

then

$$r_i < r_m < 2r_i \left[1 + f / (1 + D_e/E)^{1/2} \right] \equiv r_u. \quad (4)$$

The helium and neon data of CDRB^{2,3} provide both an example of the utility of Eq. (4) and a test of its validity. As an example of the utility, we have used only the high frequency periodicity which these data reveal, together with some intuition⁷ about the ratio r_0/r_m to obtain the bounds r_i and r_u shown in Table

I. In seeking an adequate potential function, these bounds would provide guidance. As it happens, very adequate potential functions are available for these systems^{8,9} as CDRB have shown,^{2,3} and we take the agreement between our bounds and the calculated values as evidence for the validity of Eq. (4).

Note that the "crude" approximation to Eq. (2), $\Delta\theta = \pi/kv_m$, provides a surprisingly accurate rule of thumb.

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¹Excellent reviews, are in *Advances in Chemical Physics* (Wiley, New York, 1967), Vols. X and XII, *Methods of Experimental Physics* (Academic, New York, 1968), Vol. 7A, Chaps. 3 and 4; *Collision Phenomena in Ionized Gases*, edited by Earl W. McDaniel (Wiley, New York, 1964), especially Chaps. 3 and 4.

²R. L. Champion, L. D. Doverspike, W. G. Rich and S. M. Bobbio, *Phys. Rev. A* **2**, 2327 (1970).

³S. M. Bobbio, W. G. Rich, L. D. Doverspike, and R. L. Champion, *Phys. Rev. A* **4**, 957 (1971).

⁴The problem of target size has understandably received much attention. Recent contributions of interest are those of J. R. Luoman, *J. Chem. Phys.* **53**, 129 (1970) who has shown that a rigid sphere potential is capable of reproducing low resolution scattering data rather well

with only small velocity dependence of the sphere radius; and the remarks of R. Duren and C. Schlier, *Discussions Faraday Soc.* **40**, 56 (1965), who point out that, under reasonable approximations, the rainbow impact parameter must exceed r_i , the distance at which the AR potential inflects.

⁵R. B. Bernstein, in *Advances in Chemical Physics* (Wiley, New York, 1966), Vol. X, p. 100.

⁶A similar argument is suggested by Edward A. Mason, Robert J. Munn, and Francis J. Smith in their illuminating article in *Endeavour* **30**, 91 (1971).

⁷H. F. Helbig and W. E. Ferguson, Jr. (unpublished), summarize available information on the noble gas hydride ions and indicate that for these systems $r_0/r_m \geq 0.7$.

⁸L. Wolniewicz, *J. Chem. Phys.* **43**, 1087 (1965).

⁹S. Peyerimhoff, *J. Chem. Phys.* **43**, 998 (1965).

Addendum to "Neutron Scattering from Liquid Helium II at Large Momentum Transfer and the Condensate Fraction" *

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A comparison of the recent experimental data for He II of Cowley and Woods for the shift of the peak of the distribution of the inelastically scattered neutrons from the free-particle value appropriate for large momentum transfer is made with the phenomenological theory of Kerr *et al.*

In a recent paper¹ (hereafter referred to as I), the authors presented a phenomenological theory of the large-momentum-transfer neutron-scattering experiments on liquid He II performed by Cowley and Woods.² More extensive data on these experiments have become available recently.³ Some of those data are relevant to the calculations which

were presented in I, and a further comparison of the theoretical and experimental results is given here.

