+(\omega_0) = \chi''_-(\omega_0) \quad , \quad \chi'_+(\omega_0) = -\chi'_-(\omega_0) \quad . \tag{5}$$

The calculation of the polarization direction will then be independent of nonlinear effects contained in the susceptibility χ.

In a Cartesian basis, the x-y-type loss anisotropies are represented by diagonal matrices so that at each end the reflection matrices are

$$R_1' = \begin{pmatrix} r_{1x} & 0 \\ 0 & r_{1y} \end{pmatrix} \quad , \quad R_2' = \begin{pmatrix} r_{2x} & 0 \\ 0 & r_{2y} \end{pmatrix} \quad .$$

In a circular basis these are written[4]

$$R_1 = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \quad , \quad R_2 = \begin{pmatrix} c & -d \\ -d & c \end{pmatrix} \quad ,$$

where

$a = \frac{1}{2}(r_{1x}+r_{1y})$, $b = \frac{1}{2}(r_{1x}-r_{1y})$,

$c = \frac{1}{2}(r_{2x}+r_{2y})$, $d = \frac{1}{2}(r_{2x}-r_{2y})$.

Anisotropy at each end mirror will be described by $\epsilon_1 = b/a$ and $\epsilon_2 = d/c$, where ϵ_1 and ϵ_2 may theoretically take any value included between -1 and $+1$. For example, 0° or 90° values of the skew angle between two Brewster windows,[12] i.e., the angle between the planes of incidence of the Brewster plates, correspond to possible variations of ϵ_1 and ϵ_2 from about $+11\times10^{-2}$ to -11×10^{-2}.

Equations (1)–(4) each admit the same eigenvalue equation, namely,

$$p_+^2 p_-^2[(a^2-b^2)(c^2-d^2)] - 2bd\,p_+p_- - ac\,(p_+^2+p_-^2)+1=0 \quad . \tag{6}$$

Let $\tau \equiv p_+p_-$ and $\sigma \equiv p_+/p_- = e^{j\varphi/2}$, where p_+ and p_- have the same magnitude for central tuning. Then $\tau\sigma$ is equal to p_+^2 and $\tau/\sigma = p_-^2$.

When the mode is centrally tuned in the Doppler curve ($\omega=\omega_0$), we can write, from relation (5), $\sigma = e^{j\varphi/2}$ with relative phase angle

$$\tfrac{1}{2}\varphi = (\omega_0/2c)(\chi'_- - \chi'_+)L = \chi'_-(\omega_0)(\omega_0/c)L \quad .$$

We deduce that

$$\tau = \frac{\epsilon_1\epsilon_2+\cos\frac{1}{2}\varphi \pm[(\epsilon_1\epsilon_2+\cos\frac{1}{2}\varphi)^2-(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2}}{ac(1-\epsilon_1^2)(1-\epsilon_2^2)} \quad . \tag{7}$$

If both ϵ_1 and ϵ_2 are positive, the solution with the minus sign before the radical, which requires less gain, will dominate and give rise to the laser oscillation.

In these conditions, there exists an eigenvalue, or a standing wave, only if the expression under the radical is positive, that is, if we have

$$\cos\tfrac{1}{2}\varphi \geqslant \pm[(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2} - \epsilon_1\epsilon_2 \quad . \tag{8}$$

In a helium-neon laser oscillating on a $J=1 \leftrightarrow J=2$ line with weak gain (the 6328-Å line for example), the fact that the Brewster windows have relatively high losses, $\epsilon_1=\epsilon_2\approx 11\times10^{-2}$, leads to a linear polarization for any value of the magnetic field H.[13]

The calculation of the eigenvectors of (1) gives the ratio

$$\frac{E_{1+}}{E_{1-}} = \frac{bd p_+p_- + acp_-^2 - 1}{adp_+p_- + bcp_-^2} = \frac{\epsilon_1\epsilon_2+1/\sigma}{\epsilon_2+\epsilon_1/\sigma} - \frac{1}{ac(\epsilon_2+\epsilon_1/\sigma)\tau} \quad ,$$

where E_{1+} and E_{1-} are the complex numbers associated with $\vec{E}_{1+}$ and $\vec{E}_{1-}$, the two components of the electric field $\vec{E}_1$.

