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Statistical Properties of Randomly Modulated Laser Beams

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Results were recently reported on statistical properties of phase-modulated laser light. A detailed analysis is presented here of random-phase modulation. In particular it is shown that there are two kinds of such modulation: modulation by phase fluctuations, in which the instantaneous phase is a stationary stochastic process, and modulation by phase diffusion in which the phase is a diffusion process with an always increasing variance. In the case of a Gaussian modulation we show that the optical fields in the two preceding cases are very different, and we calculate their coherence functions. This difference can be detected by an interference experiment, but more easily by photon coincidence or intensity-correlation experiments on a beat between a reference beam and the modulated beams. We present a theoretical analysis of such experiments in the two cases of modulation.

I. INTRODUCTION

In a recent paper[1] Estes, Kuppenheimer, and Narducci (EKN) reported theoretical and experimental results concerning some statistical properties of random-phase-modulated laser light. As phase modulation cannot be detected directly, they performed an experiment in which beats were created between a nonmodulated reference beam and a modulated beam. The optical field obtained by this beat was analyzed by photon-counting techniques, and the experimental results were in excellent agreement with a theoretical analysis using a "semiclassical model of phase modulation due to Glauber."[2]

In their theoretical analysis, as in their experimental setup, the counting time was supposed to be very much shorter than the coherence time of the field, and it is well known that under this condition photon-counting experiments can only give information about the probability distribution of the instantaneous light intensity. For more complete knowledge of the statistical properties of the field, other experiments are necessary. In particular, it is very important to study the optical spectrum, which describes the time evolution of the field, and this can be investigated, for example, by coincidence or intensity-correlation experiments.

Such experiments are also particularly important because there are distinct types of fields which give the same results in photon-counting studies. Indeed, we will show in the following that it is possible to obtain by phase modulation of laser light different fields which cannot be distinguished by photon-counting techniques. Examples of such fields are given in Sec. II in which we define modulation by phase fluctuation or by phase diffusion. In the first case the phase is a stationary stochastic process (sp), while in the second one the phase is a diffusion (i.e., nonstationary) process.

After showing that such fields are quite different we calculate their most important statistical properties. In particular we show that even though the photon-counting statistics are the same, all the higher-order coherence functions are different. To illustrate this point we compute some statistical properties which can be easily obtained by standard optical techniques.

In our analysis we use a classical description of the electromagnetic field which is represented by an appropriate sp. A semiclassical description, as that of EKN, is also possible and even a full quantum treatment. All these descriptions are equivalent according to the equivalence theorem of quantum optics,[3] and for many calculations the classical is the simplest and the shortest.

II. DESCRIPTION OF PHASE MODULATION OF LASER LIGHT

The complex optical field generated by an *ideal* amplitude-stabilized monomode laser is

$$Z(t) = \sqrt{i_0}\; e^{i\omega t}, \tag{1}$$

where i_0 is the light intensity and ω the angular frequency. This field is evidently purely monochromatic, and experimentally there are always slow frequency fluctuations which are neglected, at least in our theoretical discussion.

By phase modulation this field is transformed into

$$Z'(t) = \sqrt{i_0}\; e^{i[\omega t + \Phi(t) + \alpha]}, \tag{2}$$

where $\Phi(t)$ is the instantaneous phase, and α some arbitrary phase difference with $Z(t)$. In the first part of our discussion we assume, to simplify, that $i_0 = 1$ and $\alpha = 0$.

We have a randomly modulated field if $\Phi(t)$ is a sp. It is supposed, for convenience, to have a zero mean value.

Two kinds of modulation are particularly interesting, because easily obtained experimentally.

The first one, used by EKN, is obtained by reflecting the field $Z(t)$ on a (randomly) vibrating mirror, as shown in Fig. 1. If the position of the mirror (for example, the distance to an arbitrary reference system) is $R(t)$ and if the motion of the mirror is slow enough to neglect relativistic and quantum effects, the instantaneous phase of the field is proportional to $R(t)$. Thus the modulated beam can be written as

$$Z_1(t) = e^{i[\omega t + \Phi_1(t)]}, \tag{3}$$

where

$$\Phi_1(t) = X_1(t) = k_1 R(t) . \tag{4}$$

If $R(t)$ is a sp this random modulation is called *modulation by phase fluctuation.*

The second type of modulation is obtained by using a (random) vibration of one of the mirrors defining the cavity of the laser, as indicated in Fig. 2. Thus the function $R(t)$ is now the length of the source cavity. For small variations of $R(t)$ there are variations of the instantaneous frequency which are

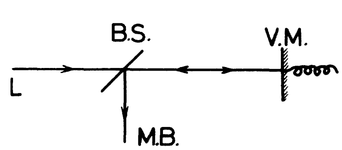

FIG. 1. Principle of modulation by phase fluctuation: L, laser beam; B.S., beam splitter; V. M., vibrating mirror; M.B., modulated beam.

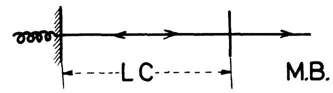

FIG. 2. Principle of modulation by phase diffusion: L. C., laser cavity; M. B., modulated beam.

proportional to $R(t)$, and the modulated beam becomes

$$Z_2(t) = e^{i[\omega t + \Phi_2(t)]}, \tag{5}$$

where

$$\Phi_2(t) = \int_0^t X_2(\theta)\, d\theta \tag{6}$$

and

$$X_2(t) = k_2 R(t) . \tag{7}$$

For random vibrations $R(t)$ this modulation is called *modulation by phase diffusion.* Indeed the instantaneous phase $\Phi_2(t)$ can be a diffusion process, i.e., a process whose variance is an always increasing function of time. In particular this is the case when $X_2(t)$ is a white noise: Thus $\Phi_2(t)$ is a Brownian motion[4] with the variance $\sigma^2_{\Phi_2} = Dt$. The quantum noise of the laser source is described by such a process.[5] In a fully quantum-mechanical description, this modulation corresponds to an harmonic oscillator with random frequency.[6]

