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Magnetic and Gravitational Energy Release by Resistive Instabilities*

Mark A. Cross† and Gerard Van Hoven
Department of Physics, University of California, Irvine, California 92664
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The resistive magnetohydrodynamic "tearing" and "gravitational-interchange" instabilities are investigated in the linear incompressible limit in the absence of heating. A periodic model is used for the initial magnetic field. The requirement that the spatial Fourier series of the perturbation must converge uniquely determines the growth rate of the instability. At the most unstable wavelength the tearing mode has a growth rate $\omega\tau_r = 3.4S^{0.57}$, where S is the magnetic Reynolds number at the Alfvén velocity and τ_r is the resistive diffusion time. The interchange mode exhibits behavior, at small wavelengths and large gravitational fields, similar to that of the magnetohydrodynamic Rayleigh-Taylor instability. The resistive mode grows for gravitational fields smaller than the equivalent Suydam threshold for infinite-conductivity instability in a sheared magnetic field. Application of these results to the solar-flare problem is briefly discussed, in the context of the temporal behavior exhibited.

I. INTRODUCTION

When one attempts to understand the instability responsible for solar flares one is led to the consideration of resistive mechanisms for the decoupling of magnetic fields and particles.[1,2] These processes allow a transformation into kinetic energy of the abundant magnetic and/or gravitational energy stored in the sunspot-connected fields of "active regions." The resistivity can be that due to collisions in the chromosphere or the higher turbulence-induced value resulting from the "flash-phase" instability.[3]

The basic paper on resistive magnetohydrodynamic instabilities of this kind was written by Furth, Killeen, and Rosenbluth.[4] This massive work treated, in various analytically solvable limits, all of the possible linear resistive instabilities of a basic sheared magnetic field configuration. In addition, the relevance of the theory to laboratory experiments was discussed. The formulation and the solutions were expressed in a unique system of normalized variables which requires careful attention in interpretation. The sheer magnitude of the work and the specialized notation have generated errors, of partial quotation or misquotation, in later citations of the results obtained.

Furth *et al.*[4] were most successful in explaining the driving forces of the instability. However, in view of the manifold possible equilibrium configurations and of the many approximations needed to make the problem tractable analytically, they were not able to describe the general parameter dependence of the instability growth rates.

Wesson[5] treated these instabilities by solving their differential equations on a computer for several different equilibrium configurations of magnetic field and resistivity. He found that the growth rate of the tearing mode depends on wavelength in a different way from that described by Furth *et al.*, but showed that their central approximation is valid. He also investigated the gravitational interchange mode under conditions in which the tearing mode is stable.

Barston[6] put analytic limits on the growth rate in the case of uniform resistivity and uniform shear. Van Hoven and Cross[7] were able to remove the uniform shear condition in the case of the tearing mode. In this paper we investigate the tearing and gravitational modes by a new method which involves a computer search for the unique value of the growth rate which brings about the convergence of the Fourier series describing the perturbations of a spatially periodic equilibrium configuration.

II. BASIC EQUATIONS

We use the coordinate system of Furth, Killeen, and Rosenbluth,[4] a two-dimensional sheet pinch with $\partial/\partial z = 0$. Initially the plasma contains a periodic sheared force-free magnetic field described by

$$\vec{B}_0(r) = B_0[\hat{x}\sin\beta y - \hat{z}\cos\beta y], \qquad (1)$$

where $\beta \sim \pi/a$ of Ref. 4. (This differs from our previous usage in Ref. 7, where β was equivalent to π.)

The gravitational mode requires a density gradient at the neutral sheet where B_x changes sign, so we choose the initial mass density

$$\rho_0(r) = (1 + R\sin 2\beta y)\rho_0, \qquad (2)$$

where ρ_0 and R are constants. Since neither the tearing mode nor the gravitational mode requires a temperature gradient, we assume that T_0 and the resistivity $\eta_0(T_0)$ are uniform.

