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Approximate Solution to the Superradiance Master Equation*

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An approximate solution is given to the master equation describing the cooperative spontaneous emission from a large number of excited two-level atoms. Our solution allows an easy analytical computation of the statistical properties of the radiated optical field. Comparison with previously reported computer results shows a satisfactory agreement. Furthermore, a formal justification is provided to a recent heuristic model of superradiant pulses.

I. INTRODUCTION

Many theoretical papers have recently dealt with the cooperative spontaneous emission from a collection of N excited two-level atoms. The time evolution of the system has been described by a master equation[1–5] for the reduced-density operator involving only atomic variables. An exact solution of the master equation has been derived,[5] but it is so complicated that the relevant features of the phenomenon can hardly be inferred. Computer calculations have been performed[5] for some special cases.

We present here an approximate analytical solution to the master equation. This solution allows a rather easy computation of the statistical properties of the radiated optical field, and reproduces to a good accuracy the computer results presented in Ref. 5 for $N = 200$ and 10 000. Furthermore, it provides a formal justification to an intuitive statistical model[6] of superradiant pulses which gives quantitative results for arbitrary N.

The most interesting feature of the solutions obtained in Refs. 5 and 6 is that, if the N atoms are initially all excited, relative fluctuations in the emitted light intensity $I(t)$ are very large, even at the peak of the average pulse, and do not decrease as the number N is increased. Different conclusions have recently been drawn in other papers[2,7] by using the drastic assumption that the state of the atomic system is represented at every time by a product of single-atom states. This approximation is not appropriate if the atoms are initially all excited because, in this case, correlations between different atoms are no longer negligible.[5,8] Our approach relies essentially on the assumption that N is a very large number, and gives a satisfactory approximation to the exact master equation solution because it does not destroy *a priori* the atomic correlations.

The time-dependent probability distribution $p(m, t)$ for the variable m representing half the population difference between excited and ground state has been shown in Refs. 1–5 to obey the following master equation

$$\dot{p}(m, t) = I_1 [g(m+1)\, p(m+1, t) - g(m)\, p(m, t)]. \quad (1)$$

The constant I_1 is defined in terms of cavity and atomic parameters and represents the radiation rate from a single atom in the cavity. The function $g(m)$ reads

$$g(m) = (r+m)(r-m+1), \quad (2)$$

where $r = N/2$.

Once $p(m, t)$ is known, the average of any ordered product of field operators is evaluated by using a theorem stated in Ref. 4. In particular, the average intensity $\langle I(t) \rangle$ and the associated relative variance $\Delta(t)$ are computed as follows:

$$\langle I(t) \rangle = I_1 \sum_{m=-r}^{r} g(m)\, p(m, t), \quad (3)$$

$$\Delta(t) = \langle I(t) \rangle^{-2} \sum_{m=-r}^{r} g(m)\, g(m-1)\, p(m, t) - 1. \quad (4)$$

In Ref. 6, Eqs. (3) and (4) are evaluated, in the particular case of a fully excited initial state, by taking the rate-equations intensity[4] for large N,

given by

$$I(t, I_0) = \frac{I_0 e^{NI_1 t}}{[1 + (I_0/I_1 N^2)\, e^{NI_1 t}]^2} \,, \tag{5}$$

where I_0 is the initial intensity $I(0)$, and averaging over I_0. The distribution function $p(I_0)$ that brings "quantum fluctuations" into the model is chosen as

$$p(I_0) = (1/\bar{I}_0)\, e^{-I_0/\bar{I}_0}, \tag{6}$$

where $\bar{I}_0 = NI_1$. The function $p(I_0)$ represents for large N a good approximation to the Bose-Einstein distribution typical of the fully excited initial state.

II. APPROXIMATE SOLUTION OF THE MASTER EQUATION

Since in any physical situation the number N is very large, we transform Eq. (1) into a differential equation by putting

$$g(m+1)\,p(m+1) = g(m)\,p(m) + \frac{\partial(gp)}{\partial m} \,. \tag{7}$$

This leads to

$$\frac{\partial p}{\partial t} = I_1 \frac{\partial(gp)}{\partial m} \,. \tag{8}$$

The approximation (7) requires gp to be a function with negligible higher-order derivatives. Keeping p constant and using Eq. (2), we verify that the right-hand side of Eq. (7) is equal to the left-hand side times $[1 + (r^2 - m^2 + r + m)^{-1}]$. This corrective factor is everywhere negligible for large N except for an extremely small region around $m = r$. For physical reasons we may expect the probability distribution p to be a smooth function, slowly decreasing from its peak value. Save for unrealistic initial states, e.g., δ functions, the approximation (7) should therefore hold with an acceptable accuracy. It could be noted that, by retaining the term $\frac{1}{2}(\partial^2/\partial m^2)(gp)$ in Eq. (7), one can allow δ-like initial conditions. However, the solution of the resulting second-order partial differential equations would be as unmanageable as the exact solution of Ref. 5.