It is obvious that if the mirrors in the two modulations *have the same motion* described by the function $R(t)$, *the two fields* $Z_1(t)$ *and* $Z_2(t)$ *are quite different.* But the most interesting question is to know if it is possible to associate to every function $X_1(t)$ another function $X_2(t)$ so that the optical fields obtained, $Z_1(t)$ and $Z_2(t)$, are identical. If this were true, we could conclude that the two kinds of modulation are not different. But we will show in the following that this is not always true: *There are fields obtained by phase diffusion which cannot be obtained by phase fluctuation.*

Defining the instantaneous phases of the fields $\Phi_1(t)$ and $\Phi_2(t)$ by Eqs. (4) and (6), we consider that the fields are equivalent if it is possible to find two different modulation functions $X_1(t)$ and $X_2(t)$ such that

$$X_1(t) = \int_0^t X_2(\theta)\, d\theta + \varphi_0 , \tag{8}$$

where φ_0 is an arbitrary phase.

In the case of nonrandom modulation this equation can be generally satisfied, and this implies that $X_2(t)$ is the derivative of $X_1(t)$. As the modulation is obtained by the vibration of one mirror, we see that if $X_1(t)$ is proportional to the position $R(t)$, $X_2(t)$ becomes proportional to the velocity $v(t) = dR/dt$ of the mirror. But evidently there are cases in which $X_1(t)$ has no derivative, at least mathematically. This occurs, for example, if $X_1(t)$ is a clipped signal with two values only and has sudden changes at time instants t_i.

In the case of random modulation the difference is much stronger, and not at all a mathematical artifice. This fact can be shown very directly. To simplify the discussion let us consider only the case of a zero-mean stationary random modulation. Thus, $X_1(t)$ or $X_2(t)$ are zero-mean second-order stationary sp.[7] This means that $\langle X(t)\rangle = 0$ and $\langle X^2(t)\rangle = \sigma^2$, which are finite and independent of t. Hence, we deduce that $X(t)$ has a finite correlation function,

$$\Gamma_X(\tau) = \langle X(t)\, X(t-\tau)\rangle \ . \tag{9}$$

From Eq. (4) we see that $\Phi_1(t)$ is also a zero-mean stationary sp. On the contrary, the phase $\Phi_2(t)$ given by Eq. (6) is evidently zero mean but not stationary. Moreover let us now show that its variance

$$\sigma^2_{\Phi_2} = \langle \Phi_2^2(t)\rangle = \int_0^t \int_0^t \Gamma_{X_2}(\theta - \theta')\, d\theta\, d\theta' \tag{10}$$

is not always finite. First, we will express this variance in terms of the power spectrum $\gamma_{X_2}(\nu)$ of $X_2(t)$, which is the Fourier transform of $\Gamma_{X_2}(t)$. Thus we obtain from Eq. (10)[8]

$$\sigma^2_{\Phi_2}(t) = \int_{-\infty}^{+\infty} \gamma_{X_2}(\nu)[(\sin\pi\nu t)/\pi\nu]^2\, d\nu \ . \tag{11}$$

If $\gamma_{X_2}(0) \neq 0$, which can be very easily obtained experimentally with noise sources, we deduce from Eq. (11) that for large values of t

$$\sigma^2_{\Phi_2}(t) \simeq t^2 \gamma_{X_2}(0) \int_{-\infty}^{+\infty} [(\sin\pi\nu t)/\pi\nu]^2\, d\nu \ \simeq \gamma_{X_2}(0)\, t \ . \tag{12}$$

Thus, this variance is always increasing, and $\Phi_2(t)$ is a diffusion process which is neither a stationary process, nor a second-order process. Therefore, it is impossible to find a stationary function $X_1(t)$ such that $\Phi_1(t) = \Phi_2(t)$, and the fields obtained by phase fluctuation and phase diffusion cannot be identical. A particular and important example of diffusion process is evidently the Brownian motion, obtained if $\gamma_{X_2}(\nu) = D$; then, for every t

$$\sigma^2_{\Phi_2}(t) = Dt \ , \tag{13}$$

and D is the diffusion constant of the phase.

Nevertheless, there are cases in which the two modulated beams can be identical, but the discussion of this point needs more mathematical arguments, using some particular properties of stationary sp. This discussion is presented in Appendix A, and we give here the main results.

Case a. If $X_2(t)$ has a second-order primitive $P_{X_2}(t)$, Eq. (8) can be verified, and the two modulations are equivalent. Moreover, as for nonrandom functions, $X_2(t)$ is the derivative in the quadratic mean sense of $X_1(t)$. This condition can be stated as a condition for the power spectrum $\gamma_{X_2}(\nu)$ of $X_2(t)$ in the vicinity of the frequency $\nu = 0$. Indeed, if we write

$$\gamma_{X_2}(\nu) = \nu^m f(\nu) \ , \tag{14}$$

where $f(0) \neq 0$, we show in Appendix A[9,10] that $X_2(t)$ has a second-order primitive if, and only if,

$$m > 1 \ . \tag{15}$$

In particular this condition implies that

$$\gamma_{X_2}(0) = 0 \ . \tag{16}$$

Case b. If $X_2(t)$ is a second-order stationary process but without second-order primitive, i.e., if

$$-1 < m < +1 \ , \tag{17}$$

the variance of $\Phi_2(t)$ for large values of t can be written as

$$\sigma^2_{\Phi_2}(t) = \alpha t^{1-m} \ . \tag{18}$$

This shows that $\Phi_2(t)$ is a true diffusion process, with ever increasing variance, and Eq. (8) cannot be verified, because $X_1(t)$ is stationary.