Therefore, we have

$$\frac{E_{1+}}{E_{1-}} = \sigma\,\frac{-j\sin\frac{1}{2}\varphi(\epsilon_1+\epsilon_2/\sigma)\pm(\epsilon_1+\epsilon_2/\sigma)[(\epsilon_1\epsilon_2+\cos\frac{1}{2}\varphi)^2-(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2}}{\epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2(\sigma+1/\sigma)} = e^{j(\varphi+K_1)/2} \quad ,$$

in which

$$\cos\tfrac{1}{2}K_1 = \frac{-\epsilon_2\sin^2\frac{1}{2}\varphi \pm(\epsilon_1+\epsilon_2\cos\frac{1}{2}\varphi)(\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi-\sin^2\frac{1}{2}\varphi)^{1/2}}{\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} \quad ,$$

$$\sin\tfrac{1}{2}K_1 = \frac{-\sin\frac{1}{2}\varphi\{\epsilon_1+\epsilon_2\cos\frac{1}{2}\varphi \pm\epsilon_2(\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi-\sin^2\frac{1}{2}\varphi)^{1/2}\}}{\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} \quad . \tag{9}$$

The reflected $\vec{E}'_1$ vector eigensolution of (2) is obtained directly by the product $\vec{E}'_1 = R_1\vec{E}_1$. Specifically, we have

$$\frac{E'_{1+}}{E'_{1-}} = \frac{1}{\sigma}\,\frac{\epsilon_1\cos\frac{1}{2}\varphi+\epsilon_2-\sigma(\epsilon_1+\epsilon_2\cos\frac{1}{2}\varphi)\pm(\epsilon_1+\epsilon_2\sigma)(\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi-\sin^2\frac{1}{2}\varphi)^{1/2}}{\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} = e^{j(K'_1-\varphi)/2} \quad ,$$

with

$$\cos\tfrac{1}{2}K'_1 = \frac{\epsilon_2\sin^2\frac{1}{2}\varphi \pm(\epsilon_1+\epsilon_2\cos\frac{1}{2}\varphi)(\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi-\sin^2\frac{1}{2}\varphi)^{1/2}}{\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} \quad ,$$

$$\sin\tfrac{1}{2}K'_1 = \frac{-\sin\frac{1}{2}\varphi[\epsilon_1+\epsilon_2\cos\frac{1}{2}\varphi \mp\epsilon_2(\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi-\sin^2\frac{1}{2}\varphi)^{1/2}]}{\epsilon_1^2+\epsilon_2^2+2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} \quad . \tag{10}$$

Corresponding expressions for eigenvectors $\vec{E}_2$ and $\vec{E}'_2$ are given by (9) and (10), in which the subscripts are interchanged.

In all cases only one sign before the radical is valid: It will be imposed by the oscillating mode, that is, by the mode which requires the less gain defined by (7).

We can easily prove the following useful relations:

$$\tfrac{1}{2}K'_1 - \tfrac{1}{2}K_2 = \tfrac{1}{2}K'_2 - \tfrac{1}{2}K_1 = \tfrac{1}{2}\varphi \quad .$$

III. INTERPRETATION AND DISCUSSION

When Eq. (8) is verified we conclude that there exists in the laser cavity a stationary wave of polarization defined by E_+/E_-.

The direction of polarization with respect to the initial direction for $H=0$ varies as a function of magnetic field H and increases until $H=H_c$. Here the critical field H_c is derived from Eq. (8) and defined implicitly by

$$\cos\tfrac{1}{2}\varphi_m = \pm[(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2} - \epsilon_1\epsilon_2 ,$$

where φ_m corresponds to the maximum of φ.

The corresponding values of $\frac{1}{2}K_1$ and $\frac{1}{2}K_2$ are then defined by Eqs. (9) and can be written

$$\cos\tfrac{1}{2}K_{1m} = \epsilon_2, \quad \sin\tfrac{1}{2}K_{1m} = (\epsilon_1 + \epsilon_2\cos\tfrac{1}{2}\varphi_m)/\sin\tfrac{1}{2}\varphi_m ;$$

$$\cos\tfrac{1}{2}K_{2m} = \epsilon_1, \quad \sin\tfrac{1}{2}K_{2m} = (\epsilon_2 + \epsilon_1\cos\tfrac{1}{2}\varphi_m)/\sin\tfrac{1}{2}\varphi_m .$$

In particular we deduce

$$\tfrac{1}{2}K_{1m} + \tfrac{1}{2}K_{2m} = \pi - \tfrac{1}{2}\varphi_m ,$$

$$\tfrac{1}{2}K_{1m} + \tfrac{1}{2}K'_{1m} = \tfrac{1}{2}K_{2m} + \tfrac{1}{2}K'_{2m} = \pi .$$