The theory in I was based on a generalized mean-field form for the density response function, from which the scattering cross section is obtained by the fluctuation-dissipation theorem. The expres-

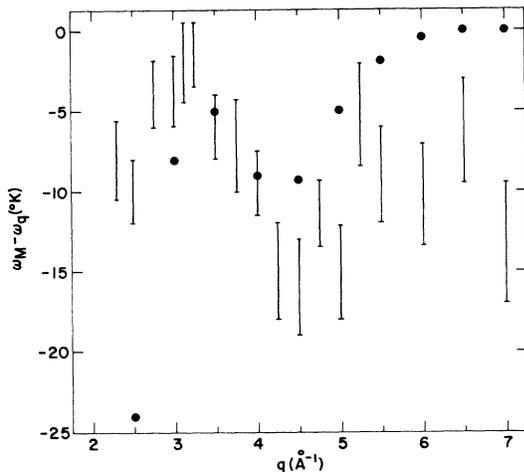


FIG. 1. Frequency (in units of temperature) of the peak of the scattering minus the free-particle recoil frequency $\hbar q^2/2M$ as a function of wave-vector transfer q . The theoretical results are given by the filled circles. The experimental points are taken from Ref. 3.

sion assumed for the density response function had two unknown functions in it; these were determined by requiring that the zeroth, first, and third moments of the approximate cross section be identical with the exact moments.

The cross section $S(q, \omega)$ obtained by this procedure had two distinguishing features in common with the experimental results. First, for a given wave vector q in the range $3 \leq q \leq 9 \text{ \AA}^{-1}$, the frequency ω_M of the position of the peak⁴ in the cross section is at a slightly lower frequency than the free-particle recoil frequency $\omega_q = \hbar q^2/2M$. Secondly, the width of the peak showed oscillations as a function of q . In I most attention was paid to the oscillations in the width of the peak.

In Ref. 3 extensive data on the shift $\omega_M - \omega_q$ are given. Here we give a more detailed comparison with experiment of our calculations for this shift than was given in I.

The width and shift have also been calculated by Sears⁵ by expanding the cross section in a Gram-Charlier series. The first four coefficients of this series can be calculated from the moments of the cross section.

The results in Ref. 3 show that oscillations also occur in the magnitude of the shift. In Fig. 1, the experimental results and the calculation from the theory of I for $\omega_M - \omega_q$ are given. The calculated results assume a condensate fraction of 6% which

was found in I to be the optimum value, and are based on the values of the moments calculated from the pair correlation function of Schiff and Verlet.⁶ (See I for a complete explanation of these details.)

The theory is valid only in the region of q where the strength of the one-phonon scattering has become negligible. This is the case for $q \geq 3 \text{ \AA}^{-1}$, and thus the calculated point at $q = 2.5 \text{ \AA}^{-1}$ shown in Fig. 1 should not be taken very seriously.

It is evident from Fig. 1 that the calculated shift also shows oscillations as a function of q , and that for $3 \leq q \leq 6 \text{ \AA}^{-1}$, there is semiquantitative agreement with experiment. Experimentally ω_M is different from ω_q for all wave vectors up to $q = 9 \text{ \AA}^{-1}$, whereas the calculated shift vanishes for $q > 6 \text{ \AA}^{-1}$. This is because the static structure factor $S(q)$, which is required to be put into the theory, is not known beyond $q = 6 \text{ \AA}^{-1}$, either from experiment or theory, and thus in the calculations $S(q)$ has been set to its asymptotic limit of unity for these larger values of q . Doing that precludes the theory from showing a shift. Any improvement in knowledge of the deviations of $S(q)$ from unity at these q values would lead to an improvement in the calculated value of the shift $\omega_M - \omega_q$. A careful measurement of $S(q)$ for values of q beyond 6 \AA^{-1} is highly desirable.

The structure in both the widths and the shifts is produced in the theory by forcing the approximate expression for the cross section to satisfy the low-order moment relations. These moment relations themselves possess variation as a function of q which then shows up in the properties of $S(q, \omega)$. This structure in the moments is characteristic of high-density liquids with a strong short-range repulsion in the interparticle potential.

Although we were unable to make calculations for nonzero temperatures and in particular for He I, due to our lack of knowledge of the momentum distribution function, it is worth mentioning that it was stated in I that similar oscillations in the width function would also exist in He I. This prediction has been confirmed by the experiments of Cowley and Woods.³

In conclusion, the theory put forth in I possesses all the qualitative features of the experimental results, and in restricted ranges of the wave vector the agreement is semiquantitative. We feel that this strengthens the conclusions reached in I concerning the magnitude of the condensate fraction and the desirability of performing inelastic scattering experiments at very large values of momentum transfer.