In this case optical fields obtained by phase fluctuation or phase diffusion are completely different, and it becomes interesting to study their differences. In all the following we suppose that Eq. (17) is verified, i.e., that $\Phi_2(t)$ is a true phase diffusion process.

The case of the Browniian motion is evidently obtained for $m = 0$, and Eq. (18) becomes Eq. (13).

If $-1 < m < 0$, the power spectrum $\gamma_{X_2}(\nu)$ is infinite for $\nu = 0$. The limiting case $m = -1$ is evidently the famous flicker noise with $1/\nu$ spectrum. This kind of noise cannot be considered in this study, because its variance is infinite, and it is not described by a second-order sp.

If $0 < m < 1$, $\gamma_{X_2}(0) = 0$, but its derivative is infinite, while if Eq. (15) is true this derivative is equal to zero. Thus it is clear that the character of diffusion process is strongly connected with the structure of the power spectrum in the vicinity of the frequency $\nu = 0$.

III. COHERENCE FUNCTIONS OF FIELDS OBTAINED BY GAUSSIAN PHASE MODULATIONS

We know that the statistical properties of optical fields are completely defined by the whole set of coherence functions, which are the moments of the complex field $Z(t)$. Particularly, the first-order coherence function is

$$\Gamma_Z(\tau) = \langle Z(t)\, Z^*(t-\tau)\rangle \ , \tag{19}$$

the Fourier transform of which is the optical spectrum of the fields. In the case of optical modulation of laser field defined by Eq. (2), calculation of the coherence functions is generally extremely hard to perform. But this becomes quite possible if we assume that the instantaneous phase $\Phi(t)$ is a Gaussian sp. In the case of modulation by phase fluctuation or phase diffusion, this assumption is evidently true if, and only if, $X_1(t)$ [or $X_2(t)$] are also Gaussian sp which is evidently frequently the case for modulation by physical noise. In this section we take for granted this assumption.

Using Eq. (2) we obtain

$$\Gamma_Z(\tau) = e^{i\omega\tau}\langle e^{i\Delta\Phi(t,\tau)}\rangle , \tag{20}$$

where

$$\Delta\Phi(t,\tau) = \Phi(t) - \Phi(t-\tau) , \tag{21}$$

which is obviously also a Gaussian random variable. Hence we deduce[11] that

$$\Gamma_Z(\tau) = e^{i\omega\tau} e^{-\sigma^2_{\Delta\Phi}(\tau)/2} , \tag{22}$$

where $\sigma^2_{\Delta\Phi}(\tau)$ is the variance of $\Delta\Phi(t, \tau)$. It is independent of t, because $X_1(t)$ [or $X_2(t)$] are stationary.

Moreover it is possible to show that all the higher-order coherence functions can be expressed in terms of the first-order one. The detail of the calculation was given in Ref. 11 and the result is

$$\begin{aligned}\Gamma_Z(t_1, \ldots, t_{2n}) &= \langle Z(t_1)\cdots Z(t_n) Z^*(t_{n+1})\cdots Z^*(t_{2n})\rangle \\ &= \Gamma_Z(t_1 - t_{n+1})\cdots\Gamma_Z(t_n - t_{2n}) \\ &\times \prod_{\substack{p:1,n \\ q:p+1,n}} \left| \frac{\Gamma_Z(t_p - t_{n+q})\,\Gamma_Z(t_q - t_{n+p})}{\Gamma_Z(t_p - t_q)\,\Gamma_Z(t_{n+p} - t_{n+q})}\right| .\end{aligned} \tag{23}$$

Therefore, we obtain from Eqs. (22) and (23) that all the statistical properties of Gaussian modulated fields are given by $\Gamma_Z(\tau)$ or by $\sigma^2_{\Delta\Phi}(\tau)$. Thus the comparison of the fields is deduced from a study of $\sigma^2_{\Delta\Phi}(\tau)$.

A. Modulation by Phase Fluctuations

In this case we use Eq. (4) and (21), and we obtain

$$\sigma^2_{\Delta\Phi_1}(\tau) = 2\left[\sigma^2_{X_1} - \Gamma_{X_1}(\tau)\right] , \tag{24}$$

and the first-order coherence function becomes

$$\Gamma_{Z_1}(\tau) = e^{i\omega t}\, e^{-[\sigma^2_{X_1} - \Gamma_{X_1}(\tau)]} . \tag{25}$$

From Eq. (24) we deduce that

$$\sigma^2_{\Delta\Phi_1}(\tau) \leq 4\sigma^2_{X_1} , \tag{26}$$

which gives, from Eq. (22),

$$|\Gamma_{Z_1}(\tau)| \geq e^{-2\sigma^2_{X_1}} . \tag{27}$$

This property is very important, and means that even for large τ, there is always a correlation between $Z_1(t)$ and $Z_1(t-\tau)$. This fact could be verified experimentally. Indeed we know that the mean light intensity obtained in an interference experiment is[12]

$$I_\tau = 2[\Gamma_Z(0) + \mathrm{Re}\,\Gamma_Z(\tau)] , \tag{28}$$

which is in our case

$$I_\tau = 2[1 + |\Gamma_{Z_1}(\tau)| \cos\omega\tau] . \tag{29}$$

Thus Eqs. (27) and (29) show that the interference pattern is decreasing but never vanishing. The visibility of the interference fringes is evidently a function of σ_{X_1}. For a small modulation, this visibility is almost equal to one, while it becomes zero if there is a strong phase-fluctuation modulation.