Two interesting cases appear:

Case (*a*):

$$\epsilon_1 > \epsilon_2 \geq 0 ;$$

Case (*b*):

$$\epsilon_2 < 0, \quad \epsilon_1 > |\epsilon_2| .$$

Case (*a*). The first case corresponds, for example, to a cavity with one plate at each end, having the same plane of incidence, but inserted at different angles to the laser axis. We obtain different maximum rotations θ_1 and θ_2 at each end:

$$\theta_{1m} = \tfrac{1}{4}\varphi_m + \tfrac{1}{2}\cos^{-1}\epsilon_2, \quad \theta_{2m} = \tfrac{1}{4}\varphi_m + \tfrac{1}{2}\cos^{-1}\epsilon_1 .$$

In the same way, the angles of the reflected beams θ'_1 and θ'_2 are given by

$$\theta'_{1m} = -\tfrac{1}{4}\varphi_m + \tfrac{1}{2}\pi - \tfrac{1}{2}\cos^{-1}\epsilon_2 ,$$

$$\theta'_{2m} = -\tfrac{1}{4}\varphi_m + \tfrac{1}{2}\pi - \tfrac{1}{2}\cos^{-1}\epsilon_1 .$$

For $\epsilon_2 = 0$ (particular case considered by Greenstein) we have

$$\cos\tfrac{1}{2}\varphi_m = (1-\epsilon_1^2)^{1/2}, \quad \sin\tfrac{1}{2}\varphi_m = \epsilon_1 ,$$

and

$$\theta_{1m} = \tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}\epsilon_1, \quad \theta'_{1m} = \tfrac{1}{4}\pi - \tfrac{1}{2}\sin^{-1}\epsilon_1 ;$$

$$\theta_{2m} = \tfrac{1}{2}\sin^{-1}\epsilon_1 + \tfrac{1}{2}\cos^{-1}\epsilon_1 ,$$

$$\theta'_{2m} = \tfrac{1}{2}\pi - \tfrac{1}{2}\sin^{-1}\epsilon_1 - \tfrac{1}{2}\cos^{-1}\epsilon_1 .$$

For the limiting value $\epsilon_1 = 1$ we have

$$\theta_{1m} = \tfrac{1}{2}\pi, \quad \theta'_{1m} = 0, \quad \theta_{2m} = \theta'_{2m} = \tfrac{1}{4}\pi .$$

Thus the theoretical maximum rotation of the polarization vector is $\frac{1}{2}\pi$ on the $\epsilon_1 = 1$ mirror, in agreement with Greenstein's result, but exactly $\frac{1}{4}\pi$ on the other mirror whatever the value of ϵ_1.

Case (*b*). The second case corresponds, for example, to a cavity with one plate at each end, having orthogonal planes of incidence, but inserted at different angles to the laser axis. In this case mirror 1 always determines the polarization for a $J=1 \rightarrow J=2$ line in a zero magnetic field. The preceding equations are then written

$$\theta_{1m} = \tfrac{1}{4}\varphi_m + \tfrac{1}{2}\pi - \tfrac{1}{2}\cos^{-1}|\epsilon_2| , \quad \theta'_{1m} = -\tfrac{1}{4}\varphi_m + \tfrac{1}{2}\cos^{-1}|\epsilon_2| ;$$

$$\theta_{2m} = \tfrac{1}{4}\varphi_m + \tfrac{1}{2}\cos^{-1}\epsilon_1, \quad \theta'_{2m} = -\tfrac{1}{4}\varphi_m + \tfrac{1}{2}\pi - \tfrac{1}{2}\cos^{-1}\epsilon_1 .$$

If we have $\epsilon_1 = 1$, then $\cos\frac{1}{2}\varphi_m = |\epsilon_2|$ and

$$\theta_{1m} = \tfrac{1}{2}\pi, \quad \theta'_{1m} = 0 ;$$

$$\theta_{2m} = \tfrac{1}{2}\cos^{-1}|\epsilon_2| , \quad \theta'_{2m} = \tfrac{1}{2}\pi - \tfrac{1}{2}\cos^{-1}|\epsilon_2| .$$

As the anisotropy ratio ϵ_2 approaches -1, the angle of maximum rotation of the polarization on mirror 2 is given by $\theta_{2m} \rightarrow 0$ and $\theta'_{2m} \rightarrow \frac{1}{2}\pi$. In all cases the difference $\Delta\theta_m = \theta_{1m} - \theta_{2m} = \frac{1}{4}(K_{1m} - K_{2m})$ is defined by

$$\cos\tfrac{1}{2}(K_{1m} - K_{2m}) = [(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2} + \epsilon_1\epsilon_2 .$$

At the limit, if $\epsilon_2 \rightarrow -\epsilon_1$ with $\epsilon_1 = \epsilon$, then we have $\cos\frac{1}{2}(K_{1m} - K_{2m}) = 1 - 2\epsilon^2$, and for $\epsilon \rightarrow 1$, $\Delta\theta_m \rightarrow \frac{1}{2}\pi$.