The equilibrium configuration should have $\vec{v}_0 = 0$; Ohm's law $[\vec{E} + (\vec{v}/c)\times\vec{B} = \eta\vec{J}]$ and Maxwell's equations then require

$$\frac{\partial \vec{B}_0}{\partial t} = \frac{\eta_0 c^2}{4\pi}\frac{\partial^2 \vec{B}_0}{\partial y^2}. \qquad (3)$$

Previous work[4,6] assumes $\partial B_{0x}/\partial y = \text{constant}$, which leads to $B_{0x}\to\infty$ as $y\to\infty$, a difficult boundary consition. There is controversy over whether the tearing mode exists in this case.[7] We choose to satisfy Eq. (3) with the initial magnetic field of Eq. (1), which retains its form during slow resistive decay. We can ignore changes in $\vec{B}_0$ during the growth of the instability if

$$\omega \gg \eta_0 c^2\beta^2/4\pi \sim 1/\tau_r, \qquad (4)$$

where ω is the instability growth rate and τ_r is the resistive diffusion time.

We linearize the magnetohydrodynamic equations, considering perturbations of the form

$$B_{1y}(r,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} B_y(n)\cos n\beta y\,\sin kx\, e^{\omega t} \qquad (5)$$

and

$$v_{1y}(r,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} v_y(n)\sin n\beta y\,\cos kx\, e^{\omega t}, \qquad (6)$$

where the sums run over even n for B_{1y} and odd n for v_{1y}. (More general functions of y were investigated, but only the above form was found to grow exponentially.)

We assume incompressibility, then use the continuity equation and Eq. (2) to get the density perturbation

$$\rho_1(r,t) = -v_{1y}(r,t)\rho_0'(y)/\omega, \qquad (7)$$

where a prime denotes $\partial/\partial y$. There will be no resistivity perturbation if we neglect compressibility and collisional heating.

The linearized equation of motion is

$$\omega\rho_0(y)\vec{v}_1 = \frac{\vec{J}_1\times\vec{B}_0}{c} + \frac{\vec{J}_0\times\vec{B}_1}{c} - \nabla P_1 + \rho_1\vec{g}, \qquad (8)$$

with $\vec{g} = g\hat{y}$. We take the z component of the curl of Eq. (8) to obtain

$$-\omega\rho_0(y)(kv_{1y} + v_{1x}') - \omega v_{1x}\rho_0'(y) = (B_0/4\pi)\sin\beta y[(\beta^2 - k^2)B_{1y} + B_{1y}''] + (gk/\omega)v_{1y}\rho_0'(y). \qquad (9)$$

Maxwell's equations and Ohm's law lead to

$$\omega B_{1y} = -kB_0 v_{1y}\sin\beta y + (\eta_0 c^2/4\pi)(B_{1y}'' - k^2 B_{1y}) \qquad (10)$$

which, if it is used to eliminate B_{1y}'' from Eq. (9), produces a system of coupled equations equivalent to Eqs. (13) and (14) of Ref. 4.

III. METHOD OF SOLUTION

We multiply Eq. (9) by $\sin(n\beta y)$ and Eq. (10) by $\cos(n\beta y)$ and then integrate from $y = -\pi/\beta$ to $y = +\pi/\beta$ to obtain

$$\frac{4\pi\omega\rho_0}{k}(k^2 + n^2\beta^2)v_y(n) = \frac{-4\pi gk\rho_0\beta R}{\omega}[v_y(n+2) + v_y(n-2)] + \tfrac{1}{2}B_0\{[k^2 + \beta^2 n(n-2)]B_y(n-1) - [k^2 + \beta^2 n(n+2)]B_y(n+1)\} \qquad (11)$$

and

$$\omega B_y(n) = \frac{-\eta_0 c^2}{4\pi}(k^2 + n^2\beta^2)B_y(n) + \tfrac{1}{2}kB_0[v_y(n-1) - v_y(n+1)]. \qquad (12)$$