These properties can be established more precisely. The modulation function $X_1(t)$ is a zero-mean stationary sp. We know that if this process is wide sense ergodic[13]

$$\lim_{\tau\to\infty} \Gamma_{X_1}(\tau) = 0 .$$

Hence we deduce from Eq. (25) that

$$\lim_{\tau\to\infty} \Gamma_{Z_1}(\tau) = e^{-\sigma^2_{X_1} + i\omega t} , \tag{30}$$

which gives, by using Eq. (29), the structure of the interference pattern for large τ, and shows, by Fourier transformation, that the optical spectrum of the modulated beam necessarily has a discrete component on the frequency ω. Thus, the power of the optical field, which is equal to unity because $|Z(t)|^2 = 1$, has a part $e^{-\sigma^2_{X_1}}$ distributed on the frequency ω and another part with a continuous distribution. The shape of the latter distribution depends evidently on the structure of the correlation function $\Gamma_{X_1}(\tau)$.

B. Modulation By Phase Diffusion

In this case, Eq. (11) [see Eq. (38) of Ref. 11], gives

$$\sigma^2_{\Delta\Phi_2}(\tau) = \int_{-\infty}^{+\infty} \gamma_{X_2}(\nu)\left[(\sin\pi\nu t)/\pi\nu\right]^2 d\nu , \tag{31}$$

and by the procedure of Appendix A we deduce that for large values of τ

$$\sigma^2_{\Delta\Phi_2}(\tau) \simeq \alpha\tau^{1-m} , \quad -1 < m < +1 . \tag{32}$$

Thus we have

$$\lim_{\tau\to\infty} \Gamma_{Z_2}(\tau) = 0 , \tag{33}$$

and the spectrum of the optical field has no discrete component at the frequency ω. The shape of this spectrum is evidently a function of $\gamma_{X_2}(\nu)$, the power spectrum of the modulation function $X_2(t)$.

IV. DISCUSSION OF EXPERIMENTAL METHODS FOR STUDYING MODULATED BEAMS

We have seen in Sec. III that fields obtained by phase fluctuation or phase diffusion can be studied by interference techniques, and that the interference patterns are quite different. Nevertheless these methods are not very practicable, because they need in general very long optical delays. The same problem appears in the study of the bandwidth of laser sources, which cannot in general be obtained by interference spectroscopy.

Thus, it seems interesting to use photon coincidence or counting techniques.[14] But these methods cannot be used directly, because, even with phase modulation, the optical field given by Eq. (2) has a constant light intensity, and the effect of the modulation cannot be detected. Therefore, the only way to obtain information on phase fluctuation is to perform, as in the experiment of EKN, an optical beat between the modulated beam and a reference beam. By this method, phase fluctuations are transformed into intensity fluctuations which can be analyzed by photon-statistics techniques.

The optical field obtained by superimposing the reference beam $Z(t)$, Eq. (1), and the modulated beam $Z'(t)$, Eq. (2), is

$$Z_j(t)=\sqrt{i_0}\; e^{i\omega t}\;(1+e^{i[\Phi_j(t)+\alpha]})\,, \tag{34}$$

where $j=1$ or 2 for modulations by phase fluctuation, Eq. (4), or by phase diffusion, Eq. (6).

The instantaneous light intensity of this beat is $|Z_j(t)|^2$ or

$$I_j(t)=2i_0\{1+\cos[\Phi_j(t)+\alpha]\}\,. \tag{35}$$

In their experiment EKN have used a modulation by phase fluctuation $\Phi_1(t)$, but with $\alpha=0$. This means that without modulation there is no constant phase difference between the reference and the modulated beam. Moreover, they used only one laser and a Michelson interferometer with one randomly vibrating mirror. The advantage of this procedure is that Eq. (35) is still valid, even if the laser source is not ideal but has a finite bandwidth due to frequency instabilities: In the interference pattern the frequency fluctuations of the laser source were suppressed.

This is no longer possible in the case of modulation by phase diffusion, because the source itself is modulated, and we are obliged to use two independent sources for the beat. In this case Eqs. (34) and (35) are valid if the fluctuations or the sources are small enough compared to the modulation. There are other methods to cancel fluctuations of the source, and we will discuss this problem in another paper.

Moreover, we see [from Eq. (35)] a significant difference between the two modulations. We have shown that $\Phi_1(t)$ is stationary, while the phase $\Phi_2(t)$ of a diffusion process cannot be stationary (see Appendix B). The same property is valid for $I_j(t)$ which is stationary for phase fluctuation but not for phase diffusion. Thus the experimental study of this case needs in general methods adapted for nonstationary sp, and in particular ensemble averages instead of time averages. These methods will be discussed elsewhere, but we may remark that in the present paper we have calculated ensemble and not time averages. The situation is exactly the same when we use a quantum description of the field.[15]

Now let us calculate some statistical properties of $I_j(t)$ for the two possible modulations. In the following calculations, as in the previous ones, the modulation is supposed to be Gaussian.

A. Mean Value and Variance of the Intensity

From Eq. (35) we deduce the mean value of $I_j(t)$

$$\langle I_j(t)\rangle=2i_0[1+\cos\alpha\,\langle\cos\Phi_j(t)\rangle-\sin\alpha\,\langle\sin\Phi_j(t)\rangle]\,. \tag{36}$$

As $\Phi_j(t)$ is a zero-mean Gaussian sp with variance $\sigma_j(t)$, we have $\langle\sin\Phi_j(t)\rangle=0$ and $\langle\cos\Phi_j(t)\rangle=e^{-\sigma_j^2(t)/2}$. Thus the mean value of the intensity is

$$\langle I_j(t)\rangle=2i_0[1+\cos\alpha\, e^{-\sigma_j^2(t)/2}]\,. \tag{37}$$

In the case of modulation by phase fluctuations $\sigma_1^2(t)=\sigma_1^2$, independent of t. If $\sigma_1=0$, we have the classical interference formula, and if $\sigma_1\to\infty$, the effect of modulation cancels the coherence between the beating beams. In the case of modulation by phase diffusion, $\sigma_2^2(t)$ is an always increasing function of time, and the factor $\cos\alpha\, e^{-\sigma_2^2(t)/2}$ tends to zero.