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of another (K. S. S.) by the National Science Foundation under Grant No. GP-11054.

¹W. C. Kerr, K. N. Pathak, and K. S. Singwi, Phys. Rev. A **2**, 2416 (1970).

²R. A. Cowley and A. D. B. Woods, Phys. Rev. Let-

ters 21, 787 (1968).

³R. A. Cowley and A. D. B. Woods, *Can. J. Phys.* 49, 177 (1971).

⁴Here we use the same procedure to locate the peak in

$S(q, \omega)$ that was used in Ref. 3, namely, the mean of the points where $S(q, \omega)$ has fallen to half its maximum value.

⁵V. F. Sears, *Phys. Rev. A* 1, 1699 (1970).

⁶D. Schiff and L. Verlet, *Phys. Rev.* 160, 208 (1967).

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Approximate Solution to the Superradiance Master Equation*

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An approximate solution is given to the master equation describing the cooperative spontaneous emission from a large number of excited two-level atoms. Our solution allows an easy analytical computation of the statistical properties of the radiated optical field. Comparison with previously reported computer results shows a satisfactory agreement. Furthermore, a formal justification is provided to a recent heuristic model of superradiant pulses.

I. INTRODUCTION

Many theoretical papers have recently dealt with the cooperative spontaneous emission from a collection of N excited two-level atoms. The time evolution of the system has been described by a master equation¹⁻⁵ for the reduced-density operator involving only atomic variables. An exact solution of the master equation has been derived,⁵ but it is so complicated that the relevant features of the phenomenon can hardly be inferred. Computer calculations have been performed⁵ for some special cases.

We present here an approximate analytical solution to the master equation. This solution allows a rather easy computation of the statistical properties of the radiated optical field, and reproduces to a good accuracy the computer results presented in Ref. 5 for $N=200$ and 10 000. Furthermore, it provides a formal justification to an intuitive statistical model⁶ of superradiant pulses which gives quantitative results for arbitrary N .

The most interesting feature of the solutions obtained in Refs. 5 and 6 is that, if the N atoms are initially all excited, relative fluctuations in the emitted light intensity $I(t)$ are very large, even at the peak of the average pulse, and do not decrease as the number N is increased. Different conclusions have recently been drawn in other papers^{2,7} by using the drastic assumption that the state of the atomic system is represented at every time by a product of single-atom states. This approximation is not appropriate if the atoms are initially all excited because, in this case, correlations between

different atoms are no longer negligible.^{5,8} Our approach relies essentially on the assumption that N is a very large number, and gives a satisfactory approximation to the exact master equation solution because it does not destroy *a priori* the atomic correlations.

The time-dependent probability distribution $p(m, t)$ for the variable m representing half the population difference between excited and ground state has been shown in Refs. 1-5 to obey the following master equation

$$\dot{p}(m, t) = I_1 [g(m+1)p(m+1, t) - g(m)p(m, t)]. \quad (1)$$

The constant I_1 is defined in terms of cavity and atomic parameters and represents the radiation rate from a single atom in the cavity. The function $g(m)$ reads

$$g(m) = (r+m)(r-m+1), \quad (2)$$

where $r = N/2$.

Once $p(m, t)$ is known, the average of any ordered product of field operators is evaluated by using a theorem stated in Ref. 4. In particular, the average intensity $\langle I(t) \rangle$ and the associated relative variance $\Delta(t)$ are computed as follows:

$$\langle I(t) \rangle = I_1 \sum_{m=-r}^r g(m) p(m, t), \quad (3)$$

$$\Delta(t) = \langle I(t) \rangle^{-2} \sum_{m=-r}^r g(m) g(m-1) p(m, t) - 1. \quad (4)$$

In Ref. 6, Eqs. (3) and (4) are evaluated, in the particular case of a fully excited initial state, by taking the rate-equations intensity⁴ for large N ,