With respect to the symmetry of the system and the nonreciprocity of the Faraday effect, we show that it is theoretically possible to obtain maximum rotations of the polarization vectors anywhere between 0 and $\frac{1}{2}\pi$, with ϵ_1 and ϵ_2 varying between -1 and $+1$.

In reality, for the most common neon-laser lines with weak gain the maximum deviation $\Delta\theta_m$ between the polarization at each end of the cavity will be small (for the 3.39-μ line, with one plate of quartz at the Brewster angle presenting a $\epsilon \approx 11 \times 10^{-2}$ loss, one can theoretically obtain a deviation $\Delta\theta_m \approx 3°$).

We also notice that for the maximum rotations we have

$$\tfrac{1}{2}(\theta_{1m} + \theta'_{1m}) = \tfrac{1}{2}(\theta_{2m} + \theta'_{2m}) = \tfrac{1}{4}\pi ,$$

and the question arises whether it is not possible in all cases to define an average eigenvector of the system, of maximum rotation $\frac{1}{4}\pi$ for $H = H_c$, in agreement with the result given by the self-consistent-field theory of Sargent, Lamb, and Fork.

IV. EXISTENCE OF AVERAGE POLARIZATION VECTOR

Let us see if there exists a single eigenvector for the laser system. We postulate the existence of such a vector $\vec{E}$ at a point of the coordinate z between the mirrors M_1 and M_2 on the axis of the laser. First we introduce the matrices defining this point and which permit the calculation of the eigenvector.

By analogy with the preceding notation we shall write

$$P_1 = \begin{pmatrix} p_{+1} & 0 \\ 0 & p_{-1} \end{pmatrix} , \qquad P_2 = \begin{pmatrix} p_{+2} & 0 \\ 0 & p_{-2} \end{pmatrix} .$$

The eigenvector for the wave propagating toward mirror 1 is defined by

$$P_1R_1P_1\vec{E} = \vec{E} \ . \qquad (11)$$

In the same way, for the wave propagating toward mirror 2 we have

$$P_2R_2P_2\vec{E} = \vec{E} \ . \qquad (12)$$

The uniqueness of the solution is assured if we have the commutator

$$[P_1R_1P_1,\ P_2R_2P_2] \equiv 0 \ , \qquad (13)$$

that is, if we have

$$\frac{bc}{ad}\,\frac{p_{+1}p_{-1}}{p_{+2}\,p_{-2}} = \frac{p_{+1}^2 - p_{-1}^2}{p_{+2}^2 - p_{-2}^2} \ .$$

Letting $\sigma_1 = p_{+1}/p_{-1} = e^{j\varphi_1/2}$, and $\tau_1 = p_{+1}p_{-1}$, and using similar expressions for σ_2 and τ_2, this condition reduces to

$$\epsilon_2/\epsilon_1 = \sin\varphi_2/\sin\varphi_1 \ . \qquad (14)$$

This is a necessary and sufficient condition for the existence of the eigenvector denoted by $\vec{E}_L$ and defined by

$$\left(\frac{E_+}{E_-}\right)_L = \frac{ap_{-1}^2 - 1}{bp_{+1}p_{-1}} = \frac{1}{\epsilon_1\sigma_1} - \frac{1}{b\,\tau_1} \ ,$$

where

$$\tau_1 = \frac{\cos\frac{1}{2}\varphi_1 \pm (\epsilon_1^2 - \sin^2\frac{1}{2}\varphi_1)^{1/2}}{a(1-\epsilon_1^2)} \ .$$

Then we have

$$(E_+/E_-)_L = \pm\, e^{\mp jK_L/2} \ ,$$

with

$$\sin\tfrac{1}{2}K_L = (\sin\tfrac{1}{2}\varphi_1)/\epsilon_1 = (\sin\tfrac{1}{2}\varphi_2)/\epsilon_2 \ . \qquad (15)$$