In order to calculate the variance, we must obtain $\langle I_j^2(t)\rangle$. This is performed in Appendix C, and the result is

$$\sigma_{I_j}^2(t)=4i_0^2[\tfrac{1}{2}(1-e^{-2\sigma_j^2(t)}) - \cos^2\alpha\, e^{-\sigma_j^2(t)}\,(1-e^{-\sigma_j^2(t)})]\,. \tag{38}$$

If there is no modulation $[\sigma^2(t)=0]$, we obviously have no intensity fluctuations. In the case of modulation by phase fluctuations, the variance of the intensity is maximum if $\alpha=\frac{1}{2}(2k+1)\pi$, which is the best value for observing the intensity fluctuations. If $\sigma_j^2(t)\to\infty$, we obtain $\sigma_I^2=2i_0^2$.

B. Photon-Counting Experiments

It is well known that the time instants $\{t_i\}$ at which photoelectrons are emitted by a detector immersed in an optical field constitute a Poisson-compound sp.[16] Thus the probability p_n to obtain n events in the time interval $[t-T, t]$ is given by

$$p_n = \langle e^{-M} M^n/n! \rangle , \tag{39}$$

where M is deduced from the instantaneous light intensity $I(t)$ by

$$M = \alpha \int_{t-T}^{t} I(\theta)\, d\theta . \tag{40}$$

Hence, we obtain from p_n information on the probability distribution of the integrated light intensity. If $T \ll t_c$, which is the correlation time of $I(t)$, we have approximately $M \simeq \alpha I T$, and the effect of integration disappears. In this case we obtain only information on the probability distribution of the random variable $I(t)$. On the other hand, if T is not so small, it is possible to obtain information on the time evolution of $I(t)$, but the calculation of the distribution of M is extremely hard to perform.

By assuming $T \ll t_c$ and taking $a = 2\alpha i_0 T$, we deduce from Eqs. (35) and (39)

$$p_n(t) = (2\pi)^{-1/2}[\sigma(t)]^{-1} \int_{-\infty}^{+\infty} e^{-a[1+\cos(\varphi+\alpha)]} \times \{a^n[1+\cos(\varphi+\alpha)]^n/n!\}\, e^{-\varphi^2/2\sigma^2(t)}\, d\varphi , \tag{41}$$

which is due to the Gaussian distribution of the random variable $\Phi(t)$.

From $\alpha = 0$ this equation must give the same distribution as Eq. (23) of EKN. The algebraic structures of the two equations are very different, and we do not report here the long calculation which shows this identity. Nevertheless for numerical calculation we have performed a direct computer program of Eq. (41), and numerical results are given in the Table I. It is clear that the description of the optical field by an appropriate sp gives a much shorter analysis than the semiclassical calculations of EKN.

The discussion of Eq. (41) is not very simple without a numerical calculation of $p_n(t)$. Nevertheless, we can obtain directly the variance of the photocounts, which is an interesting feature concerning the distribution.

From Eqs. (39) and (40) we deduce that

$$\sigma_N^2 = \alpha T \langle I \rangle + (\alpha T)^2 \sigma_I^2 , \tag{42}$$

where $\langle I \rangle$ and σ_I^2 are the mean value and the variance of the intensity $I(t)$, respectively. The first term in Eq. (42) is the mean value $\langle N \rangle$, and is due to the Poisson distribution; the second one comes from the fluctuations of the light intensity. Thus, we deduce from Eqs. (37) and (38) that

TABLE I. Comparison between some results of quantum calculation (Q.C.) bye EKN and classical calculation (C.C.) by using Eq. (41). The parameter $|\alpha_0|^2$ of EKN is equal to $4\alpha i_0 T$ in our notations.

	$\lvert\alpha_0\rvert^2=1.80$, $\sigma^2=1.15$			$\lvert\alpha_0\rvert^2=1.35$, $\sigma^2=1.55$	
n	Q.C.	C.C.	n	Q.C.	C.C.
0	0.274	0.27449	0	0.404	0.40409
1	0.318	0.31861	1	0.331	0.33068
2	0.226	0.22611	2	0.174	0.17380
3	0.115	0.11509	3	0.066	0.06603
4	0.0458	0.045690	4	0.0196	0.019614
5	0.0149	0.014868	5	0.0048	0.004781
6	0.0041	0.004099	6	0.0010	0.0009877
7	0.00099	0.0009803	7	0.00018	0.0001771
8	0.00021	0.0002070	8	0.00003	0.0000280
9	0.00004	0.0000391	9	0.000004	0.0000040
10	0.000007	0.0000067			
	$\lvert\alpha_0\rvert^2=1.71$, $\sigma^2=2.08$			$\lvert\alpha_0\rvert^2=1.38$, $\sigma^2=2.84$	
n	Q.C.	C.C.	n	Q.C.	C.C.
0	0.366	0.36737	0	0.476	0.47465
1	0.307	0.30685	1	0.299	0.29950
2	0.191	0.19098	2	0.148	0.14870
3	0.0893	0.088892	3	0.0553	0.055675
4	0.0330	0.032789	4	0.0164	0.016555
5	0.0101	0.0099911	5	0.0040	0.0040698
6	0.0026	0.0025905	6	0.00084	0.0008515
7	0.00059	0.00058419	7	0.00015	0.0001549
8	0.00011	0.00011652	8	0.000024	0.0000249
9	0.00002	0.00002083	9	0.000004	0.0000036

$$\sigma_N^2 = u^2\left(e^{-\sigma_j^2(t)} - 1\right) + u + a + \tfrac{1}{2}a^2\left(1 - e^{-2\sigma_j^2(t)}\right) , \quad (43)$$

where $u = a\cos\alpha\, e^{-\sigma_j^2(t)/2}$ and $a = 2\alpha i_0 T$. We note in this expression that for $\sigma_j^2 = 0$, $\sigma_N^2 = \langle N\rangle$ because in this case we obtain a Poisson distribution. On the other hand if $\sigma_j^2 \to \infty$, which can appear in the case of modulation by phase diffusion, $\sigma_N^2 = \langle N\rangle(1 + \frac{1}{2}\langle N\rangle)$. This value gives an indication on the bunching effect of the photoelectrons which appears also in coincidence experiments described in the following. Moreover, for a given value of σ_j^2 it is possible to calculate the value of α which gives the maximum or the minimum of fluctuations of N. There is no general result on this point.