In Eq. (11) we have used $\nabla\cdot\vec{v} = 0 = k\,v_x(n) + n\beta v_y(n)$. $B_y(n)$ can be eliminated by using Eq. (12), after which Eq. (11) reduces to

$$\left[\frac{4p}{\alpha^2 S^2}(\alpha^2 + n^2\pi^2) + F_+(n) + F_-(n)\right]v_y(n) - \left[F_+(n) + \frac{2G}{p}\right]v_y(n+2) + \left[F_-(n) + \frac{2G}{p}\right]v_y(n-2) = 0, \qquad (13)$$

where

$$p = 4\pi^3\omega/\eta_0 c^2\beta^2, \qquad (14)$$

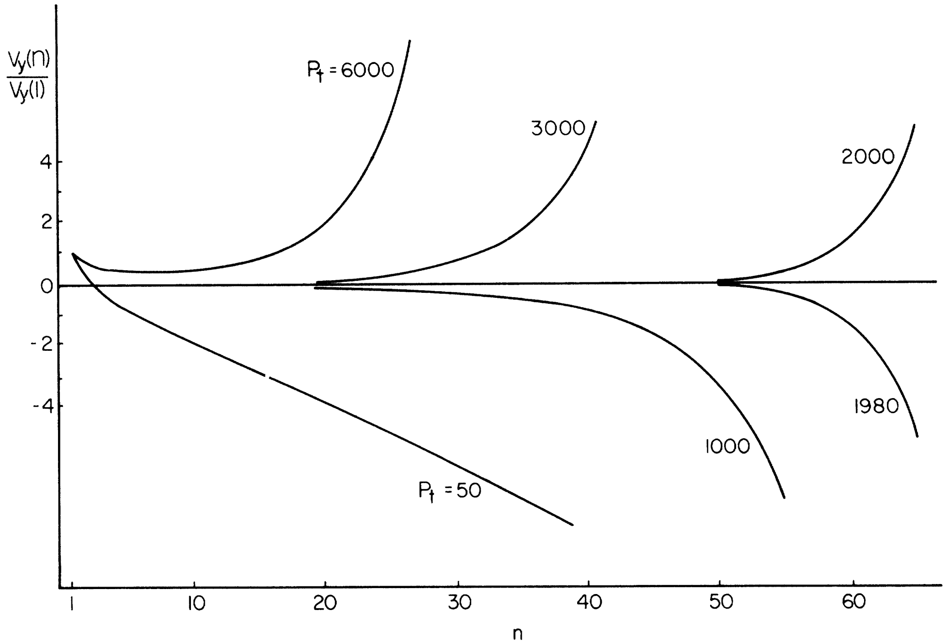

FIG. 1. Convergence of the coefficients of the Fourier series when $S=10^5$, $\alpha=0.1\pi$, and $G=0$.

$$\alpha = \pi k/\beta, \tag{15}$$

$$S = 4\pi^2 B_0/\beta\, \eta_0 c^2 (4\pi\rho_0)^{1/2}, \tag{16}$$

$$F_\pm(n) = \left[\frac{\alpha^2}{\pi^2} + n(n\pm 2)\right] \Big/ \left[\frac{p}{\pi^2} + \frac{\alpha^2}{\pi^2} + (n\pm 1)^2\right], \tag{17}$$

$$G = -8\pi^3 g\rho_0 R/B_0^2\, \beta \tag{18}$$

form the normalized set of parameters used by other authors.[4–7]

The Fourier expansion has decoupled Eqs. (9) and (10). Our trial-and-error method of solution begins by choosing a test growth rate p_t. We then calculate the coefficients of the Fourier series using Eq. (13). Starting with $n=1$ we note that $v_y(-1) = -v_y(1)$ and find that

$$\frac{v_y(3)}{v_y(1)} = \left(\frac{4p}{\alpha^2 S^2}(\alpha^2+\pi^2) + F_+(1) + 2F_-(1) + \frac{2G}{p}\right) \Big/ \left(F_+(1) + \frac{2G}{p}\right), \tag{19}$$

after which we can calculate the ratio of each higher coefficient to $v_y(1)$ by using successively higher n's in Eq. (13). In general $v_y(n)$ gets larger as $n\to\infty$ so the series does not converge. However, convergence is approached as p_t approaches some critical value p_c which can be calculated as accurately as desired. Since the perturbation functions must have a convergent Fourier series, we conclude that p_c is the growth rate of the instability. The Fourier coefficients $v_y(n)$ are shown in Fig. 1 for several trial values of p_t when $\alpha = 0.1\pi$, $G=0$, and $S=10^5$. Convergence is effected as p_t approaches $p_c = 1996$.