The general integral of Eq. (8) is given by

$$p(m, t) = \Phi\left(\frac{r - m + 1}{r + m}\, e^{-t/\tau}\right) g^{-1}(m), \tag{9}$$

where $\tau = (NI_1)^{-1}$, and Φ is an arbitrary function that is specified by choosing the initial conditions and by normalizing p according to

$$\int_{-r}^{+r} p\, dm = 1 \,. \tag{10}$$

It should be stressed that Eq. (8) admits solutions which do not conserve the normalization condition (10), whereas Eq. (1) always grants a time-independent normalization. We shall see however that the solutions we will discuss show negligible deviations from the condition (10) if the ratio t/τ is not too small.

Taking now as initial condition

$$p(m, 0) = \delta(m - k), \tag{11}$$

the solution can be written as

$$p_k(m, t) = \frac{g(k)}{g(m)}\, \delta\left(\frac{r + 1 - r[(r - m + 1)/(r + m)]\, e^{-t/\tau}}{1 + [(r - m + 1)/(r + m)]\, e^{-t/\tau}} - k\right). \tag{12}$$

Except for the case $m = r$, this function satisfies Eq. (10) for any time t.

We must emphasize that the initial condition (11) clearly violates our assumption (7). Therefore the expression (12), though being a solution of Eq. (8), cannot be considered as an approximate solution of the master equation (1). Since Eq. (8) is linear in p, we may however obtain a good approximation of master equation solutions by a superposition of the type

$$p(m, t) = \int_{-r}^{+r} f(k)\, p_k(m, t)\, dk \,, \tag{13}$$

where $f(k)$ is a normalized and well-behaved function. The probability distribution thus derived corresponds to the initial condition $p(m, 0) = f(m)$.

The solution (12) has another useful property. By inserting it into Eq. (3), we find the following average intensity

$$\langle I_k(t) \rangle = I_1 (2r + 1)^2 \frac{[(r - k + 1)/(r + k)]\, e^{t/\tau}}{\{1 + [(r - k + 1)/(r + k)]\, e^{t/\tau}\}^2} \,. \tag{14}$$

This expression turns out to be a solution of the rate equations introduced in Ref. 4, with the initial conditions $I(0) = I_1 g(k)$; $m(0) = k$. The average intensity $\langle I(t) \rangle$ related to the initial distribution $f(k)$ simply reads

$$\langle I(t) \rangle = \int_{-r}^{+r} f(k)\, I_k(t)\, dk \,. \tag{15}$$

III. "FULLY EXCITED" STATE

We can now apply our solution to calculate the time evolution of population inversion and intensity moments starting from specific initial states.

In Refs. 4 and 5 particular interest was devoted to the "fully excited" initial state defined by $p(m, 0) = \delta_{mr}$. As pointed out before, this case can be described in our approximation by smearing out the initial δ function with a suitable probability distribution $f(k)$ corresponding to a very high excitation of the atomic system. If we now compare the average intensity derived from Eqs. (15) and (14) with that derived from Eqs. (5) and (6), we realize that the heuristic model of Ref. 6 produces a particular solution of the master equation (8) with an initial

distribution $f(k)$ given by

$$f(k)=\left(\frac{2r}{r+k}\right)^{2}\exp\left(-2r\,\frac{r-k+1}{r+k}\right)\ . \tag{16}$$

This function is well behaved in the whole range $-r\leq k\leq r$. Furthermore, for large r, it is sensibly different from zero only if k is very close to r, and gives an average value $m(0)=r-1$.