Finally, it is clear that Eqs. (41)–(43) are valid for the two kinds of modulation discussed in this paper, because the only property used in the calculation was the Gaussian property of the phase $\Phi_j(t)$. The only difference is always that for modulation by phase diffusion $\sigma_2^2(t)$ can become infinite. Thus photon-counting experiments do not give direct means to distinguish the two kinds of fields, and other experiments are more appropriate.

C. Coincidence or Intensity-Correlation Experiments

As we have previously noted, the most significant difference between optical fields obtained by the two kinds of modulation is the structure of their coherence function, and particularly the limit for $\tau \to \infty$, as shown by Eqs. (30) and (33). These coherence functions can be measured by means of interference techniques, but not very easily. Now we will show that coincidence or correlation experiments can also be used for this purpose.

The aim of such experiments is to measure the correlation function of the light intensity $I_j(t)$ of the optical beat. As this light intensity is in general not stationary, the result is a covariance function

$$\Gamma_{I_j}(t, \tau) = \langle I_j(t) I_j(t-\tau)\rangle .$$

There are many ways of carrying out such experiments: true coincidences,[17] lifetime measurements,[18] shift register correlations,[19] etc.; but we will not discuss and compare these methods here.

Starting from Eq. (35) the calculation of the covariance function is performed in Appendix C, and the result is

$$\Gamma_{I_j}(t,\tau) = 4i_0^2\big[1 + \cos\alpha\left(e^{-\sigma_j^2(t)/2} + e^{-\sigma_j^2(t-\tau)/2}\right) + \tfrac{1}{2}\cos 2\alpha\, e^{-[\sigma_j^2(t) + \sigma_j^2(t-\tau) + 2\Gamma_j(t,\tau)]/2} + \tfrac{1}{2}e^{-\sigma_{\Delta\Phi_j}^2(\tau)/2}\big] . \quad (44)$$

In this expression $\sigma_j^2(t)$ is the variance of $\Phi_j(t)$, $\Gamma_j(t,\tau)$ its covariance function, and $\sigma_{\Delta\Phi_j}^2(\tau)$ the variance of $\Delta\Phi_j = \Phi_j(t) - \Phi_j(t-\tau)$, as in Eq. (22).

First, let us consider the case of modulation by phase fluctuations, where $\sigma_1^2(t) = \sigma_1^2$ and $\Gamma_1(t,\tau) = \Gamma_{X_1}(\tau)$, the correlation function of the stationary phase $X_1(t)$. From Eq. (44) we deduce that

$$\Gamma_{I_1}(\tau) = 4i_0^2\{1 + 2\cos\alpha\, e^{-\sigma_1^2/2} + \tfrac{1}{2}[\,|\Gamma_Z(\tau)| + \cos 2\alpha\, e^{-\sigma_1^2} e^{-\Gamma_{X_1}(\tau)}]\} . \quad (45)$$

It is easy to express this correlation function only in terms of the coherence function $\Gamma_Z(\tau)$ by using Eq. (25). This expression becomes very simple if $\cos 2\alpha = 0$, and for $\alpha = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$ we obtain

$$\Gamma_{I_1}(\tau) = 4i_0^2[1 \pm \sqrt{2}\, e^{-\sigma_1^2/2} + \tfrac{1}{2}|\Gamma_Z(\tau)|\,] . \quad (46)$$

Equations (45) and (46) show that it is possible to obtain $|\Gamma_Z(\tau)|$ from measurements of $\Gamma_{I_1}(\tau)$. As $\Gamma_Z(\tau) = e^{i\omega t}|\Gamma_Z(\tau)|$, we conclude that the first-order coherence function can be measured by coincidence or intensity-correlation experiments. In the case of thermal light, the solution of the same problem is not so simple, because the modulus of $\Gamma_Z(\tau)$ does not give its phase unambiguously.[20–21]

Moreover, we deduce from Eq. (45) that

$$\lim_{\tau\to\infty} \Gamma_{I_1}(\tau) = \langle I_1\rangle^2 , \quad (47)$$

which shows that the spectrum of $I_1(t)$ has no discrete component, as the spectrum of $Z_1(t)$ [see Eq. (30)].

Finally, an interesting parameter concerning the distributions of the photoelectrons is their bunching, or Hanbury, Brown, and Twiss effect. It can be described by the parameter h defined by[22]

$$h = \Gamma_I(0)/\Gamma_I(\infty) . \quad (48)$$

As $\Gamma_I(0) = \langle I^2\rangle = \sigma_I^2 + \langle I\rangle^2$, we deduce from Eq. (42) that

$$h = 1 + \sigma_{I_1}^2/\langle I_1\rangle^2 , \quad (49)$$

which can be directly obtained from Eqs. (37) and (38).