We have inverted the convergent Fourier series to obtain the perturbations as functions of y. $B_{1y}(y)$ and v_{1y} exhibit the qualitative form of Fig. 3 in Ref. 5. For large S we find that $B_{1y}(y) \sim |\sin\beta y|$ and $v_{1y}(y) \sim -A|y|/y$, where A is a positive constant.

IV. TEARING MODE

When $G=0$ we obtain the tearing mode. Figure 2 displays the dependence of the growth rate p on wave number α for several values of S. This result agrees qualitatively with the results of Wesson[5] and with our proof[7] that p can be positive only when $0<\alpha<\pi$. Furth *et al.*[4] appear to find several different relations between p and α: $p \propto (S/\alpha)^{2/5}$ for moderate $\alpha<1$, and $p=(\alpha S)^{2/3}$ for small α in their Appendix D.[4,7] The latter result agrees with our Fig. 2 and Wesson's Fig. 1. Also, the Appendix D assumption $p \gg \alpha^2$ is similar to our restriction $p \gg \pi^2$, which follows from Eqs. (4) and (14).

For every S there is a fastest-growing wavelength, which we have plotted in Fig. 3. In Fig. 4 we plot the growth rate at this most unstable wavelength vs S. The straight-line part of Fig. 4 is fit fairly well by

$$p \cong 3.4 S^{0.57} \quad (S > 10^3), \tag{20}$$

the exponent being very close to $\frac{4}{7}$. This is to be compared with the oft quoted result $p \propto S^{0.4}$, and the Appendix D result $p \propto S^{0.5}$ of Ref. 4.

In order to apply these results to the solar-flare problem, we take Spitzer's form[8] for the resistivity

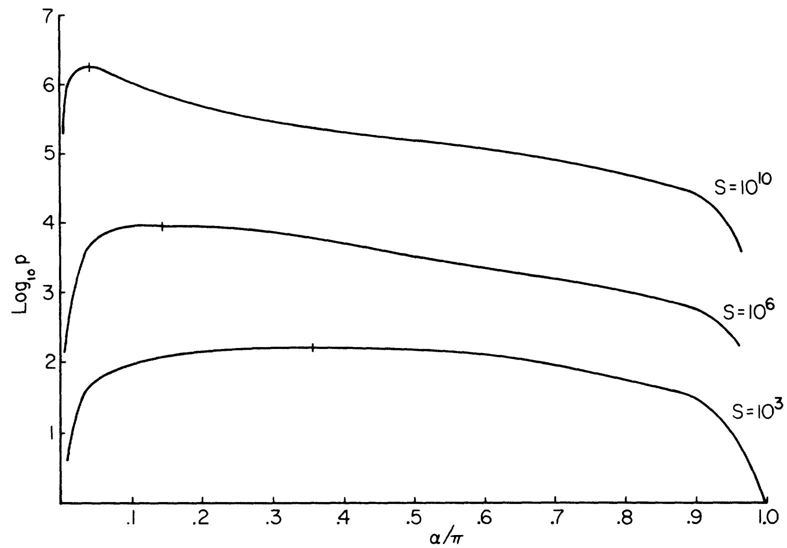

FIG. 2. Wavelength dependence of the growth rate of the tearing mode.

$$\eta c^2 = 1.28 \times 10^{13} T^{-3/2} \ln\lambda \text{ (esu)}, \qquad (21)$$

with $\ln\lambda \sim 10$. Using Eqs. (14) and (16) to return Eq. (20) to the original physical variables,

$$1/\omega \cong 9.2\,[\rho_0^2/(\eta c^2)^3 \beta^{10} B_0^4]^{1/7}, \qquad (20')$$

we can find the e-folding time of the instability. Typical solar-flare parameters,[1,2] $T \sim 10^4 - 10^6$, $B_0 \sim 10^2 - 10^3$, $1/\beta \sim 10^5$, and $n_e \sim 10^{10} - 10^{15}$, then lead to $1/\omega \sim 0.1$–180 sec.