The corresponding time-dependent probability distribution calculated with (12) and (13) is

$$p(m,t)=\left(\frac{2r}{r+m}\right)^{2}\exp\left(-\frac{t}{\tau}-2r\frac{r-m+1}{r+m}\,e^{-t/\tau}\right). \tag{17}$$

It is easy to see that $p(m,t)$, starting from the very narrow initial distribution, broadens and exhibits a maximum which moves towards $m=-r$ as time elapses. For $t\to\infty$ it becomes again a sharply peaked function centered now around $m=-r$. Equation (17) thus gives an analytic form to the computer graphs reported for the fully excited initial state in Fig. 1 of Ref. 5.

We have to mention that $p(m,t)$, as given by Eq. (17), is not a normalized function for short times. This can be seen by carrying out the variable transformation $x=(r-m+1)(r+m)^{-1}e^{-t/\tau}$. The normalization integral is then

$$\int_{-r}^{+r}p(m,t)\,dm=\int_{N^{-1}e^{-t/\tau}}^{\infty}Ne^{(-Nx)}\,dx,$$

which is practically equal to one only for times $t>2\tau$. We shall normalize the probability distribution by putting the lower limit of integration always equal to zero. Of course, our results will not be valid for times shorter than 2τ, but this loss of information is not relevant for large N, since the intensity pulse shows a peak at a time $t_p\approx\tau\ln N$ with a half-width duration $\approx\tau$.

It could seem that there is some arbitrariness in our approximate solution stemming from the choice of the initial distribution. However, in order to approximate with a good accuracy the case of the fully excited initial state, $p(m,0)$ must reproduce the Bose-Einstein probability distribution for the radiated intensity at $t=0$.[9] The expression (17) deduced from the model of Ref. 6 satisfies also this criterion.

We have derived from Eq. (17) the average intensity $\langle I(t)\rangle$ and its relative variance $\Delta(t)$, defined in Eqs. (3) and (4), and the first two moments of the quantity $n(t)=N/2-m(t)$. This quantity represents the number of photons emitted during the time interval between 0 and t. The expressions read

$$\langle I(t)\rangle=I_1N^2[(1+z)H(z)-z]\,, \tag{18}$$

$$\Delta(t)=\frac{2z-z^2+z^3+(6z+6z^2-z^3)H(z)-6[(1+z)H(z)]^2}{6[(1+z)H(z)-z]^2}\,, \tag{19}$$

$$\langle n(t)\rangle=N[1-H(z)]\,, \tag{20}$$

$$\langle\Delta n^2(t)\rangle=\langle n^2(t)\rangle-\langle n(t)\rangle^2=N^2[z-(z+H(z))H(z)]\,, \tag{21}$$

where $z=Ne^{-t/\tau}$, and $H(z)$ is defined by

$$H(z)=ze^{z}\int_{z}^{\infty}y^{-1}e^{-y}\,dy\ .$$

Equations (19), (20), and (21) show that relative fluctuations of $I(t)$ and $n(t)$ are functions of the ratio $\langle n(t)\rangle/N$ only.

A comparison has been performed with the computer graphs reported in Ref. 5 for $N=200$. In this case the peak of the average intensity occurs at the time $t_p=5.62\tau$. We have focused our attention to the region $2\tau\leq t\leq 12\tau$, which includes practically the whole pulse. Deviations from the computer results are within 1% for $\langle I(t)\rangle$ and $\langle n(t)\rangle$. As expected, larger errors are found for the variances $\Delta(t)$ and $\langle\Delta n^2(t)\rangle$, however, they do not exceed 10% up to $t=8\tau$. For larger N, an even better agreement is likely to be found.

IV. SUPERRADIANT INITIAL STATE

We shall now shortly discuss the second example treated in Ref. 5, which is characterized by an initial state centered around $m=0$, with a probability distribution

$$p(m,0)=2^{-N}\binom{N}{N/2-m}\ . \tag{22}$$

A good approximation for large N is

$$p(m,0)=[2/(\pi N)]^{1/2}e^{-2m^2/N}\,. \tag{23}$$

By inserting Eq. (23) into Eq. (15), the average intensity is given by

$$\langle I(t)\rangle=I_1N^2\frac{e^{t/\tau}}{(1+e^{t/\tau})^2}\left[1+O\left(\frac{1}{N}\right)+\cdots\right]\ . \tag{24}$$

Apart from corrective terms of the order $1/N$ we have obtained the rate-equation intensity, as derived from Eq. (14) with $m(0)=0$. A similar conclusion has been reached in Ref. 5 through computer calculations.

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