Let us now consider the case of modulation by phase diffusion. As $\sigma_2^2(t)$ is always increasing, we will study the situation where t is sufficiently large to assume that $\sigma_2^2(t)$ is infinite. So we obtain from Eq. (44)

$$\Gamma_{I_2}(\tau) = 4i_0^2[1 + \tfrac{1}{2}e^{-\sigma_{\Delta\Phi_2}^2(\tau)/2}] , \quad (50)$$

and with Eq. (22)

$$\Gamma_{I_2}(\tau) = 4i_0^2[1 + \tfrac{1}{2}|\Gamma_Z(\tau)|\,] . \quad (51)$$

This relation shows a very simple relation between the first-order coherence function of the field and the correlation function of the light intensity of the beat. Moreover, we can notice that this expression has the same kind of structure as the correlation function obtained for general Gaussian fields studied in Ref. 22. For example, in the case of thermal fields we have

$$\Gamma_I(\tau)=\langle I\rangle^2[1+|\tilde{\Gamma}_Z(\tau)|^2], \tag{52}$$

where $\tilde{\Gamma}_Z(\tau)$ is the normalized coherence function.

Finally, the parameter of the bunching effect deduced from Eq. (51) is

$$h=1,5 \tag{53}$$

while for Gaussian fields we have $2\leq h\leq 3$. This coefficient appears also in the study of the fluctuations of the photoncounts, because, as we have already shown,

$$\sigma_N^2=\langle N\rangle(1+\tfrac{1}{2}\langle N\rangle). \tag{54}$$

Thus, it appears clearly that coincidence experiments constitute the best methods to study modulated beams, and to distinguish modulation by phase fluctuation or by phase diffusion.

In conclusion, we can indicate that the results of this study can be applied to many problems. Artificial modulations of the two kinds can be used for the verification of the theoretical result, as performed by EKN and it would be particularly convenient to measure coherence and correlation functions.

But it is well known that the optical noise of a laser is a phase noise, and we have not yet information to decide if this noise is a modulation by phase fluctuation or diffusion.[23–26] Finally, in some problems of propagation of laser beams in random media, there appear phase-modulation problems, for which the methods of this study can be applied.

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APPENDIX A

Let us consider a zero-mean second-order stationary sp $X(t)$. Its variance can be expressed in terms of the power spectrum by

$$\sigma_X^2=\int_{-\infty}^{+\infty}\gamma_X(\nu)\,d\nu<+\infty. \tag{A1}$$

In particular, if we write $\gamma_X(\nu)$ as

$$\gamma_X(\nu)=\nu^m f(\nu), \tag{A2}$$

where $f(0)\neq 0$, Eq. (A1) means that

$$m>-1. \tag{A3}$$

Thus, the power spectrum can be infinite for $\nu=0$, but only with the condition of Eqs. (A2) and (A3).

For our discussion it is necessary to introduce two linear filters[9] D and P defined by their transfer function $G_D(\nu)$ and $G_P(\nu)$,

$$G_D(\nu)=2\pi i\nu, \tag{A4}$$

$$G_P(\nu)=1/2\pi i\nu. \tag{A5}$$

From Eq. (A4) we see that the filter D associates to every function $X(t)$ its "derivative" $D_X(t)$. For stochastic processes, this derivative is meant in the quadratic mean sense.[10] Conversely the filter P, which is the inverse filter of D, associates to $X(t)$ its "primitive" in the quadratic mean sense $P_X(t)$. For a stationary sp $X(t)$, the derivative or primitive does not always exist. In particular $P_X(t)$ does exist if and only if

$$\int_{-\infty}^{+\infty}[\gamma_X(\nu)/4\pi^2\nu^2]\,d\nu<+\infty. \tag{A6}$$

As Eq. (A1) is verified, this new condition concerns only the values of $\gamma_X(\nu)$ for $\nu=0$. More precisely, if we use Eq. (A2) we see that $P_X(t)$ does exist if $m>1$.

In conclusion we have three cases: (a) $m\leq -1$: $X(t)$ is not a second-order sp; (b) $-1<m<1$: $X(t)$ is second order, but without primitive; (c) $m>1$: $X(t)$ and $P_X(t)$ are second-order sp.

Moreover let us show that if $P_X(t)$ does exist we have, as for the nonstochastic process, the relation between integral and primitive, i.e.,

$$\int_{t_0}^{t_1}X(\theta)\,d\theta=P_X(t_1)-P_X(t_0). \tag{A7}$$

As we have introduced the linear filter $G_P(\nu)$, it is interesting to use the harmonic expansion of the sp $X(t)$ which is

$$X(t)=\int_{-\infty}^{+\infty}dx(\nu)\,e^{2\pi i\nu t}. \tag{A8}$$

Thus we have

$$P_X(t)=\int_{-\infty}^{+\infty}(1/2\pi i\nu)\,dx(\nu)\,e^{2\pi i\nu t}. \tag{A9}$$

By integration of Eq. (A8) and using Eq. (A9) we obtain Eq. (A7) directly.

We can now study more precisely the sp $\Phi_2(t)$ defined by Eq. (6) and describing the instantaneous phase in the case of a modulation by phase diffusion. The question is to know exactly in which case $\Phi_2(t)$ is a diffusion process with an always increasing variance or not.

If $X_2(t)$ has a second-order primitive $P_{X_2}(t)$, we can use Eq. (A7), and thus

$$\Phi_2(t)=P_{X_2}(t)-P_{X_2}(0). \tag{A10}$$

We obtain directly the variance of $\Phi_2(t)$ by

$$\sigma_{\Phi_2}^2(t)=2[\sigma_{P_2}^2-\Gamma_{P_2}(t)]\leq 4\sigma_{P_2}^2, \tag{A11}$$

where $\sigma_{P_2}^2$ and $\Gamma_{P_2}(\tau)$ are, respectively, the variance and correlation function of the stationary second-order sp $P_{X_2}(t)$. This equation shows that $\sigma_{\Phi_2}^2(t)$ is bounded, and thus $\Phi_2(t)$ is not a true diffusion process. Moreover, even if $\Phi_2(t)$ is not stationary, it is clear from Eq. (A10) that $\Phi_2(t)+P_{X_2}(0)$ is stationary. This means that

$$X_1(t)=\int_0^t X_2(\theta)\,d\theta+P_{X_2}(0) \qquad \text{(A12)}$$

is stationary, and therefore, Eq. (8) can be verified. In this case the two modulations are equivalent, and, as for nonrandom functions, $X_2(t)$ is the derivative $D_{X_1}(t)$ of $X_1(t)$.