The observed optical development time for a solar flare is usually in the range 1–100 sec. However, the Type-III radio bursts which often accompany flares exhibit characteristic times of the order of one second, which may be more indicative of the initiation of the flash phase. Under favorable circumstances, when the resistivity is high, we find that the tearing mode instability and a solar flare can grow at similar rates. Recent work by Coppi and Friedland[3] investigates conditions under which the resistivity would be greater than the collisional value given by Eq. (21). This would cause the tearing mode to develop even more quickly so that this linear instability could easily

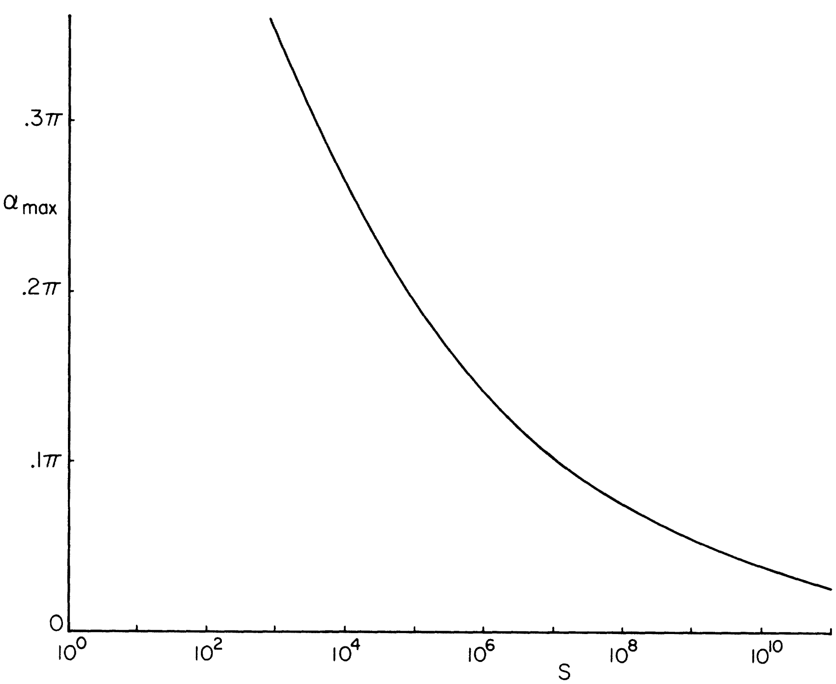

FIG. 3. Most unstable wavelength of the tearing mode as a function of S.

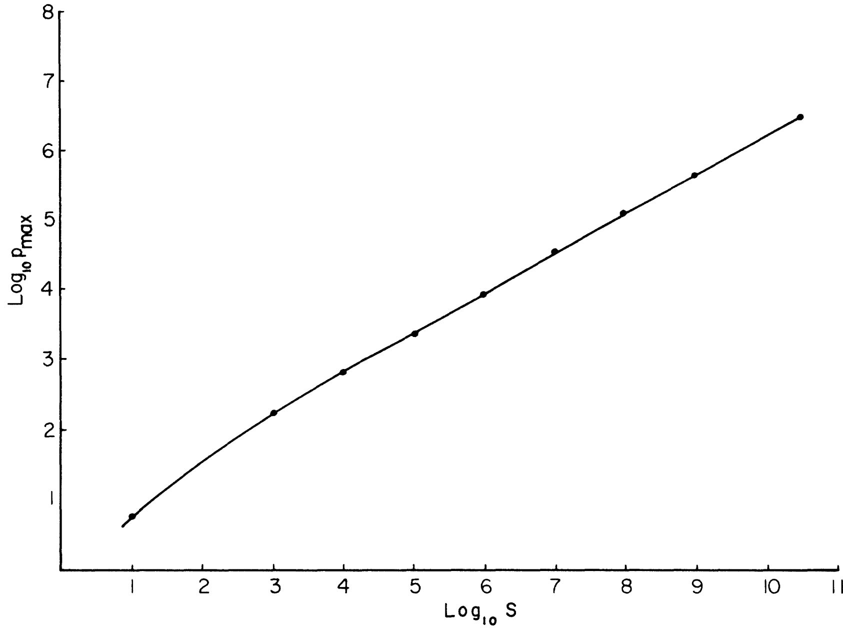

FIG. 4. Growth rate of the most unstable wavelength of the tearing mode.