So there exists a unique eigenvector of the system, the maximum rotation of which is given by

$$\sin\tfrac{1}{2}K_L = 1 \ ,$$

that is, by

$$\tfrac{1}{2}K_{Lm} = \tfrac{1}{2}\pi, \quad \theta_m = \tfrac{1}{4}\pi \ .$$

Since the Faraday effect adds up for both directions of propagation, we can write

$$\varphi_1 + \varphi_2 = \varphi$$

and verify that

$$\cos\tfrac{1}{2}\varphi_m = [(1-\epsilon_1^2)(1-\epsilon_2^2)]^{1/2} - \epsilon_1\epsilon_2 \ .$$

The position of $\vec{E}_L$, called the "Lamb's vector," is given by

$$z_1 + z_2 = L \ ,$$

with

$$\epsilon_2 \sin\tfrac{1}{2}\varphi_1 = \epsilon_1 \sin\tfrac{1}{2}\varphi_2 \ ,$$

and will depend on the values of the anisotropy ratios ϵ_1 and ϵ_2.

Three cases arise:

Case (*a*). If ϵ_1 and ϵ_2 are of the same sign, then $\frac{1}{2}\varphi_1$ and $\frac{1}{2}\varphi_2$ are also of the same sign. Since $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2 = \frac{1}{2}\varphi$, we deduce that $z_1 < L$ and $z_2 < L$. Thus there exists an eigenvector in the cavity; the "Lamb's vector" is real.

Case (*b*). For $\epsilon_2 = 0$, we have $\frac{1}{2}\varphi_2 = 0$, and $z_2 = 0$; here the "Lamb's vector" is situated on the isotropic mirror. We again meet Greenstein's particular case.

Case (*c*). For $\epsilon_1/\epsilon_2 < 0$, $\frac{1}{2}\varphi_1$ and $\frac{1}{2}\varphi_2$ are of opposite signs, and if $|\epsilon_1| > |\epsilon_2|$, we shall have $z_1 - z_2 = L$. The average eigenvector is then outside the cavity; the "Lamb's vector" is virtual.

Whatever the anisotropy of the system, there exists an average polarization vector $\vec{E}_L$ whose rotation, with respect to the initial direction, i.e., for $H = 0$, is given by

$$\theta_L = \tfrac{1}{4}K_L \ ,$$

with

$$\sin\tfrac{1}{2}K_L = \sin\tfrac{1}{2}\varphi_1/\epsilon_1 = \sin\tfrac{1}{2}\varphi_2/\epsilon_2 \ .$$

Remark. Inasmuch as the rotations of the incident and reflected eigenvectors are given on the mirrors, respectively, by

$$\theta_1 = \tfrac{1}{4}(\varphi + K_1) \ , \quad \theta_1' = -\tfrac{1}{4}(\varphi - K_1') \ ,$$

we obtain $\frac{1}{2}(\theta_1 + \theta_1') = \frac{1}{8}(K_1 + K_1')$ and $\frac{1}{2}(\theta_2 + \theta_2') = \frac{1}{8}(K_2 + K_2')$. Since we have

$$\cos\tfrac{1}{2}(K_2 + K_2') = \cos\tfrac{1}{2}(K_1 + K_1') = \frac{\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi - 2\sin^2\frac{1}{2}\varphi}{\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1\epsilon_2\cos\frac{1}{2}\varphi} \ ,$$

$$\cos\tfrac{1}{2}\varphi = [(1-\epsilon_1^2\sin^2\tfrac{1}{2}K_L)(1-\epsilon_2^2\sin^2\tfrac{1}{2}K_L)]^{1/2} - \epsilon_1\epsilon_2\sin^2\tfrac{1}{2}K_L \ ,$$

$$\sin\tfrac{1}{2}\varphi = \epsilon_1\sin\tfrac{1}{2}K_L(1-\epsilon_2^2\sin^2\tfrac{1}{2}K_L)^{1/2} + \epsilon_2\sin\tfrac{1}{2}K_L(1-\epsilon_1^2\sin^2\tfrac{1}{2}K_L)^{1/2} \ .$$

By comparison with expressions (15), we deduce that

$$\tfrac{1}{2}(K_1 + K_1') = \tfrac{1}{2}(K_2 + K_2') = K_L \ ,$$

and so

$$\theta_L = \tfrac{1}{2}(\theta_1 + \theta_1') = \tfrac{1}{2}(\theta_2 + \theta_2') \ ;$$

that is, the average rotation of the eigenvectors defined at mirror 1 by (1) and (2), or at mirror 2 by (3) and (4), coincides with the rotation of Lamb's average vector defined here.