Now let us consider the case where $X_2(t)$ has no second-order primitive, and then let us calculate the variance of $\Phi_2(t)$.

Using Eqs. (11) and (14) this variance can be written as

$$\sigma^2_{\Phi_2}(t)=\int_{-\infty}^{+\infty}\nu^m f(\nu)[(\sin\pi\nu t)/\pi\nu]^2 d\nu\ , \qquad \text{(A13)}$$

where $f(0)\neq 0$ and $m>-1$. By putting $2\alpha=2-m$ and $x=\pi\nu t$, we obtain

$$\sigma^2_{\Phi_2}(t)=\pi^{2\alpha-3}J(t)\,t^{2\alpha-1}\ , \qquad \text{(A14)}$$

where

$$J(t)=\int_{-\infty}^{+\infty} f\left(\frac{x}{\pi t}\right)\left(\frac{\sin x}{x^\alpha}\right)^2 dx\ . \qquad \text{(A15)}$$

This integral can be divergent for $x\to 0$ or $x\to\infty$. In the neighborhood of $x=0$ we have

$$J_0\simeq f(0)\int_{-\epsilon}^{+\epsilon}[(\sin x)/x^\alpha]^2 dx$$

$$\simeq f(0)\int_{-\epsilon}^{+\epsilon} x^{2(1-\alpha)}\,dx\ , \qquad \text{(A16)}$$

which is convergent if

$$2\alpha<3 \quad\text{or}\quad m>-1\ . \qquad \text{(A17)}$$

This condition, equivalent to Eq. (A3), is satisfied because $X_2(t)$ is a second-order sp.

The integral (A15) is convergent for $x\to\infty$, because Eq. (A1) is verified. Thus $\sigma^2_{\Phi_2}(t)$ is bounded for every t. Now let us calculate the asymptotic value for large t. As $J(t)$ is convergent, we have

$$J(\infty)\simeq f(0)\int_{-\infty}^{+\infty}[(\sin x)/x^\alpha]^2 dx\ , \qquad \text{(A18)}$$

which is convergent if $-1<m<+1$, i.e., if $X_2(t)$ is second order without primitive sp. Thus, we can write

$$\sigma^2_{\Phi_2}(t)\simeq\pi^{-(m+1)}J(\infty)\,t^{1-m}\ , \qquad \text{(A19)}$$

which shows that $\Phi_2(t)$ is a diffusion process. The Brownian motion is a particular case obtained if $m=0$ and $f(\nu)=D$.

APPENDIX B

We assume that $X_2(t)$ has no primitive. If $\Phi_2(t)$ could become stationary as in Eq. (A10), it would be possible to find a random variable F such that

$$F(t)=\Phi_2(t)+F \qquad \text{(B1)}$$

is stationary. From Eq. (6) we deduce that $\Phi_2(0)=0$, and thus

$$F(0)=F\ . \qquad \text{(B2)}$$

Hence we obtain

$$\Phi_2(t)=F(t)-F(0)\ , \qquad \text{(B3)}$$

where $F(t)$ is stationary, which means that

$$\sigma^2_{\Phi_2}(t)=2\,[\sigma_F^2-\Gamma_F(t)]\leq 4\sigma_F^2\ , \qquad \text{(B4)}$$

which is impossible, because of Eq. (18). Thus, $\Phi_2(t)$ cannot become stationary.

APPENDIX C

From Eq. (35) we have

$$\Gamma_I(t,\tau)=\langle I(t)I(t-\tau)\rangle$$

$$=4i_0^2\,[1+\langle\cos\psi(t)\rangle+\langle\cos\psi(t-\tau)\rangle$$

$$+\langle\cos\psi(t)\cos\psi(t-\tau)\rangle] \qquad \text{(C1)}$$

with

$$\psi(t)=\Phi(t)+\alpha\ . \qquad \text{(C2)}$$

As $\Phi(t)$ is Gaussian and zero mean,

$$\langle\cos\psi(t)\rangle=\cos\alpha\ e^{-\sigma^2(t)/2}\ . \qquad \text{(C3)}$$

Moreover we have

$$\langle\cos\psi(t)\cos\psi(t-\tau)\rangle$$

$$=\tfrac{1}{2}\{\langle\cos[\psi(t)+\psi(t-\tau)]\rangle+\langle\cos[\psi(t)-\psi(t-\tau)]\rangle\}\ . \qquad \text{(C4)}$$

The mean values can be written as

$$\langle\cos[\psi(t)+\psi(t-\tau)]\rangle=\langle\cos[\Phi(t)+\Phi(t-\tau)+2\alpha]\rangle$$

$$=\cos 2\alpha\,\langle\cos[\Phi(t)+\Phi(t-\tau)]\rangle$$

$$=\cos 2\alpha\ e^{-[\sigma^2(t)+\sigma^2(t-\tau)+2\Gamma(t,\tau)]/2}\ , \qquad \text{(C5)}$$

and

$$\langle\cos[\psi(t)-\psi(t-\tau)]\rangle=\langle\cos[\Phi(t)-\Phi(t-\tau)]\rangle$$

$$=e^{-\sigma^2_{\Delta\Phi}(\tau)/2}\ . \qquad \text{(C6)}$$

By addition of the different terms we obtain Eq. (44).

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