account for a range of solar-flare time scales without extreme assumptions concerning plasma parameters or magnetic scale lengths.[2]

V. GRAVITATIONAL-INTERCHANGE MODES

In this section we investigate the gravitational instability of Furth *et al.*[4] and its relation to the well-known magnetohydrodynamic Rayleigh-Taylor instability.[9] We will consider both finite and infinite conductivity.

When resistivity is present, Ref. 4 found that the gravitational instability had an infinite sequence of possible growth rates, corresponding to a series of eigenvalues of their parameter Λ. Our trial and error solution of Eq. (13) reaches a similar conclusion. The coefficients of the Fourier series for the four fastest growth rates are shown in Fig. 5 for the case $S = 100$, $\alpha = 10\pi$, and $G = 5$. The Fou-

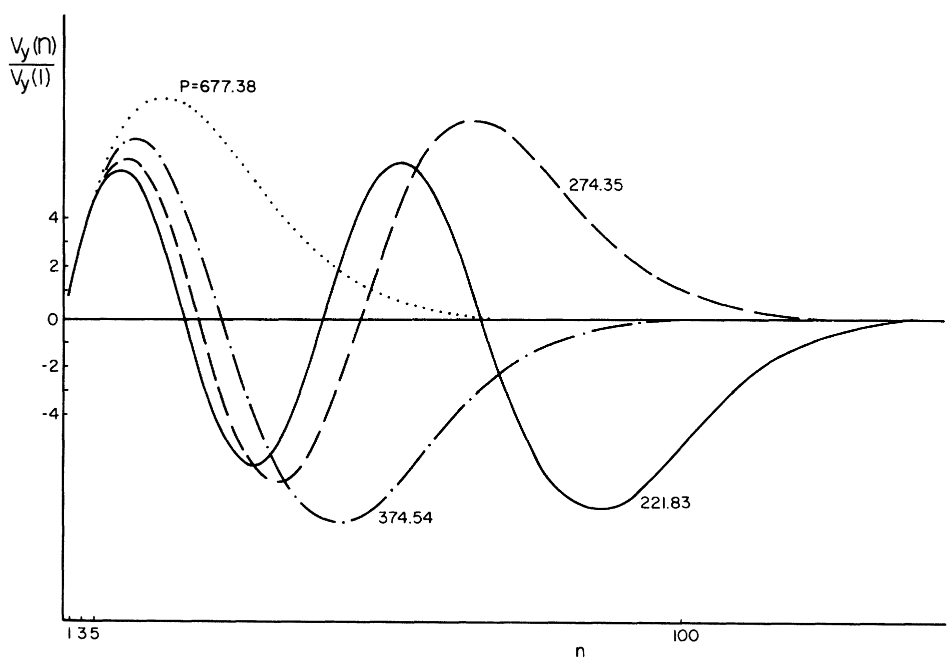

FIG. 5. Convergent Fourier series for four fastest growth rates of the gravitational instability when $S = 10^3$, $\alpha = 10\pi$, and $G = 5$.