V. CONCLUSION

In a single-mode laser, subject to an axial magnetic field, with different anisotropic x-y-type losses at each mirror, there always exists in the locking region ($H < H_c$) for central tuning an average polarization vector, real or virtual, which obeys the self-consistent-field theory of Sargent, Lamb, and Fork. Nevertheless, taking into account the localization of the losses in a highly anisotropic system, the polarization vectors at each end of the cavity are different and the angle between them can reach the theoretical value $\frac{1}{2}\pi$. The self-consistent-field theory and the resonance-condition treatment for a round-trip pass appear to be complementary. The nonlinear effects, which have been ignored in the calculation of the polarization, will reappear in the expression of the intensities,[13] which one can evaluate, at first approximation, by the self-consistent method.[1]

[1]M. Sargent III, W. E. Lamb, Jr., and R. L. Fork, Phys. Rev. 164, 436 (1967); 164, 450 (1967).

[2]W. J. Tomlinson and R. L. Fork, Phys. Rev. 164, 466 (1967).

[3]W. Culshaw and J. Kannelaud, Phys. Rev. 141, 228 (1966).

[4]H. Greenstein, Phys. Rev. 178, 585 (1969).

[5]H. De Lang, Ph.D. thesis (University of Utrecht, 1966) (unpublished).

[6]M. I. D'yakonov, Zh. Eksperim. i Teor. Fiz. 49, 1169 (1965) [Sov. Phys. JETP 22, 812 (1966)].

[7]H. Pelikan, Z. Physik 201, 523 (1967).

[8]B. L. Zhelnov and V. S. Smirnov, Opt. i Spektroskopiya 24, 355 (1968)[Opt. Spectry. (USSR) 24, 185 (1968)].

[9]G. Durand, Ph.D. thesis (University of Paris, 1966) (unpublished).

[10]C. H. F. Velzel, Phys. Letters 23, 72 (1966).

[11]W. Van Haeringen, Phys. Rev. 158, 256 (1967).

[12]R. Dändliker, J. Opt. Soc. Am. 58, 1062 (1968).

[13]A. Le Floch and P. Brun, Compt. Rend. 269, 23 (1969).

PHYSICAL REVIEW A VOLUME 4, NUMBER 1 JULY 1971

Density of States and Mobility of an Electron in a Random System of Hard-Core Scatterers

Michael H. Coopersmith

Department of Physics and Center for Advanced Studies, University of Virginia, Charlottesville, Virginia 22901

(Received 20 November 1970)

The mobility and density of states as functions of the energy are computed from the mobility and free energy reported previously by Neustadter and Coopersmith. The results are shown to be equivalent to the recently published calculations by Eggarter and Cohen based on a semiphenomenological model. Some difficulties with both theories are noted and commented upon.

I. INTRODUCTION

The purpose of this paper is to provide a connection between the mobility transition reported by Neustadter and Coopersmith[1] and the model of Eggarter and Cohen[2] for the calculation of the density of states and mobility of an electron in a gas (random system) of hard-core scatterers. When the free energy is computed in a manner analogous to the mobility in Ref. 1, the density of states (which is related to the partition function by an inverse Laplace transform) will be shown to exhibit a tail which is identical to that found in Ref. 2. To avoid further confusion regarding nomenclature, we shall follow Cohen[3] and refer to the average mobility which is a function of temperature as the conductivity. The quantity $\mu(E)$ which occurs in the Kubo-Greenwood formula[4] will be referred to as the mobility. The reader should note that other papers by Cohen and co-workers, notably Ref. 2, are not consistent with this and use the word mobility to refer to both quantities.

The details of the calculation are contained in a number of works by Coopersmith and Neustadter[5-7] and will not be repeated here. However, we shall have to refer to the details as the interpretation of the final results is somewhat different from the original calculations. We begin with a consideration of some of these details.

II. FREE ENERGY AND CONDUCTIVITY (PRELIMINARIES)

The Helmholtz free energy for the system under consideration is given by

$$-\beta F = \ln \operatorname{tr} e^{-\beta \mathcal{H}}, \tag{1}$$

where

$$\mathcal{H} = T_e + V = \frac{p^2}{2m} + \sum_{j=1}^{N} + v(r_{je}),$$

$v(r)$ is the hard-core potential,

$$v(r) = \infty, \quad r < a$$
$$= 0, \quad r > 0 \tag{2}$$

m is the electron mass, and β is the inverse temperature ($\beta = 1/kT$). In the original treatment by the author,[5] the free energy was evaluated using the