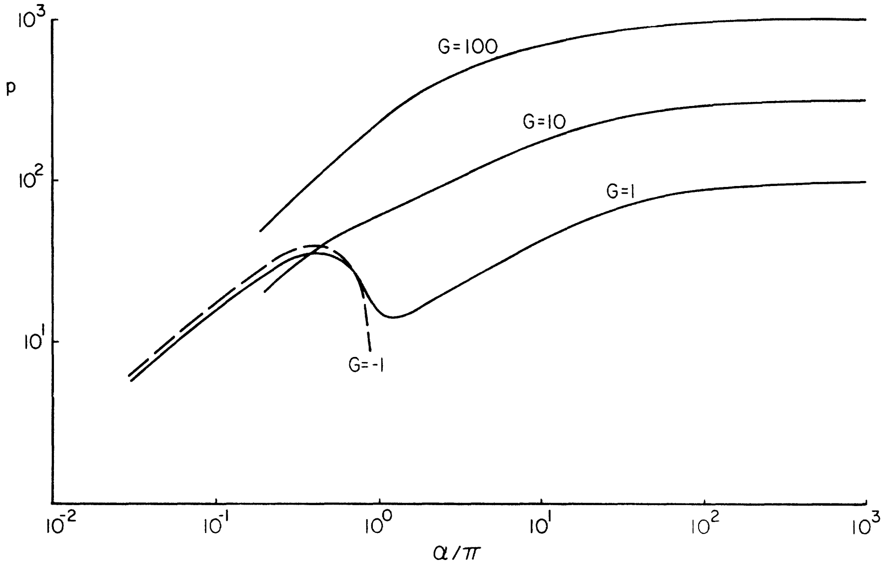

FIG. 6. Wavelength dependence of the growth rate of the gravitational instability when $S=100$.

rier coefficients $v_y(n)$ all have the same sign for the fastest-growing mode, but they cross the axis one more time for each higher-order (more-slowly-growing) mode. The following results concern the fastest growing mode only.

We wish to study the dependence of the growth rate on the parameters α, S, and G. We begin by choosing a particular S, such as $S=100$, and plotting a family of curves $p(\alpha)$ for different G in Fig. 6. The structure at the left-hand side of Fig. 6 is the tearing mode, which is dominant for small G and α. The most unstable region, $\alpha \gg 1$, has p independent of α. This agrees with the results of Appendix C of Furth *et al.*[4]

We can eliminate the parameter α by restricting our attention to the asymptotic parts of the curves on the right-hand side of Fig. 6, for which p is independent of α. This maximum growth rate has the variation $p=SG^{1/2}$, which is the same as that of Appendix C of Ref. 4. Examining the definitions of the dimensionless parameters, Eqs. (13) through (17), we see that the growth rate $p=SG^{1/2}$ is equivalent to

$$\omega=(2g\beta R)^{1/2}=\left(g\frac{d\ln\rho}{dy}\right)^{1/2}_{y=0}, \qquad (22)$$

where β is the inverse of the transverse scale length and $R\approx 1$ defines the density gradient in Eq. (2). This form for the growth rate was also found in work done by Coppi.[1,10]

The magnetohydrodynamic Rayleigh-Taylor gravitational instability has plasma moving across the boundary between denser matter and its supporting magnetic field. The perturbation we have used, Eq. (6), has $v_y=0$ at the $y=0$ boundary. This difference leads us to look for an even velocity mode, described by the perturbation

$$B_{1y}(r,t)=\frac{1}{2}\sum_{-\infty}^{\infty} B_y(n)\,\sin n\beta y\,\sin kx\,e^{\omega t}, \qquad (23)$$

$$v_{1y}(r,t)=\frac{1}{2}\sum_{-\infty}^{\infty} v_y(n)\,\cos n\beta y\,\cos kx\,e^{\omega t}, \qquad (24)$$

for which Eq. (13) is unchanged. Since we now have $v_y(1)=v_y(-1)$, Eq. (19) becomes

$$\frac{v_y(3)}{v_y(1)}=\frac{4p(\alpha^2+\pi^2)/\alpha^2S^2+F_+(1)-2G/p}{F_+(1)+2G/p}. \qquad (25)$$

In the range investigated ($0.1\le S\le 10^6$, $0.1\le G<\infty$) we find that the perturbation in Eqs. (23) and (24) has a short-wavelength growth rate, $p=SG^{1/2}$, which is the same as that for the other perturbation symmetry in Eqs. (5) and (6). We conclude that in the case of the resistive gravitational instability both modes exhibit the same maximum growth rate.

We can investigate the classical magnetohydrodynamic Rayleigh-Taylor instability, which has infinite conductivity, by dropping the resistive terms in Eq. (10). We find that this leads to the minor change of replacing Eq. (17) with

$$F_\pm(n)=[\alpha^2/\pi^2+n(n\pm 2)]\pi^2/p. \qquad (26)$$

Our Eqs. (14) and (16) have p and $S\sim\eta^{-1}$. Although this is a poor choice of dimensionless parameters when $\eta\to 0$, we are able to retain these choices because p and S now occur only in the ratio p/S. One could alternatively define new dimensionless parameters which do not contain η.

In Fig. 7 we compare the infinite conductivity growth rate with the resistive growth rate when $S=100$. (Identically shaped curves are obtained when $S=0.1$ and $S=10^4$, but p changes in magni-

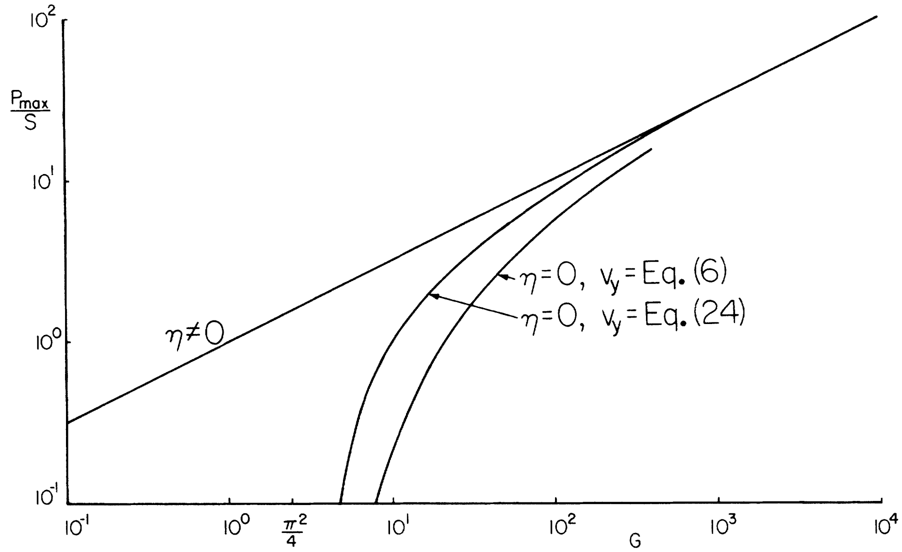

FIG. 7. Finite and zero resistivity growth rates of the gravitational instability.

tude to preserve $p \propto S$.) The infinite conductivity Rayleigh-Taylor mode is only unstable when $G > \pi^2/4 = B_{0x}''/B_{0x}$ as indicated by the gravitational analog of the Suydam criterion. When resistivity is present the instability grows for all positive G.

The gravitational interchange instability has also been proposed as a mechanism for solar flares.[1] In Eq. (22) we find that a transverse scale length on the order of T kilometers gives an e-folding time $1/\omega \sim 1.9T^{1/2}$ sec which can be short enough to act as a trigger,[1] as measured by the Type-III flash, or long enough to contribute to the optical output when the width is greater. There is a difficulty, however, in conceiving how the mass inversion could be assembled, and whether this configuration is stable on the time scale required for its formation.[1]

VI. CONCLUSION

We have tried to clarify some aspects of linear resistive instabilities in a sheared magnetic field. For $G < \pi^2/4$ the resistive gravitational interchange mode is unstable but the magnetohydrodynamic Rayleigh-Taylor mode is stable. At large G there is little difference between the gravitational modes, which all grow at the rate $p = SG^{1/2}$.

Our tearing mode growth rate at the most unstable wavelength, Eq. (20), is offered as a replacement for the oft quoted $p \approx S^{0.4}$ of Ref. 4. Although this previous result has the advantage of being purely analytical, it ignores the wavelength dependence of p.

We have shown that the linear growth rate of the tearing mode can be large enough to apply to a solar flare. It is known that a solar flare heats plasma from roughly 10^6 °K to more than 10^{10} °K. A likely source of this heat, the ηJ^2 term in the energy equation, has been ignored in almost all discussions of the problem.[3] The tearing mode should be investigated with both nonlinear effects and plasma heating included. Perhaps this will shed light on the disagreement over which instability drives solar flares, and on the mechanism for the transition between the flash phase and the main phase.

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†Present address: Physics Department, Grambling College, Grambling, La.

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